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A Calculation Approach to Scalarization for Polyhedral Sets by Means of Set Relations

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Dedicated to the memory of Professor Hang-Chin Lai.

Abstract. In this paper, we focus on certain functions as scalarization for six types of set relations and discuss calculation algorithms for them between polyhedral sets, while those between polytopes have been already investigated. A major difference between polyhedral sets and polytopes is in boundedness. Polyhedral sets are no longer necessarily bounded. Methods for calculating types (1), (2), (4), (6) are easily available by a similar way to existing ideas. However, those for types (3) and (5), which are actually the most famous and long-standing types, require some technical ways approaching to the value of them by using the fact that finitely generatedness and polyhedrality coincide and can be algorithmically switched in finite-dimensional spaces. As a result, we show all types are reduced to a finite number of linear programming problems. Also, we demonstrate our methods through an example and give detailed calculation process.

1. Introduction

What is the importance of scalarization? Scalar is treated as a quantifier of important properties in a vector space such as norm, volume and angle. In set optimization, we are usually required to find a set which precedes others to solve given problems by using a preference relation of sets called a set relation. The concept of set relations was originally stated with six types in [12], and there are some variations of set relations having been studied (e.g., [9]) based on the pointwise ordering of vectors; we use the original six types in this paper. However, it is quite difficult to see which set of two is preferred because a set may consist of infinitely many elements. If scalars characterize set relations, set-to-set comparisons would get far simplified. This is a reason why we should consider scalarization as a characterization of set relations.

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Gerstewitz's scalarization function (see [4, Section 2.3]) is one of well-defined scalarization for vectors with which many researchers have produced various types of characterization functions to scale sets. These functions have been used to describe optimality [1, 7, 10, 13], well-posedness (setness) [3, 6, 17] and so on. Recently, several interesting expressions [5, 15, 16] like oriented distance types or minimax types are considered.

We deal with ones in [13] involving the original six set relations. These functions have been studied as characterizations of set relations in [1, 5, 14] and applied to fuzzy theory in [8]. However, there are few studies on concrete calculation process of the values of the scalarization functions whereas this is of great importance because some authors usually describe many properties on set-valued maps and set optimization by scalar for simplification (see [11]). As a technical approach to calculation of the scalarization functions, polytopes (or, equivalently, bounded polyhedral sets) are tested in [16] by using a minimax pointwise form. It says the values of the functions are obtained by solving a finite number of linear programming problems if the set relations are given between polytopes.

Our aim of the paper is to expand the applicable range of calculation methods in [16] from polytope cases to (not always bounded) polyhedral cases. In Section 2, some basic notions are given. Section 3 is divided into three parts. First, we recall known results in Section 3.1. Our main results and a calculation example are described in Sections 3.2 and 3.3, respectively.

2. Preliminaries

Throughout this paper, let X be a real topological vector space. We write the set of all subsets of X excluding the empty set $\emptyset$ as $\mathcal{P}(X)$. The topological interior, convex hull and convex conical hull of a set $A \subset X$ are denoted by $\operatorname{int} A$, $\operatorname{co} A$ and $\operatorname{cone} A$, respectively.

2.1. Set relations and scalarization functions

Let C be a convex cone in X with $\operatorname{int} C \neq \emptyset$. Then we define the binary relation $\leq_C$ on X induced by C as follows: $x \leq_C y$ if $y - x \in C$ for $x, y \in X$. Since C is a convex cone, this relation $\leq_C$ has reflexivity and transitivity, which means C is a preorder.

Let us define some binary relations between two sets using the relation $\leq_C$ and scalarization functions for sets.

Definition 2.1 (Set relations, [12]). For $A, B \in \mathcal{P}(X)$,

(i) $A \leq_C^{(1)} B \overset{\text{def}}{\iff} \forall\, a \in A,\ \forall\, b \in B,\ a \leq_C b \iff A \subset \bigcap_{b \in B}(b - C)$;

(ii) $A \leq_C^{(2)} B \overset{\text{def}}{\iff} \exists\, a \in A$ s.t. $\forall\, b \in B,\ a \leq_C b \iff A \cap \left(\bigcap_{b \in B}(b - C)\right) \neq \emptyset$;

(iii) $A \leq_C^{(3)} B \overset{\text{def}}{\iff} \forall\, b \in B,\ \exists\, a \in A$ s.t. $a \leq_C b \iff B \subset A + C$;

(iv) $A \le_C^{(4)} B \overset{\text{def}}{\iff} \exists\, b \in B$ s.t. $\forall\, a \in A,\ a \le_C b \iff \left(\bigcap_{a\in A}(a+C)\right) \cap B \neq \emptyset$;

(v) $A \le_C^{(5)} B \overset{\text{def}}{\iff} \forall\, a \in A,\ \exists\, b \in B$ s.t. $a \le_C b \iff A \subset B - C$;

(vi) $A \le_C^{(6)} B \overset{\text{def}}{\iff} \exists\, a \in A,\ \exists\, b \in B$ s.t. $a \le_C b \iff A \cap (B - C) \neq \emptyset$.

Definition 2.2 (Scalarization functions, [13])**.** Let $A, B \in \mathcal{P}(X)$ and $k \in \operatorname{int} C$. For each $i = 1, \dots, 6$, we define a scalarization function $E_{C,k}^{(i)} \colon \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\pm\infty\}$ by

$$E_{C,k}^{(i)}(A, B) := \inf\{t \in \mathbb{R} \mid A \le_C^{(i)} (B + tk)\}.$$

These scalarization functions measure the difference between two given sets with respect to each set relation. By definition, one can easily check that the following inequalities hold:

$$E_{C,k}^{(1)}(A,B) \ge E_{C,k}^{(2)}(A,B) \ge E_{C,k}^{(3)}(A,B) \ge E_{C,k}^{(6)}(A,B),$$
$$E_{C,k}^{(1)}(A,B) \ge E_{C,k}^{(4)}(A,B) \ge E_{C,k}^{(5)}(A,B) \ge E_{C,k}^{(6)}(A,B).$$

2.2. Polyhedral set and finitely generated set

In this part, we introduce basic concepts of polyhedral set and finitely generated set. Let X^* be the topological dual space of X and A° the negative polar cone of $A \subset X$. The set of all $m \times n$ real matrices is written as $M^{m\times n}$.

Definition 2.3 (Polyhedral set)**.** A set $A \subset X$ is said to be polyhedral if $A = \{x \in X \mid \langle p_i, x\rangle \le q_i, i = 1, \dots, m\}$ for some $p_1, \dots, p_m \in X^*$ and $q_1, \dots, q_m \in \mathbb{R}$. In particular, $A \subset \mathbb{R}^n$ is polyhedral if $A = \{x \in \mathbb{R}^n \mid Px \le q\}$ for some $P \in M^{m\times n}$ and $q \in \mathbb{R}^m$.

Definition 2.4 (Finitely generated set)**.** A set $A \subset X$ is said to be finitely generated if $A = \operatorname{co} V + \operatorname{cone} W$ for some finite sets $V, W \subset X$.

We remark that a cone $C \subset X$ is polyhedral if $C = \{x \in X \mid \langle p_i, x\rangle \le 0, i = 1, \dots, m\}$ for some $p_1, \dots, p_m \in X^*$ and is finitely generated if $C = \operatorname{cone} W$ for some finite set $W \subset X$.

The polyhedrality and the finitely generatedness of a set, in fact, coincide with each other in a finite-dimensional space. In the next section, we utilize the transformation of a polyhedral form into a finitely generated form to obtain our main results. Therefore, we introduce here a detailed technique for the transformation, where the following Fourier–Motzkin elimination plays an important role.

Proposition 2.5 (Fourier–Motzkin elimination, e.g., see [2])**.** *Let*

$$\sum_{j=1}^{n} p_{ij} x_j \le q_i \quad \text{for } i = 1, \dots, m$$

be a system of linear inequalities with variables $x_1, \ldots, x_n$*. Then, we can eliminate the variable* x_1 *and turn the system into another one*

$$\sum_{j=2}^{n} p'_{ij} x_j \leq q'_i \quad \text{for } i = 1, \ldots, m'$$

with variables $x_2, \ldots, x_n$ *such that both systems have the same solutions over the remaining variables. In particular,* $q_i = 0$ *for* $i = 1, \ldots, m$ *implies* $q'_i = 0$ *for* $i = 1, \ldots, m'$*.*

Next, we mention two propositions, the proofs of which show concrete steps of transformation leading the procedure of Theorem 3.4 and the example described later.

Proposition 2.6. [18, Theorem 1.3] *A cone* $C \subset \mathbb{R}^n$ *is polyhedral if and only if it is finitely generated.*

Proof. Assume that C is a finitely generated cone. Then, there exist $w_1, \ldots, w_m \in \mathbb{R}^n$ such that

$$C = \left\{ x \in \mathbb{R}^n \;\middle|\; x = \sum_{i=1}^{m} \mu_i w_i, \mu_i \geq 0, i = 1, \ldots, m \right\}.$$

By using the Fourier–Motzkin elimination, we can eliminate the variables $\mu_1, \ldots, \mu_m$ from the system

$$x_j = \sum_{i=1}^{m} \mu_i w_{ij},\ j = 1, \ldots, n \quad \text{and} \quad \mu_i \geq 0,\ i = 1, \ldots, m$$

(where x_j and w_{ij} are the j-th element of x and w_i, respectively) and turn it into a system of homogeneous linear inequalities with variables $x_1, \ldots, x_n$. This means C is a polyhedral cone.

Conversely, assume that C is a polyhedral cone. Then, $C = \{x \in \mathbb{R}^n \mid Px \leq \mathbf{0}\}$ for some $P \in M^{m \times n}$. Now, we define a finitely generated cone $D := \{x \in \mathbb{R}^n \mid x = P^{\mathrm{T}}\mu, \mu \geq \mathbf{0}\}$ and deduce $C = D^\circ$. Since D is a closed convex cone, we have $D = D^{\circ\circ} = C^\circ$ by the bipolar theorem. Hence, C° is finitely generated. As we already know that a finitely generated cone is also a polyhedral cone, it follows C° is polyhedral. From the above argument (the polar of any polyhedral cone is finitely generated), we conclude that $C = C^{\circ\circ}$ is finitely generated. $\square$

Proposition 2.7. [18, Theorem 1.2] *A set* $A \subset \mathbb{R}^n$ *is polyhedral if and only if it is finitely generated.*

Proof. Assume that A is a finitely generated set. Then,

$$A = \left\{ x \in \mathbb{R}^n \;\middle|\; x = \sum_{i \in I} \lambda_i v_i + \sum_{j \in J} \mu_j w_j, \sum_{i \in I} \lambda_i = 1, \lambda_i, \mu_j \geq 0, i \in I, j \in J \right\}$$

for finite sets $\{v_i \mid i \in I\}, \{w_j \mid j \in J\} \subset \mathbb{R}^n$. By using the Fourier–Motzkin elimination, we can eliminate the variables λ_i, μ_j ($i \in I$, $j \in J$) and deduce that A is a polyhedral set.

Conversely, assume that A is polyhedral: $A = \{x \in \mathbb{R}^n \mid \langle p_i, x\rangle \le q_i, i \in I\}$ for finite sets $\{p_i \mid i \in I\} \subset \mathbb{R}^n$ and $\{q_i \mid i \in I\} \subset \mathbb{R}$. Consider a polyhedral cone

$$C_A := \left\{ \begin{pmatrix} x \\ r \end{pmatrix} \in \mathbb{R}^{n+1} \;\middle|\; -r \le 0, \langle p_i, x\rangle - q_i r \le 0, i \in I \right\}.$$

By Proposition 2.6, C_A is finitely generated. Hence,

$$C_A = \left\{ \begin{pmatrix} x \\ r \end{pmatrix} \in \mathbb{R}^{n+1} \;\middle|\; \begin{pmatrix} x \\ r \end{pmatrix} = \begin{pmatrix} \sum_{j\in J} \mu_j w_j \\ \sum_{j \in J} \mu_j d_j \end{pmatrix}, \mu_j \ge 0, j \in J \right\}$$

for finite sets $\{w_j \mid j \in J\} \subset \mathbb{R}^n$ and $\{d_j \mid j \in J\} \subset \mathbb{R}$. Since $r \ge 0$, we have $d_j \ge 0$ for all $j \in J$, and thus $J = J^+ \cup J^0$ where $J^+ := \{j \in J \mid d_j > 0\}$ and $J^0 := \{j \in J \mid d_j = 0\}$. Putting $v_i := (1/d_i)w_i$ and $\lambda_i := \mu_i d_i$ for $i \in J^+$, we obtain

$$C_A = \left\{ \begin{pmatrix} x \\ r \end{pmatrix} \in \mathbb{R}^{n+1} \;\middle|\; \begin{pmatrix} x \\ r \end{pmatrix} = \begin{pmatrix} \sum_{i\in J^+} \lambda_i v_i + \sum_{j \in J^0} \mu_j w_j \\ \sum_{i \in J^+} \lambda_i \end{pmatrix}, \lambda_i, \mu_j \ge 0, i \in J^+, j \in J^0 \right\}.$$

Therefore, we deduce

$$\begin{aligned} A &= \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{pmatrix} x \\ 1 \end{pmatrix} \in C_A \right\} \\ &= \left\{ x \in \mathbb{R}^n \;\middle|\; x = \sum_{i \in J^+} \lambda_i v_i + \sum_{j \in J^0} \mu_j w_j, \sum_{i \in J^+} \lambda_i = 1, \lambda_i, \mu_j \ge 0, i \in J^+, j \in J^0 \right\}, \end{aligned}$$

which means A is finitely generated. □

3. Calculation methods of the scalarization functions

In this section, we discuss how to compute values of the six types of scalarization functions under certain assumptions. Consider a Euclidean space $\mathbb{R}^n$. Assume that C is a polyhedral cone defined as $C := \{x \in \mathbb{R}^n \mid \langle p_l, x\rangle \le 0, l = 1, \dots, m\}$ where $p_1, \dots, p_m \in \mathbb{R}^n$ and let $k \in \operatorname{int} C$.

3.1. Previous results

Proposition 3.1. [16] *Let $A, B \subset \mathbb{R}^n$. Then the following equalities hold:*

(i) $E^{(1)}_{C,k}(A, B) = \sup_{a\in A} \sup_{b \in B} \max_{l=1,\dots,m} \left\langle \dfrac{p_l}{\langle p_l, k\rangle}, a - b \right\rangle$;

(ii) $E^{(2)}_{C,k}(A,B) = \inf_{a\in A} \sup_{b\in B} \max_{l=1,\dots,m} \left\langle \frac{p_l}{\langle p_l, k\rangle}, a-b \right\rangle$;

(iii) $E^{(3)}_{C,k}(A,B) = \sup_{b\in B} \inf_{a\in A} \max_{l=1,\dots,m} \left\langle \frac{p_l}{\langle p_l, k\rangle}, a-b \right\rangle$;

(iv) $E^{(4)}_{C,k}(A,B) = \inf_{b\in B} \sup_{a\in A} \max_{l=1,\dots,m} \left\langle \frac{p_l}{\langle p_l, k\rangle}, a-b \right\rangle$;

(v) $E^{(5)}_{C,k}(A,B) = \sup_{a\in A} \inf_{b\in B} \max_{l=1,\dots,m} \left\langle \frac{p_l}{\langle p_l, k\rangle}, a-b \right\rangle$;

(vi) $E^{(6)}_{C,k}(A,B) = \inf_{a\in A} \inf_{b\in B} \max_{l=1,\dots,m} \left\langle \frac{p_l}{\langle p_l, k\rangle}, a-b \right\rangle$.

A set $A \subset \mathbb{R}^n$ is called a polytope if $A = \operatorname{co} V$ for some finite set $V \subset \mathbb{R}^n$. It is obvious that any polytope is a bounded finitely generated set (and also a bounded polyhedral set by Proposition 2.7).

Proposition 3.2. [16] *Let $A, B \subset \mathbb{R}^n$ be polytopes defined as $A := \operatorname{co}\{a_1, \dots, a_\alpha\}$, $B := \operatorname{co}\{b_1, \dots, b_\beta\}$. For each $h \in \mathbb{N}$, define $I(h) := \{1, \dots, h\}$ and $\Delta^h := \{(\lambda_1, \dots, \lambda_h) \in \mathbb{R}^h \mid \sum_{i=1}^h \lambda_i = 1, \lambda_i \geq 0, i \in I(h)\}$. Then,*

(i) $E^{(1)}_{C,k}(A,B) = \max_{i\in I(\alpha)} \max_{j\in I(\beta)} \max_{l\in I(m)} \left\langle \frac{p_l}{\langle p_l,k\rangle}, a_i - b_j \right\rangle$;

(ii) $E^{(2)}_{C,k}(A,B) = \inf\{t \in \mathbb{R} \mid \langle p_l, k\rangle t + \sum_{i=1}^\alpha \langle p_l, -a_i\rangle \lambda_i \geq \max_{j\in I(\beta)} \langle p_l, -b_j\rangle, l \in I(m),$ *for some* $\lambda \in \Delta^\alpha\}$;

(iii) $E^{(3)}_{C,k}(A,B) = \max_{j\in I(\beta)} \inf\{t \in \mathbb{R} \mid \langle p_l, k\rangle t + \sum_{i=1}^\alpha \langle p_l, -a_i\rangle \lambda_i \geq \langle p_l, -b_j\rangle,$ $l \in I(m),$ *for some* $\lambda \in \Delta^\alpha\}$;

(iv) $E^{(4)}_{C,k}(A,B) = \inf\{t \in \mathbb{R} \mid \langle p_l, k\rangle t + \sum_{j=1}^\beta \langle p_l, b_j\rangle \mu_j \geq \max_{i\in I(\alpha)} \langle p_l, a_i\rangle, l \in I(m),$ *for some* $\mu \in \Delta^\beta\}$;

(v) $E^{(5)}_{C,k}(A,B) = \max_{i\in I(\alpha)} \inf\{t \in \mathbb{R} \mid \langle p_l, k\rangle t + \sum_{j=1}^\beta \langle p_l, b_j\rangle \mu_j \geq \langle p_l, a_i\rangle, l \in I(m),$ *for some* $\mu \in \Delta^\beta\}$;

(vi) $E^{(6)}_{C,k}(A,B) = \inf\{t \in \mathbb{R} \mid \langle p_l, k\rangle t + \sum_{i=1}^\alpha \langle p_l, -a_i\rangle \lambda_i + \sum_{j=1}^\beta \langle p_l, b_j\rangle \mu_j \geq 0,$ $l \in I(m),$ *for some* $\lambda \in \Delta^\alpha, \mu \in \Delta^\beta\}$.

This proposition reveals that the problem to calculate each scalarization function can be decomposed into a finite number of linear programming problems when A and B are polytopes.

3.2. Main results

In this part, we deal with a new case where A and B are polyhedral sets. This paper is a direct generalization of [16] since any polytope is a polyhedral set. Henceforth, let A, B be defined as $A := \{x \in \mathbb{R}^n \mid P_A x \leq q_A\}$, $B := \{x \in \mathbb{R}^n \mid P_B x \leq q_B\}$ where $P_A \in M^{\alpha \times n}$, $P_B \in M^{\beta \times n}$, $q_A \in \mathbb{R}^\alpha$, $q_B \in \mathbb{R}^\beta$.

By Proposition 3.1, we give methods for computing types (1), (2), (4) and (6) of the scalarization functions.

Theorem 3.3. *The value* $E^{(1)}_{C,k}(A,B)$ *can be calculated by solving the following linear programming problems* LP$(1.l)$ *(*$l = 1, \ldots, m$*) with* $(x^{\mathrm{T}}, y^{\mathrm{T}}) \in \mathbb{R}^n \times \mathbb{R}^n$ *and taking the maximum of their* m *optimal values. For* $l = 1, \ldots, m$*,*

$$\mathrm{LP}(1.l) \qquad \textit{Maximize} \quad \left\langle \frac{p_l}{\langle p_l, k\rangle}, x - y \right\rangle \quad \textit{subject to} \quad P_A x \leq q_A,\ P_B y \leq q_B.$$

Next, we attain the value $E^{(2)}_{C,k}(A,B)$ *by solving* LP(2) *with* (t, x^{T})*:*

$$\mathrm{LP}(2) \qquad \begin{array}{l} \textit{Minimize} \quad t \in \mathbb{R} \quad \textit{subject to} \quad t \geq \left\langle \dfrac{p_l}{\langle p_l, k\rangle}, x \right\rangle + \sup\limits_{y \in B} \left\langle \dfrac{p_l}{\langle p_l, k\rangle}, -y \right\rangle \\ \textit{for } l = 1, \ldots, m,\ P_A x \leq q_A. \end{array}$$

Here, we need to solve the following m *linear programming subproblems to fulfill the constraint conditions of* LP(2)*:*

$$\mathrm{LP}(2.l) \qquad \textit{Maximize} \quad \left\langle \frac{p_l}{\langle p_l, k\rangle}, -y \right\rangle \quad \textit{subject to} \quad P_B y \leq q_B.$$

We remark that when the optimal value of LP$(2.l)$ *is infinite for some* l*, so is that of* LP(2) *since the feasible set of it is empty.*

The values $E^{(4)}_{C,k}(A,B)$ *and* $E^{(6)}_{C,k}(A,B)$ *are similarly obtained.*

Finally, let us consider methods for computing types (3) and (5) by converting polyhedral sets into finitely generated sets.

Theorem 3.4. *The value* $E^{(3)}_{C,k}(A,B)$ *can be calculated by the following algorithm.*

Step 1. *By using Propositions* 2.6 *and* 2.7*, convert the polyhedral cone* C *and polyhedral sets* A*,* B *into the following forms:*

- $A = \operatorname{co} V_A + \operatorname{cone} W_A$ *for finite sets* $V_A, W_A \subset \mathbb{R}^n$*;*
- $B = \operatorname{co} V_B + \operatorname{cone} W_B$ *for finite sets* $V_B, W_B \subset \mathbb{R}^n$*;*
- $C = \operatorname{cone} W_C$ *for a finite set* $W_C \subset \mathbb{R}^n$*.*

Step 2. *For each $w' \in W_B$, consider the equation $\sum_{w \in W_A \cup W_C} x_w w = w'$. If all the equations have a solution $\{x_w\}_{w \in W_A \cup W_C} \subset \mathbb{R}_+$, we have* $\operatorname{cone} W_B \subset \operatorname{cone} W_A + \operatorname{cone} W_C$ *and go to* Step 3. *If not, we see* $\operatorname{cone} W_B \not\subset \operatorname{cone} W_A + \operatorname{cone} W_C$ *and conclude* $E^{(3)}_{C,k}(A, B) = +\infty$.

Step 3. *Solve the linear programming problems*

$$\mathrm{LP}(3.v) \qquad \begin{aligned} &\textit{Minimize} \quad t \in \mathbb{R} \quad \textit{subject to} \quad t \geq \left\langle \frac{p_l}{\langle p_l, k \rangle}, x - v \right\rangle \\ &\textit{for } l = 1, \ldots, m, \ P_A x \leq q_A \end{aligned}$$

for $v \in V_B$ to take the maximum of their optimal values, which comes equal to $E^{(3)}_{C,k}(A, B)$.

Proof. We shall prove the following statements:

(i) If $\operatorname{cone} W_B \not\subset \operatorname{cone} W_A + \operatorname{cone} W_C$, $E^{(3)}_{C,k}(A, B) = +\infty$;

(ii) If $\operatorname{cone} W_B \subset \operatorname{cone} W_A + \operatorname{cone} W_C$, $E^{(3)}_{C,k}(A, B) = E^{(3)}_{C,k}(A, V_B)$.

(i) Let $D := \operatorname{cone} W_A + \operatorname{cone} W_C$. Then, we have $x \notin D$ for some $x \in \operatorname{cone} W_B$. Since D is a closed convex cone, by the separation theorem there exists nonzero $p \in \mathbb{R}^n$ such that $\langle p, x \rangle > 0 \geq \langle p, y \rangle$ for all $y \in D$. By the compactness of $\operatorname{co} V_A$, it holds $\operatorname{co} V_A + D \subset sk + D$ for some $s \in \mathbb{R}$. Now, fix $t \in \mathbb{R}$ and $z \in \operatorname{co} V_B$. As $\langle p, x \rangle > 0$, it follows that there exists $s' > 0$ such that $s' \langle p, x \rangle > \langle p, x - z - tk + sk \rangle$. This implies $z + s'x + tk - sk \notin D$ and hence $B + tk \not\subset sk + D$. Therefore, we obtain $B + tk \not\subset A + C$ for all $t \in \mathbb{R}$, which means $E^{(3)}_{C,k}(A, B) = +\infty$.

(ii) Let $t \in \mathbb{R}$. It is sufficient to prove $B + tk \subset A + C \iff V_B + tk \subset A + C$. The necessity of this equivalence is clear. Assume that $V_B + tk \subset A + C$. Then, by the convexity of $A + C$, we have $\operatorname{co} V_B + tk \subset A + C$. Since $\operatorname{cone} W_B \subset \operatorname{cone} W_A + \operatorname{cone} W_C$, it follows $\operatorname{co} V_B + \operatorname{cone} W_B + tk \subset A + \operatorname{cone} W_A + C + \operatorname{cone} W_C$, and thus we obtain $B + tk \subset A + C$.

Step 3 is based on the above statements (ii) and (iii) of Proposition 3.1. □

We can get the value $E^{(5)}_{C,k}(A, B)$ in a similar way.

3.3. Example

As the last part of the paper, we show a calculation sample to demonstrate how it goes with our methods.

To begin with, let

$$P_A := \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \quad q_A := \begin{pmatrix} 2 \\ 2 \\ -3 \\ 4 \\ 1 \\ 1 \end{pmatrix}, \quad P_B := \begin{pmatrix} 1 & 1 & -3 \\ -1 & -1 & 2 \\ -1 & -1 & -1 \\ 2 & -3 & 0 \\ -3 & 2 & 0 \end{pmatrix}, \quad q_B := \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 1 \end{pmatrix},$$

$$p_1 := \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \quad p_2 := \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \quad p_3 := \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \quad k := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and define $A := \{x \in \mathbb{R}^3 \mid P_A x \leq q_A\}$, $B := \{x \in \mathbb{R}^3 \mid P_B x \leq q_B\}$, $C := \{x \in \mathbb{R}^3 \mid \langle p_l, x\rangle \leq 0, l = 1, 2, 3\}$. Note that B is not a polytope as opposed to A because $\{x \in \mathbb{R}^3 \mid x_1 = x_2 = x_3 \geq 1\} \subset B$, that is, B is not compact.

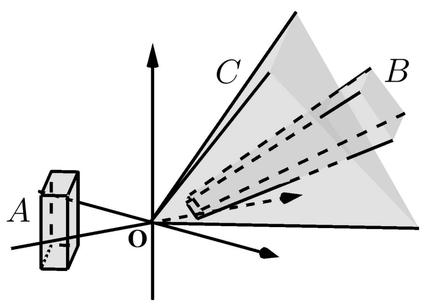

Figure 3.1: Illustration of the sets A, B and cone C.

The value $E^{(1)}_{C,k}(A, B)$ is given by solving LP$(1.l)$ with variables $(x^{\mathrm{T}}, y^{\mathrm{T}}) \in \mathbb{R}^3 \times \mathbb{R}^3$ for $l = 1, 2, 3$.

$$\text{LP}(1.l) \qquad \text{Maximize} \quad \left\langle \frac{p_l}{\langle p_l, k\rangle}, x - y \right\rangle \quad \text{subject to} \quad \begin{pmatrix} P_A & \mathbf{0} \\ \mathbf{0} & P_B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} q_A \\ q_B \end{pmatrix}.$$

The numerical result is indicated in Table 3.1. Here, the symbol Val$(\cdot)$ stands for the optimal value of each specified problem. We derive $E^{(1)}_{C,k}(A, B) = \max\{7/2, -8/3\} = 7/2 > 0$ and hence $A \not\leq^{(1)}_C B$ because assuming $A \leq^{(1)}_C B$ implies $E^{(1)}_{C,k}(A, B) \leq 0$, a contradiction.

Type (1)	LP(1.1)	LP(1.2)	LP(1.3)
Val(·)	$\mathbf{7/2}$	$\mathbf{7/2}$	$\mathbf{-8/3}$
x_1	$-5/2$	$-1/2$	$-5/2$
x_2	$-1/2$	$-5/2$	$-1/2$
x_3	1	1	-1
y_1	$5/4$	$1/2$	1
y_2	$1/2$	$5/4$	$1/3$
y_3	$1/4$	$1/4$	$2/3$

Table 3.1: The optimal solutions and optimal values of LP($1.l$).

In order to obtain $E^{(2)}_{C,k}(A,B)$, we have to solve the two kinds of linear programming problems below.

LP($2.l$) Maximize $\left\langle \frac{p_l}{\langle p_l, k\rangle}, -x \right\rangle$ subject to $P_B x \leq q_B$,

LP(2) Minimize $t \in \mathbb{R}$ subject to $t \geq \left\langle \frac{p_l}{\langle p_l, k\rangle}, x \right\rangle + \mathrm{Val}(\mathrm{LP}(2.l))$

for $l = 1, 2, 3$, $P_A x \leq q_A$.

Table 3.2 shows $E^{(2)}_{C,k}(A,B) = -1/2 < 0$. From this outcome, we deduce $A \leq^{(2)}_C B$ owing to the following property of the set relations: For each $i = 1, \ldots, 6$, $A \leq^{(i)}_C (B + tk)$ for some $t \in \mathbb{R}$ implies $A \leq^{(i)}_C (B + t'k)$ for all $t' \in (t, +\infty)$.

Type (2)	LP(2.1)	LP(2.2)	LP(2.3)	LP(2)
Val(·)	$1/2$	$1/2$	$-2/3$	$\mathbf{-1/2}$
x_1	$5/4$	$1/2$	1	$-3/2$
x_2	$1/2$	$5/4$	$1/3$	$-3/2$
x_3	$1/4$	$1/4$	$2/3$	-1
t	-	-	-	$-1/2$

Table 3.2: The optimal solutions and optimal values of LP($2.l$) and LP(2).

Finally, we consider type (3) by following Theorem 3.4. Let

$$
\begin{aligned}
&a_1 := (-1/2, -5/2, 1)^{\mathrm{T}}, && a_2 := (-5/2, -1/2, 1)^{\mathrm{T}}, && a_3 := (-1, -3, 1)^{\mathrm{T}},\\
&a_4 := (-3, -1, 1)^{\mathrm{T}}, && a_5 := (-1/2, -5/2, -1)^{\mathrm{T}}, && a_6 := (-5/2, -1/2, -1)^{\mathrm{T}},\\
&a_7 := (-1, -3, -1)^{\mathrm{T}}, && a_8 := (-3, -1, -1)^{\mathrm{T}}, && \\
&b_1 := (1, 1/3, 2/3)^{\mathrm{T}}, && b_2 := (1/3, 1, 2/3)^{\mathrm{T}}, && b_3 := (5/4, 1/2, 1/4)^{\mathrm{T}},\\
&b_4 := (1/2, 5/4, 1/4)^{\mathrm{T}}, && b_5 := (6, 9, 5)^{\mathrm{T}}, && b_6 := (9, 6, 5)^{\mathrm{T}},\\
&b_7 := (6, 4, 5)^{\mathrm{T}}, && b_8 := (4, 6, 5)^{\mathrm{T}}, && \\
&c_1 := (1, 1, 0)^{\mathrm{T}}, && c_2 := (1, 0, 1)^{\mathrm{T}}, && c_3 := (0, 1, 1)^{\mathrm{T}}.
\end{aligned}
$$

Step 1. By using Propositions 2.6 and 2.7, we have $A = \mathrm{co}\{a_1, \dots, a_8\} + \mathrm{cone}\{\mathbf{0}\}$, $B = \mathrm{co}\{b_1, \dots, b_4\} + \mathrm{cone}\{b_5, \dots, b_8\}$ and $C = \mathrm{cone}\{c_1, c_2, c_3\}$.

Step 2. It is clear that $\mathrm{cone}\{b_5, \dots, b_8\} \subset \mathrm{cone}\{\mathbf{0}\} + \mathrm{cone}\{c_1, c_2, c_3\}$.

Step 3. For $j = 1, \dots, 4$, consider $\inf_{x\in A} \max_{l=1,2,3} \left\langle \frac{p_l}{\langle p_l, k\rangle}, x - b_j \right\rangle$, that is,

$$
\mathrm{LP}(3.j) \qquad \begin{aligned} &\text{Minimize} \quad t \in \mathbb{R} \quad \text{subject to} \quad t \geq \left\langle \frac{p_l}{\langle p_l, k\rangle}, x - b_j \right\rangle \\ &\text{for } l = 1, 2, 3,\ P_A x \leq q_A. \end{aligned}
$$

According to Table 3.3, $E^{(3)}_{C,k}(A, B) = \max\{-5/3, -5/4\} = -5/4 < 0$. Also, we conclude $A \leq^{(3)}_C B$ in analogy with the result of type (2).

Type (3)	LP(3.1)	LP(3.2)	LP(3.3)	LP(3.4)
Val$(\cdot)$	$\mathbf{-5/3}$	$\mathbf{-5/3}$	$\mathbf{-5/4}$	$\mathbf{-5/4}$
x_1	$-7/6$	$-11/6$	$-9/8$	$-15/8$
x_2	$-11/6$	$-7/6$	$-15/8$	$-9/8$
x_3	-1	-1	-1	-1
t	$-5/3$	$-5/3$	$-5/4$	$-5/4$

Table 3.3: The optimal solutions and optimal values of LP$(3.j)$.

4. Conclusion

In this paper, we have given a new approach to getting values of the scalarization functions for set relations with finitely many linear programming problems. We have investigated a calculation method for the scalarization between polyhedral sets, and it is a natural extension of [16]. As shown in Section 3.3 through an example, one can calculate the values of the functions by following the algorithms stated in Theorems 3.3 and 3.4.

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