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Some Remarks on Dynamical System of Solenoids

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Abstract. We show that a solenoid is a dynamical object and we express its complexity by a number of different entropy-like quantities in Hurley's sense. Some relations between these entropy-like quantities are presented. We adopt the theory of Carathéodory dimension structures introduced axiomatically by Pesin to a case of a solenoid. Any of the above mentioned entropy-like quantities determines a particular Carathéodory structure such that its upper capacity coincides with the considered quantity. We mimic a definition of the local measure entropy, introduced by Brin and Katok for a single map, to a case of a solenoid. Lower estimations of these quantities by corresponding local measure entropies are described.

1. Introduction and preliminaries

A solenoid was defined in mathematics by Vietoris [18] in the late 1920s as an inverse limit space over a circle map and generalized by McCord [11], Williams [20], Smale [15] and others. A solenoid can be presented either in an abstract way, as an inverse limit, or in a geometric way, as nested intersections of solid tori.

Solenoids are compact metrizable spaces that have many unexpected properties. These are connected spaces, but they are neither locally nor path connected spaces. Therefore, they appear in a natural way in continuum theory (see [1, 5, 6]). In the context of smooth dynamics, inverse limits of branched manifolds were considered by Williams [20] and studied as hyperbolic attractors by Smale in his celebrated paper [15]. The reader may find many aspects of solenoids in the recent paper by Sullivan [17].

An inverse limit construction is a powerful tool, therefore it is applied in many branches of mathematics, e.g., in differential geometry (solenoids appear as total spaces of a fiber bundle projection onto a closed manifold with a profinite structure group) or in measure theory. In group theory, the inverse limit construction is related to adding machine and odometers. The inverse limit of a branched covering space mappings of Riemann sphere admits an invariant subspace which is laminated and admits transverse invariant measure (e.g., see papers of Sullivan [16], Lyubich and Minsky [9]).

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In this paper we study a sequence $f_\infty = (f_n \colon X_n \to X_{n-1})_{n=1}^\infty$ of continuous epimorphisms of compact metric spaces X_n, called bonding maps. We assume that all spaces X_n coincide with a compact metrizable space X. By *solenoid* determined by f_∞, we mean the inverse limit

$$X_\infty = \varprojlim X_k = \{(x_k)_{k=0}^\infty : x_{k-1} = f_k(x_k)\}.$$

A solenoid is both a metric space and a dynamical object of a complicated structure. Its complexity stems from the dynamics of bonding maps and can be measured by topological and measure-theoretic entropies which are crucial in understanding classical and generalized dynamical systems. In recent years, many different entropy-like invariants for a single map, based on preimage structure of a map, have been formulated and intensively studied (see [3, 4, 7, 8, 12]).

In the first part of the paper, we generalize a number of different entropy-like invariants (such as: inverse image entropy, preimage relation entropy, point entropy). These invariants were studied by Hurley [4] for a single map. We consider them in the context of a solenoid. In particular, our results generalize the inequalities between entropy-like invariants obtained by Hurley [4].

For a solenoid X_∞ determined by f_∞ and a subset $Z \subset X_0$ we introduce: a branch inverse entropy $h_{\mathrm{Inv}}(f_\infty|Z)$, an inverse entropy $h_{\mathrm{inv}}(f_\infty|Z)$, a preimage relation entropy $h_{\mathrm{pre}}(f_\infty|Z)$ and a point entropy $h_{\mathrm{pt}}(f_\infty|Z)$. The entropy-like quantities are related as follows.

Theorem 1.1. *For any subset $Z \subset X_0$ the following inequalities hold*

(i) $h_{\mathrm{inv}}(f_\infty|Z) \le h_{\mathrm{pre}}(f_\infty|Z)$,

(ii) $h_{\mathrm{Inv}}(f_\infty|Z) \le h_{\mathrm{inv}}(f_\infty|Z) + h_{\mathrm{pt}}(f_\infty|Z)$.

In the second part of the paper, we investigate dynamics of solenoids from the point of view of dimension theory. Our main tool is the so-called Carathéodory structure which has been introduced axiomatically by Pesin [13]. A Carathéodory dimension structure is a generalization of the Hausdorff measure and dimension. We adopt the theory of Carathéodory structures to the case of a solenoid and prove that any entropy-like quantity mentioned above is determined by a particular Carathéodory structure. The upper capacity of the corresponding structure coincides with the considered entropy-like quantity. On the space X_0 we introduce a number of natural distinguished metrics related directly to entropy-like quantities. Each of these distinguished metrics, say a metric ρ_a, determines a dimensional-like quantity called an upper capacity $\overline{\mathrm{CP}_{\rho_a}}(\cdot)$. In particular, we get

Theorem 1.2. *For any subset $Z \subset X_0$ we have*

$$h_{\text{pre}}(f_\infty|Z) = \overline{\text{CP}}_{\rho_{\text{adj}}}(Z),$$
$$h_{\text{Inv}}(f_\infty|Z) = \overline{\text{CP}}_{\rho_b}(Z),$$
$$h_{\text{inv}}(f_\infty|Z) = \overline{\text{CP}}_{\rho_{bH}}(Z).$$

Combining Theorem 1.1 with Theorem 1.2 we get inequalities between capacities corresponding to entropy-like quantities determined by different metrics. In particular,

Corollary 1.3. *For any subset $Z \subset X_0$ the following inequality holds*

$$\overline{\text{CP}}_{\rho_{bH}}(Z) \leq \overline{\text{CP}}_{\rho_{\text{adj}}}(Z).$$

In the third part of the paper, we mimic the definition of Brin and Katok [2] of a local measure entropy to get local measure entropies for a solenoid. Thus, for a Borel probability measure ν on X_0 and a distinguished metric ρ_a we define a local measure entropy $\underline{h}^{\rho_{\rho_a}}_{\text{loc},\nu}$ of the solenoid and obtain the following estimations.

Theorem 1.4. *Let ν be a Borel probability measure on X_0. Let Z be a Borel subset of X_0 with $\nu(Z) > 0$. Then we have*

$$h_{\text{pre}}(f_\infty|Z) \geq \operatorname{ess\,inf} \underline{h}^{\rho_{\text{adj}}}_{\text{loc},\nu},$$
$$h_{\text{Inv}}(f_\infty|Z) \geq \operatorname{ess\,inf} \underline{h}^{\rho_b}_{\text{loc},\nu},$$
$$h_{\text{inv}}(f_\infty|Z) \geq \operatorname{ess\,inf} \underline{h}^{\rho_{bH}}_{\text{loc},\nu},$$

where $\operatorname{ess\,inf} \underline{h}^{\rho_a}_{\text{loc},\nu}$ *stands for the essential infimum of the local measure entropy function* $\underline{h}^{\rho_a}_{\text{loc},\nu}$.

Remark 1.5. The proof of Theorem 1.4 was inspired by Theorem 1 in [10] by Ma and Wen who related the lower measure entropy of a single continuous map $f\colon X \to X$ of a compact metric space (X, d) with a dimensional type characteristic of the dynamical system.

2. Entropy-like quantities for a solenoid

Let $\mathbb{N}$ denote the set of nonnegative integers. Fix a compact metrizable space X and assume (X_n, d_n), where $n \in \mathbb{N}$ and $X_n = X$, is a sequence of compact metric spaces. Consider a sequence of continuous epimorphisms $f_\infty = (f_n\colon X_n \to X_{n-1})_{n=1}^\infty$, called bonding maps.

By a *solenoid* determined by f_∞, we mean the inverse limit

$$X_\infty = \varprojlim X_k = \{(x_k)_{k=0}^\infty : x_{k-1} = f_k(x_k)\}.$$

Clearly, X_∞ is a compact subset of the Hilbert cube ΠX_k. A distance function d_∞ on X_∞ is given by the usual formula

$$d_\infty((x_k),(u_k)) = \sum_{k=0}^{\infty} \frac{1}{2^k} d_k(x_k,u_k).$$

Since X_∞ is uniquely determined by f_∞, we will often identify these two objects. Moreover, define a sequence of maps $(g_k)_{k=0}^{\infty}$, $g_k\colon X_k \to X_0$ by

$$g_0 = \mathrm{id}_{X_0}, \quad g_k = f_1 \circ f_2 \circ \cdots \circ f_k \quad \text{for } k \geq 1.$$

We mimic definitions of entropy-like invariants, introduced by Hurley [4] for a single map, and we define several entropy-like quantities to measure complexity of a solenoid f_∞.

Branch inverse entropy. Let $Z \subset X_0$, $z_0 \in Z$. In this approach we focus on the growth rates of inverse images $g_n^{-1}(z_0)$. It is convenient to introduce a notion of a tree.

Denote by $[z_0,z_1,z_2,\ldots,z_n]$ a finite sequence of points such that $z_k \in X_k$ for $k = 0,1,2,\ldots,n$, and $f_k(z_k) = z_{k-1}$ for $k = 1,2,\ldots,n$. Observe that every such sequence can be extended (but not uniquely) to a member of the solenoid X_∞.

The *tree* $T_n(z_0)$ of inverse images of $z_0 \in Z$, up to the order n, is the set

$$\begin{aligned} T_n(z_0) &= \bigcup_{k=0}^{n} g_k^{-1}(z_0) \\ &= \{[z_0,z_1,z_2,\ldots,z_n] : z_k \in X_k \text{ and } f_k(z_k) = z_{k-1}\}. \end{aligned}$$

We also call $T_n(z_0)$ a tree over z_0. Every sequence $[z_0,z_1,z_2,\ldots,z_n]$, a member of $T_n(z)$, is called a *branch*. The number n is called a *length* of a branch. Since the sequence f_∞ is a sequence of epimorphisms, we obtain that $T_n(z) \neq \emptyset$.

In the set $T_n = \bigcup_{z\in Z} T_n(z)$ of all trees of length n over Z, we introduce a branch metric. For two branches $b_1 = [z_0,z_1,\ldots,z_n]$ and $b_2 = [w_0,w_1,\ldots,w_n]$ of the same length, we put

$$d_b(b_1,b_2) = \max\{d_k(z_k,w_k) : k = 0,1,\ldots,n\}.$$

We say that branches $b_1,b_2 \in T_n$ are (n,ε)-separated with respect to the branch metric d_b if $d_b(b_1,b_2) \geq \varepsilon$. Denote by $s_{\mathrm{Inv}}(n,\varepsilon,Z)$ the maximal cardinality of all (n,ε,d_b)-separated subsets of T_n. From compactness of X_n, it follows that $s_{\mathrm{Inv}}(n,\varepsilon,Z)$ is finite. Therefore, we may define a *branch inverse* entropy of f_∞, restricted to Z, by the formula

$$h_{\mathrm{Inv}}(f_\infty|Z) := \lim_{\varepsilon\to 0^+} \limsup_{n\to\infty} \frac{1}{n} \log s_{\mathrm{Inv}}(n,\varepsilon,Z).$$

Inverse entropy. Let d_{bH} be the Hausdorff metric based on the metric d_b, called a *branch-Hausdorff metric*. Notice that, for $z, w \in Z$, the distance $d_{bH}(T_n(z), T_n(w)) < \varepsilon$ if and only if, for any two branches $b \in T_n(z)$ and $c \in T_n(w)$, there exist branches $b' \in T_n(w)$ and $c' \in T_n(z)$ such that

$$d_b(b, b') < \varepsilon \quad \text{and} \quad d_b(c, c') < \varepsilon.$$

We say that two trees $T_n(z)$ and $T_n(w)$ are (n, ε)-separated with respect to branch Hausdorff metric d_{bH} if $d_{bH}(T_n(z), T_n(w)) \geq \varepsilon$. By compactness of X_n, it follows that the maximal cardinality $s_{\text{inv}}(n, \varepsilon, Z)$ of (n, ε, d_{bH})-separated subset of T_n is finite. Consequently, we may define the *inverse* entropy h_{inv} of f_∞, restricted to Z, as follows

$$h_{\text{inv}}(f_\infty|Z) := \lim_{\varepsilon \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \log s_{\text{inv}}(n, \varepsilon, Z).$$

Preimage relation entropy. Another approach to the dynamics of solenoids is based on comparing behavior of branches under the action of the so-called *adjusted mappings*. In the case of a single transformation, this method has been introduced by Langevin and Walczak [8]. Let $Z \subset X_0$. Consider the set T_n of all trees of length n over Z.

For every $k = 0, 1, \ldots, n$, define a projection

$$\pi_k \colon T_n \to X_k \quad \text{by} \quad \pi_k(b) = z_k$$

for a branch $b = [z_0, z_1, \ldots, z_n]$. We say that a map $\phi\colon T_n(z) \to T_n(w)$ between trees $T_n(z)$ and $T_n(w)$ is *adjusted* if ϕ preserves the branching structure, i.e., if $b, c \in T_n(x)$ and, for some $k = 0, 1, \ldots, n$, we have $\pi_k(b) = \pi_k(c)$, then $\pi_k(\phi(b)) = \pi_k(\phi(c))$. Denote the set of adjusted mappings from $T_n(z)$ to $T_n(w)$ by $\text{adj}(z, w)$. It is clear that for any $z, w \in Z$, $\text{adj}(z, w) \neq \emptyset$. Moreover, the composition of adjusted mappings is again an adjusted mapping.

Fix $\varphi\colon X_0 \to X_0$ with $\varphi(Z) \subset Z$ for a subset $Z \subset X_0$. Let $\varphi_k = \varphi\colon X_k \to X_k$. Observe that φ induces an adjusted mapping $\phi\colon T_n \to T_n$, defined by

$$\phi([z_0, z_1, \ldots, z_n]) = [\varphi_0(x_0), \varphi_1(x_1), \ldots, \varphi_n(x_n)],$$

if and only if $f_k \circ \varphi_k = \varphi_{k-1} \circ f_k$ for every $k = 1, 2, \ldots, n$.

Let $z, w \in Z \subset X_0$. Put

$$\delta(\phi) = \max\{d_b(b, \phi(b)) : b \in T_n(z)\}$$

and

$$\delta(z, w) = \inf\{\delta(\phi) : \phi \in \text{adj}(z, w)\}.$$

Lemma 2.1. *Let* $d_{\text{adj}}(T_n(z), T_n(w)) = \max\{\delta(z,w), \delta(w,z)\}$. *Then* d_{adj} *is a distance function defined on the set* T_n *of all trees of length* n.

Proof. The coincidence axiom and symmetry of d_{adj} are clear. Take $z, w, u \in Z$. Let $C = \{\eta \circ \psi : \psi \in \text{adj}(z,w), \eta \in \text{adj}(w,u)\}$. Since the composition of adjusted maps is again an adjusted map, $C \subset \text{adj}(z,u)$. Consequently,

$$\begin{aligned}
\delta(z,u) &= \inf\{\delta(\phi) : \phi \in \text{adj}(z,u)\} \\
&\leq \inf\{\delta(\phi) : \phi \in C\} \\
&\leq \inf\{\max\{d_b(b, \psi(b)) + d_b(\psi(b), \phi(b))\} : b \in T_n(z), \phi = \eta \circ \psi \in C\} \\
&\leq \inf\{\delta(\psi) : \psi \in \text{adj}(z,w)\} + \inf\{\delta(\eta) : \eta \in \text{adj}(w,u)\} \\
&= \delta(z,w) + \delta(w,u),
\end{aligned}$$

which completes the proof. □

Let $\varepsilon > 0$ and $n \geq 1$. Denote by $s_{\text{pre}}(n, \varepsilon, Z)$ the maximal cardinality of the family of trees of length n which are $(n, \varepsilon, d_{\text{adj}})$-separated. We define the *preimage relation entropy* h_{pre} of f_∞, restricted to Z, by

$$h_{\text{pre}}(f_\infty | Z) := \lim_{\varepsilon \to 0^+} \lim_{n \to \infty} \frac{1}{n} \log s_{\text{pre}}(n, \varepsilon, Z).$$

Point entropy. Let $Z \subset X_0$. In this approach, we consider a union of trees

$$T_{\leq n}(z) = \bigcup_{k=0}^{n} T_k(z) = \bigcup_{j=0}^{n} g_j^{-1}(z),$$

where $T_k(z)$ denotes a tree over z of length k.

Recall that in the tree $T_m(z)$ we introduced the branch metric d_b as follows

$$d_b(b,c) = \max\{d(z_k, w_k) : k = 0, 1, \ldots, m\},$$

where $b = [z_0, z_1, \ldots, z_m]$, $c = [w_0, w_1, \ldots, w_m]$ and $z_0 = w_0$. Now, we define a metric d_{po} in $T_{\leq n}(z)$ as follows

$$d_{po}(b,c) = \begin{cases} d_b(b,c) & \text{if } c, b \in T_m(z), \\ |m - l| & \text{if } b \in T_k(z),\ c \in T_l(z) \text{ and } m \neq l. \end{cases}$$

Denote by $s_z(n, \varepsilon)$ the cardinality of maximal (n, ε, d_{po})-separated subset of $T_{\leq n}(z)$. Let $s_{\text{pt}}(n, \varepsilon, Z) = \sup\{s_z(n, \varepsilon) : z \in Z\}$. The following quantity is called a *point entropy* of f_∞ restricted to Z

$$h_{\text{pt}}(f_\infty | Z) := \lim_{\varepsilon \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \log s_{\text{pt}}(n, \varepsilon, Z).$$

Remark 2.2. (1) In particular, the solenoid $f_\infty = (f_n\colon X_n \to X_{n-1})_{n=1}^{\infty}$ with $(X_n, d_n) = (X_0, d_0)$ and $f_n = f_0$ for any $n \in \mathbb{N}$, coincides with the dynamical system considered by Hurley [4]. In this case, our definitions of entropy-like quantities are reduced to definitions introduced in [4].

(2) Generally, the entropies h_{pre} and h_{inv} are different. Nitecki and Przytycki [12] constructed an example of a skew product f on a unit square (see Example 3.1 in [12]), such that $h_{\text{inv}}(f) = 0$ and $h_{\text{pre}}(f) = \log 2$. So, the above mentioned entropies of f_∞, where all $f_n = f$, are respectively 0 and $\log 2$.

(3) Even if two transformations are relatively simple, their composition may be very complicated. Raith [14] constructed two interval maps $f, g\colon I \to I$ of zero topological entropy such that $h \circ g$ has positive topological entropy.

Natural distance functions. Let $Z \subset X_0$. We define several sequences of distance functions on Z. Let $z_0, w_0 \in Z$ and $n \geq 0$. Put

$$\begin{aligned} \rho_{\text{adj},n}(z_0, w_0) &= d_{\text{adj}}(T_n(z_0), T_n(w_0)), \\ \rho_{b,n}(z_0, w_0) &= \inf\{d_b(b_1, b_2) : b_1 \in T_n(z_0), b_2 \in T_n(w_0)\}, \\ \rho_{bH,n}(z_0, w_0) &= d_{bH,n}(T_n(z_0), T_n(w_0)). \end{aligned}$$

For any $a \in \{\text{adj}, b, bH\}$, let $\rho_{a,n}$ denote corresponding distance function on Z. By $B_{\rho_{a,n}}(z, \delta)$ we denote the standard ball, with respect to the metric $\rho_{a,n}$, centered at z and of radius $\delta > 0$.

Relation between entropies. Our entropy-like quantities can be expressed in language of (n, ε)-spannings sets. Using the previous notation, let $\gamma \in \{\text{Inv}, \text{inv}, \text{pre}\}$. By compactness of X_n, we get that the minimal cardinality $r_\gamma(n, \varepsilon, Z)$ of (n, ε)-spanned subset in an appropriate space is finite. Using standard arguments (e.g., [19]) we get an estimation $r_\gamma(n, \varepsilon) \leq s_\gamma(n, \varepsilon) \leq r_\gamma(n, \varepsilon/2)$. Consequently, passing to the suitable limits, we get

$$h_\gamma(f_\infty|Z) = \lim_{\varepsilon \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \log r_\gamma(n, \varepsilon, Z).$$

Theorem 2.3. *For any subset $Z \subset X_0$, the following inequalities hold*

(i) $h_{\text{inv}}(f_\infty|Z) \leq h_{\text{pre}}(f_\infty|Z)$,

(ii) $h_{\text{Inv}}(f_\infty|Z) \leq h_{\text{inv}}(f_\infty|Z) + h_{\text{pt}}(f_\infty|Z)$.

Proof. Let n be a positive integer and $\varepsilon > 0$.

(i) Take two trees $T_n(x)$, $T_n(y)$ and $c > 0$, such that

$$d_{\text{adj}}(T_n(x), T_n(y)) < c.$$

Then, $\delta(\phi) < c$ for some adjusted map $\phi\colon T_n(x) \to T_n(y)$. So, for arbitrary branch b_x of $T_n(x)$, we have that $d_b(b_x, \phi(b_x)) < c$. By the same arguments we obtain the existence of an adjusted map $\phi'\colon T_n(y) \to T_n(x)$ such that, for any branch b_y of $T_n(y)$ we obtain that $d_b(b_y, \phi'(b_y)) < c$. This means that $d_{bH}(T_n(x), T_n(y)) < c$ and $d_{\mathrm{adj}}(T_n(x), T_n(y)) \geq d_{bH}(T_n(x), T_n(y))$. Any two trees (n, ε, d_{bH})-separated with respect to d_{bH} are (n, ε)-separated with respect to d_{adj}, so $s_{\mathrm{inv}}(n, \varepsilon, Z) \geq s_{\mathrm{pre}}(n, \varepsilon, Z)$. Passing to the suitable limits, we obtain the inequality.

(ii) Let A be a maximal $(n, \varepsilon/3, d_{bH})$-separated subset of Z. Let $A_Z = \{u \in Z : T_n(u) \in A\}$. For every $u \in A_Z$, choose an $(n, \varepsilon/3, d_b)$-separated subset $M(u) \subset T_n(u)$ of maximal cardinality $s_u(n, \varepsilon/3)$. Put $M = \bigcup_{u \in A_Z} M(u)$.

Assume that Lemma 2.4 is proved. Since

$$\begin{aligned} \operatorname{card} A_Z &= \operatorname{card} A = s_{\mathrm{inv}}(n, \varepsilon/3, Z), \\ \operatorname{card} M(u) &= s_u(n, \varepsilon/3) \leq s_{\mathrm{pt}}(n, \varepsilon/3, Z), \end{aligned}$$

we have

$$\begin{aligned} r_{\mathrm{Inv}}(n, \varepsilon, Z) &\leq \operatorname{card} M \\ &\leq \operatorname{card} A_Z \times \max\{\operatorname{card} M(u) : u \in A_Z\} \\ &\leq s_{\mathrm{inv}}(n, \varepsilon/3, Z) s_{\mathrm{pt}}(n, \varepsilon/3, Z). \end{aligned}$$

Now taking logarithm and passing to the suitable limits, we obtain inequality (ii). □

Lemma 2.4. *M is (n, ε, d_b)-spanning subset of T_n.*

Proof. Choose $z \in Z$, consider a tree $T_n(z)$ and take a branch $b \in T_n(z)$. Since any maximal separated subset is a spanning subset, there exists $u \in A_Z$ such that $d_{bH}(T_n(z), T_n(u)) \leq \varepsilon/3$. Therefore, we may find a branch $b_1 \in T_n(u)$ with

$$d_b(b, b_1) \leq \frac{\varepsilon}{3}.$$

Again, since $M(u)$ is a maximal separated subset of $T_n(u)$, the set $M(u)$ is a spanning subset. Consequently, there exists a branch $b_2 \in M(u)$ such that

$$d_b(b_1, b_2) \leq \frac{\varepsilon}{3}.$$

Hence $d_b(b, b_2) \leq d_b(b, b_1) + d_b(b_1, b_2) < \varepsilon$. This completes the proof of the lemma. □

Remark 2.5. Results of Theorem 2.3 are generalizations of the inequalities between entropy-like invariants obtained by M. Hurley [4] but the proof of the theorem was inspired by the approach of [4].

3. Dynamics of solenoids via Carathéodory structures

In this section, we investigate dynamics of solenoids from the dimension theory point of view. Our main tool here is the so-called Carathéodory structure. This powerful tool has been introduced by Pesin [13].

Carathéodory dimension structure. Let Y be a nonempty set. Suppose that a cover $\mathcal{F}$ of a subset of Y and three set functions $\eta, \psi, \xi \colon \mathcal{F} \to \mathbb{R}_+$ are given. Assume that the following conditions are satisfied:

(C1) $\emptyset \in \mathcal{F}$ and $\psi(\emptyset) = \eta(\emptyset) = 0$. If $\emptyset \neq U \in \mathcal{F}$ then $\psi(U)\eta(U) > 0$.

(C2) For every $\delta > 0$ there exists $\varepsilon > 0$ such that $\eta(U) < \delta$ for any $U \in \mathcal{F}$ with $\psi(U) < \varepsilon$.

(C3) For every $\varepsilon > 0$ there exists a finite or countable subcover $\mathcal{G} \subset \mathcal{F}$ of Y such that $\psi(V) < \varepsilon$ for every $V \in \mathcal{G}$.

A system $\tau = (\mathcal{F}, \xi, \eta, \psi)$ is called a *Carathéodory structure* or shortly a *C-structure* on Y. In this paragraph we give relevant facts about Carathéodory structures.

Suppose that the C-structure $\tau = (\mathcal{F}, \xi, \eta, \psi)$ on Y is given. We will use the following notation: If $\mathcal{G} \subset \mathcal{F}$ is a finite or countable subcollection then we write $\mathcal{G} \prec \mathcal{F}$. Moreover, we put $\psi(\mathcal{G}) = \sup\{\psi(V) : V \in \mathcal{G}\}$.

Let $\alpha \in \mathbb{R}$. Following Pesin, we define an outer measure m_α on Y as follows: Take $Z \subset Y$ and $\varepsilon > 0$ and put

$$M_\alpha(Z, \varepsilon) = \inf\left\{\sum_{V \in \mathcal{G}} \xi(V)\eta(V)^\alpha : \mathcal{G} \prec \mathcal{F}, Z \subset \cup\mathcal{G} \text{ and } \psi(\mathcal{G}) < \varepsilon\right\}.$$

One can show that there exists a limit

$$m_\alpha(Z) = \lim_{\varepsilon \to 0^+} M_\alpha(Z, \varepsilon).$$

According to the general measure theory, m_α induces a σ-additive measure on Y called the α-*Carathéodory measure*.

It turns out that there exists a critical value $\alpha_C \in [-\infty, \infty]$ such that

$$m_\alpha(Z) = \begin{cases} \infty & \text{for } \alpha \leq \alpha_C, \\ 0 & \text{for } \alpha > \alpha_C. \end{cases}$$

The quantity $\dim_\tau(Z) = \alpha_C$ is called a *C-dimension* of Z with respect to the C-structure τ.

Remark 3.1. Let $Y = \mathbb{R}^n$ and $\mathcal{F}$ be a topology of Y. For every open set U put $\xi(U) = 1$, $\psi(U) = \eta(U) = \mathrm{diam}(U)$. Then, for every $Z \subset \mathbb{R}^n$, the C-dimension of Z coincides with the Hausdorff dimension

$$\dim_\tau(Z) = \dim_H(Z) \quad \text{where } \tau = (\mathcal{F}, \xi, \eta, \psi).$$

Now, suppose that the set function ψ satisfies a condition stronger than (C3). Namely, assume that

(C3') There exists $\varepsilon_0 > 0$ such that, for every $\varepsilon_0 > \varepsilon_1 > 0$, there exist $\varepsilon \in (0, \varepsilon_1)$ and a subcover $\mathcal{G} \prec \mathcal{F}$ with $\psi(V) = \varepsilon$ for every $V \in \mathcal{G}$.

Take $\alpha \in \mathbb{R}$ and $\varepsilon \in (0, \varepsilon_0)$. Proceeding as before, for any $Z \subset Y$, we define the quantity

$$R_\alpha(Z, \varepsilon) = \inf \left\{ \sum_{V \in \mathcal{G}} \xi(V)\eta(V)^\alpha : \mathcal{G} \prec \mathcal{F}, Z \subset \cup\mathcal{G} \text{ and } \psi(\mathcal{G}) = \varepsilon \right\}.$$

Due to (C3'), the quantity $R_\alpha(Z, \varepsilon)$ is well defined. It yields the existence of the limits

$$\underline{r}_\alpha(Z) = \liminf_{\varepsilon \to 0^+} R_\alpha(Z, \varepsilon) \quad \text{and} \quad \overline{r}_\alpha(Z) = \limsup_{\varepsilon \to 0^+} R_\alpha(Z, \varepsilon).$$

As before, there exist $\underline{\alpha}_C, \overline{\alpha}_C \in [-\infty, \infty]$ such that

$$\underline{r}_\alpha(Z) = \begin{cases} \infty & \text{for } \alpha \le \underline{\alpha}_C, \\ 0 & \text{for } \alpha > \underline{\alpha}_C, \end{cases} \qquad \overline{r}_\alpha(Z) = \begin{cases} \infty & \text{for } \alpha \le \overline{\alpha}_C, \\ 0 & \text{for } \alpha > \overline{\alpha}_C. \end{cases}$$

The quantity $\underline{\alpha}_C$ (resp. $\overline{\alpha}_C$) is called *lower* (resp. *upper*) *C-capacity* of Z with respect to a C-structure τ. We denote lower (resp. upper) C-capacity of Z by $\underline{\mathrm{Cap}}(Z)$ (resp. $\overline{\mathrm{Cap}}(Z)$), i.e.,

$$\underline{\mathrm{Cap}}(Z) = \underline{\alpha}_C \quad \text{and} \quad \overline{\mathrm{Cap}}(Z) = \overline{\alpha}_C. \tag{3.1}$$

Moreover, we assume that the set functions η and ψ are related as follows:

(C4) If $U, V \in \mathcal{F}$ and $\psi(U) = \psi(V)$ then $\eta(U) = \eta(V)$. In other words, η is constant on each level set $\psi^{-1}(a)$, $a \in \mathbb{R}$, of the set function ψ.

The Carathéodory structure τ is called *Carathéodory-Pesin-structure* (or *CP-structure*), if τ satisfies additionally both conditions (C3') and (C4).

Let $Z \subset X_0$ and $\varepsilon > 0$. Put

$$\Lambda(Z, \varepsilon) = \inf \left\{ \sum_{V \in \mathcal{G}} \xi(V) : \mathcal{G} \prec \mathcal{F}, Z \subset \cup\mathcal{G} \text{ and } \mathcal{G} \subset \psi^{-1}(\varepsilon) \right\}.$$

Observe that (C4) implies that, for every such $\mathcal{G}$ as above, $\eta|_{\mathcal{G}}$ is constant. Denote its value by η_ε, i.e., $\eta_\varepsilon = \eta(V)$ if $V \in \psi^{-1}(\varepsilon)$.

Lower and upper capacities determined by CP-structures have the following properties: Let $Z, S \subset Y$ then

$$\underline{\mathrm{Cap}}(S \cup Z) = \max\left\{\underline{\mathrm{Cap}}(S), \underline{\mathrm{Cap}}(Z)\right\},$$
$$\overline{\mathrm{Cap}}(S \cup Z) = \max\left\{\overline{\mathrm{Cap}}(S), \overline{\mathrm{Cap}}(Z)\right\}.$$

Moreover, the lower and upper capacities are related to Λ as follows

$$\underline{\mathrm{Cap}}(Z) = \liminf_{\varepsilon \to 0^+} \frac{-\log \Lambda(Z, \varepsilon)}{\log \eta_\varepsilon}, \tag{3.2}$$

$$\overline{\mathrm{Cap}}(Z) = \limsup_{\varepsilon \to 0^+} \frac{-\log \Lambda(Z, \varepsilon)}{\log \eta_\varepsilon}. \tag{3.3}$$

Natural Carathéodory structure for solenoids. We apply notation from Section 2. Let $Z \subset X_0$. Recall that Z can be equipped with one of the natural distance functions $\rho_{b,n}$, $\rho_{bH,n}$ or $\rho_{\mathrm{adj},n}$. Let $\rho_a = \{\rho_{a,n} : n \geq 0\}$ where $a \in \{\mathrm{adj}, b, bH\}$. Fix a natural metric $\rho_{a,n}$ on Z.

For every $z \in Z$ and $r > 0$ let $B_{\rho_{a,n}}(z, r)$ denote a ball in metric $\rho_{a,n}$. Let $\mathcal{B}_{\rho_{a,n}}(r) = \{B_{\rho_{a,n}}(z, r) : z \in Z\}$. It is clear that $\mathcal{B}_{\rho_{a,n}}(r)$ is an open cover of Z.

Fix a subset $H \subset Z$ and $\delta > 0$. Put

$$\mathcal{B}_{\rho_{a,n}}(H, \delta) = \{B_{\rho_{a,n}}(z, \delta) \cap H : z \in H\},$$
$$\mathcal{F}_{\rho_a,\delta} = \mathcal{F}_{\rho_a,\delta}(H) = \left\{\emptyset, \bigcup_{n=0}^{\infty} \mathcal{B}_{\rho_{a,n}}(H, \delta)\right\}.$$

Consider set functions $\xi, \eta, \psi \colon \mathcal{F}_\delta(H) \to \mathbb{R}$ given by

$$\xi(V) = 1, \quad \eta(V) = \exp(-n) \quad \text{and} \quad \psi(V) = n^{-1}$$

for $V = B_{\rho_{a,n}}(z, \delta) \cap H$ with $z \in H$ and $n \geq 1$. Moreover, let $\xi(V) = \psi(V) = \eta(V) = 1$ if $V = B_{\rho_{a,0}}(z, \delta)$. Then, the system

$$\tau_{\rho_a,\delta}(H) = (\mathcal{F}_{\rho_a,\delta}, \xi, \eta, \psi)$$

is a Carathéodory-Pesin structure on H determined by the dynamical system f_∞. To indicate that all objects under considerations depend on ρ_a and δ we will write $\Lambda_{\rho_a,\delta}$, $\overline{\mathrm{Cap}}_{\rho_a,\delta}$ etc. instead of Λ, $\overline{\mathrm{Cap}}$. Observe that for every $\alpha \in \mathbb{R}$,

$$\overline{r_{\rho_a,\alpha}}(H) = \limsup_{m \to \infty} R_{\rho_a,\alpha}(H, m^{-1}), \tag{3.4}$$

where in our case, $R_{\rho_a,\alpha}(Z,m^{-1})$ reduces to

$$R_{\rho_a,\alpha}(H,m^{-1}) = \inf\left\{\sum_{V\in\mathcal{G}} \exp(-m\alpha) : \mathcal{G} \prec \mathcal{B}_{\rho_{a,m}}(H,\delta) \text{ and } H \subset \cup\mathcal{G}\right\}.$$

Moreover, formulae (3.2) and (3.3) imply

$$\underline{\mathrm{Cap}}_{\rho_a,\delta}(H) = \liminf_{n\to\infty} n^{-1}\log\Lambda_{\rho_a,\delta}(H,n^{-1}),$$
$$\overline{\mathrm{Cap}}_{\rho_a,\delta}(H) = \limsup_{n\to\infty} n^{-1}\log\Lambda_{\rho_a,\delta}(H,n^{-1}).$$

Next, following Pesin [13], we obtain that there exist limit capacities

$$\underline{\mathrm{CP}}_{\rho_a}(H) = \lim_{\delta\to 0^+}\underline{\mathrm{Cap}}_{\rho_a,\delta}(H) \quad\text{and}\quad \overline{\mathrm{CP}}_{\rho_a}(H) = \lim_{\delta\to 0^+}\overline{\mathrm{Cap}}_{\rho_a,\delta}(H).$$

Consequently,

$$\underline{\mathrm{CP}}_{\rho_a}(H) = \lim_{\delta\to 0^+}\liminf_{n\to\infty}\frac{1}{n}\log\Lambda_{\rho_a,\delta}(Z,n^{-1}),$$
$$\overline{\mathrm{CP}}_{\rho_a}(H) = \lim_{\delta\to 0^+}\limsup_{n\to\infty}\frac{1}{n}\log\Lambda_{\rho_a,\delta}(Z,n^{-1}).$$

Observe that, by the definition of Λ, we get that $\Lambda_{\rho_a,\delta}(Z,n^{-1})$ is equal to the minimal cardinality of (n,δ,Z)-spanned subset in the appropriate metric space. Hence, by the correspondence between entropies h_{pre}, h_{Inv}, h_{inv} and distances ρ_{adj}, ρ_b, ρ_{bH}, it follows that

Theorem 3.2. *For any subset $Z \subset X_0$ we have*

$$h_{\mathrm{pre}}(f_\infty|Z) = \overline{\mathrm{CP}}_{\rho_{\mathrm{adj}}}(Z),$$
$$h_{\mathrm{Inv}}(f_\infty|Z) = \overline{\mathrm{CP}}_{\rho_b}(Z),$$
$$h_{\mathrm{inv}}(f_\infty|Z) = \overline{\mathrm{CP}}_{\rho_{bH}}(Z).$$

Consequently, applying Theorem 2.3(i) we get

Corollary 3.3. *For any subset $Z \subset X_0$ we have*

$$\overline{\mathrm{CP}}_{\rho_{bH}}(Z) \le \overline{\mathrm{CP}}_{\rho_{\mathrm{adj}}}(Z).$$

Local measure entropy. Suppose now, that the set X_0 is equipped with a Borel probability measure ν and a family $\rho_a = \{\rho_{a,n} : n \ge 0\}$ natural metrics on X_0, where $a \in \{\mathrm{adj}, b, bH\}$. For any for $z \in Z$ and $r > 0$, ρ_a gives rise to a sequence of dynamical balls $\{B_{\rho_a,n}(z,r)\}_{n\in\mathbb{N}}$ (i.e., the balls in the metrics $\rho_{a,n}$). Clearly, every ball $B_{\rho_a,n}(z,r)$ is a Borel subset of X_0 and its measure $\nu(B_{\rho_a,n}(z,r))$ is well defined. Following M. Brin and

A. Katok [2], for every $z \in Z$, we define a *local measure entropy* $\underline{h}^{\rho_a}_{\mathrm{loc},\nu}(z) = \underline{h}^{\rho_a}_{\mathrm{loc},\nu}(f_\infty|z)$ with respect to the dynamical system f_∞ and the Borel probability measure ν as follows

$$\underline{h}^{\rho_a}_{\mathrm{loc},\nu}(z) = \lim_{\varepsilon\to 0^+} \liminf_{n\to\infty} \frac{-\log \nu(B_{\rho_{a,n}}(z,\varepsilon))}{n}.$$

Theorem 3.4. *Let ν be a Borel probability measure on X_0. Let Z be a Borel subset of X_0 with $\nu(Z) > 0$. Then we have*

$$h_{\mathrm{pre}}(f_\infty|Z) \geq \operatorname{ess\,inf} \underline{h}^{\rho_{\mathrm{adj}}}_{\mathrm{loc},\nu},$$
$$h_{\mathrm{Inv}}(f_\infty|Z) \geq \operatorname{ess\,inf} \underline{h}^{\rho_b}_{\mathrm{loc},\nu},$$
$$h_{\mathrm{inv}}(f_\infty|Z) \geq \operatorname{ess\,inf} \underline{h}^{\rho_{bH}}_{\mathrm{loc},\nu},$$

where $\operatorname{ess\,inf} \underline{h}^{\rho_a}_{\mathrm{loc},\nu}$ *stands for the essential infimum of the local measure entropy function* $\underline{h}^{\rho_a}_{\mathrm{loc},\nu}$.

Proof. Put $\kappa = \operatorname{ess\,inf} \underline{h}^{\rho_a}_{\mathrm{loc},\nu}$. If $\kappa = 0$, then there is nothing to prove. Assume that $\kappa > 0$. Take $\varepsilon > 0$. Let $Z' \subset Z$ such that $\nu(Z') = \nu(Z)$ and $\underline{h}^{\rho_a}_{\mathrm{loc},\nu}(z) \geq \alpha$ for $z \in Z'$. Observe, that we may write $Z' = \bigcup_{k=1}^\infty Z_k$, where Z_k is given by

$$Z_k = \bigcap_{r\in(0,1/k)} \left\{ z \in Z' : \liminf_{n\to\infty} \frac{-\log \nu(B_{\rho_{a,n}}(z,r))}{n} > \kappa - \frac{1}{2}\varepsilon \right\}.$$

Since $0 < \nu(Z') \leq \sum_{k=1}^\infty \nu(Z_k)$, there exists $K \in \mathbb{N}$ with $\nu(Z_K) > 0$. Next, we may write $Z_K = \bigcup_{m=1}^\infty Z_{K,m}$, where $Z_{K,m}$ is given by

$$Z_{K,m} = \bigcap_{n\geq m} \bigcap_{r\in(0,1/K)} \left\{ z \in Z_K : \frac{-\log(B_{\rho_{a,n}}(z,r))}{n} > \kappa - \varepsilon \right\}.$$

As before, we conclude that there exists $M \in \mathbb{N}$ with $\nu(Z_{K,M}) > 0$. It means that, for every $n \geq M$, $\delta \in (0, K^{-1})$ and every $z \in Z_{K,M}$,

$$\nu(B_{\rho_{a,n}}(z,\delta)) < \exp(-(\kappa - \varepsilon)n). \tag{3.5}$$

Put $H = Z_{K,M}$. Fix $\delta \in (0, K^{-1})$. Consider CP-structure $\tau_{\rho_a,\delta}(H)$. Put $\alpha = \kappa - \varepsilon$. Then, for every $n \geq M$, (3.5) yields

$$\nu(V) \leq \nu(B_{\rho_{a,n}}(z,\delta)) < \exp(-n\alpha),$$

where $V = B_{\rho_{a,n}}(z,\delta) \cap H$ and $z \in H$. It follows that, for every finite or countable family $\mathcal{G} \subset \mathcal{B}_{\rho_{a,n}}(H,\delta)$ with $H = \cup\mathcal{G}$,

$$0 < \nu(H) \leq \sum_{V\in\mathcal{G}} \exp(-n\alpha).$$

Consequently, applying (3.4) we get $0 < \nu(H) \leq \overline{r_\alpha}(H)$. This means (see (3.1)) that for every $\varepsilon > 0$,

$$\kappa - \varepsilon = \alpha \leq \overline{\mathrm{Cap}}_{\rho_a,\delta}(H).$$

Hence

$$\operatorname{ess\,inf} \underline{h_{\mathrm{loc},\nu}^{\rho_a}} = \kappa \leq \overline{\mathrm{Cap}}_{\rho_a,\delta}(H) \xrightarrow[\delta\to 0^+]{} \overline{\mathrm{CP}}_{\rho_{\rho_a}}(H).$$

Now our theorem follows by Theorem 3.2 and the inclusion $H \subset Z$. □

4. Final remarks and open problems

We can obtain similar results in a slightly more general set up, where we do not assume that all spaces X_n coincide with a metrizable compact space X_0. We study a sequence $f_\infty = (f_n \colon X_n \to X_{n-1})_{n=1}^{\infty}$ of continuous epimorphisms of compact metric spaces (X_n, d_n). But, in this case, we have to assume that there exist a compact metric space (Y, d) and a sequence of injection $j_n \colon Y \to X_n$, where $n \in \mathbb{N}$. Put $Y_k = j_k(Y) \subset X_k$. We may identify $x \in Y_k$ with $j_k^{-1}(x)$. Moreover, we assume that

$$f_k(Y_k) = Y_{k-1} \quad \text{for every } k \in \mathbb{N}.$$

By *solenoid* determined by f_∞, we mean the inverse limit

$$X_\infty = \varprojlim X_k = \{(x_k)_{k=0}^{\infty} : x_{k-1} = f_k(x_k)\}.$$

Remark 4.1. The above definition of a solenoid is slightly more general than a classical one. If all the spaces X_n coincide with the space X_0 we put $Y = X$ and $j_k = id$. Then, our definition coincides with the classical one. This definition of a solenoid is natural from the point of view of the Pesin theory.

There are a few natural questions related to the dynamics of solenoids. To the best of our knowledge, they are still open. The first one concerns a construction of a solenoid with a given numerical value of an entropy-like quantity. Let $\gamma \in \{\mathrm{Inv}, \mathrm{inv}, \mathrm{pre}\}$.

Question 4.2 (Realization problem)**.** Let $\alpha > 0$. Is there a solenoid built over the space X with $h_\gamma(f_\infty | X) = \alpha$?

Topological theory of classical dynamical systems has its counterpart in measure-theoretic theory which yields a variational principle. Therefore, we may ask

Question 4.3. Is there a measure-theoretic counterpart of an entropy-like quantity $h_\gamma(f_\infty | X)$?

There are many classical dynamical systems with positive entropy. The existence of a horseshoe yields positive topological entropy. It is known that a diffeomorphism possessing a homoclinic point with a topological crossing (possibly with infinite order contact) has positive topological entropy. Therefore, we ask

Question 4.4. Are there geometric criteria for positive entropy of $h_\gamma(f_\infty|X)$?

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