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Numerical Methods for Solving the Time-fractional Telegraph Equation

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Abstract. A flexible numerical method for the time-fractional telegraph equation is proposed and analyzed in this paper. The solution is discretized with a new finite difference scheme in time, and a local discontinuous Galerkin (LDG) method in space. We prove that the method is unconditionally stable and convergent with order $O(h^{k+1} + (\Delta t)^{3-\alpha})$, where h, Δt, k are the space step size, time step size and degree of piecewise polynomial, respectively. Numerical experiments are carried out to illustrate the robustness, reliability, and accuracy of the method.

1. Introduction

In recent years fractional partial differential equations (FPDEs) have gained more and more attention since many phenomena could be modeled by these equations in science and engineering [23]. Due to the important applications of FPDEs in engineering and science, there are many scholars to design and develop numerical methods for the FPDEs. The existed methods solving the FPDEs include finite difference methods [1, 6, 8, 10, 11, 15, 21, 24, 27, 30, 32, 38, 39, 47, 51, 53], finite element methods [9, 12, 13, 19, 20, 25, 37, 52], spectral methods [4, 7, 26, 29, 31, 48], discontinuous Gakerkin methods [17, 40, 41]. Some other numerical methods are also very effective, such as homotopy perturbation method and the variational method, for details the readers can refer to [14, 16, 28, 34, 44, 46, 50]. Although some numerical methods for FPDEs have been proposed and developed, it is still meaningful and challenging to construct higher order numerical methods to solve these equations.

In this paper we consider the following time-fractional telegraph equation of order α $(1 < \alpha < 2)$

$$
\begin{aligned}
&\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \frac{\partial^{\alpha-1} u(x,t)}{\partial t^{\alpha-1}} - \rho u_{xx} = f(x,t), \quad (x,t) \in (a,b) \times (0,T], \\
&u(x,0) = u_0(x), \quad \frac{\partial u(x,0)}{\partial t} = u_1(x), \qquad x \in [a,b],
\end{aligned} \tag{1.1}
$$

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where $\partial^\alpha u(x,t)/\partial t^\alpha$, $\partial^{\alpha-1} u(x,t)/\partial t^{\alpha-1}$ are Caputo fractional derivatives with respect to t, ρ is an positive constant, x and t are the space and time variables, and the periodic boundary condition considered in this paper. The Caputo fractional derivative $\partial^\theta u(x,t)/\partial t^\theta$ is defined as [36]

$$\frac{\partial^\theta u(x,t)}{\partial t^\theta} = \frac{1}{\Gamma(m-\theta)} \int_0^t \frac{\partial^m u(x,s)}{\partial s^m} \frac{ds}{(t-s)^{1-m+\theta}}, \quad m-1<\theta<m,$$

where $\Gamma(\cdot)$ is the Gamma function.

The telegraph equations are hyperbolic partial differential equations which can be applied in some fields, such as signal analysis, wave propagation, random walk theory and so on. Some researchers have discussed the time fractional telegraph equations [22]. Beghin and Orsingher [2] studied the fractional telegraph equation and showed that the fundamental solution could be expressed as the density of the composition. Orsingher and Beghin [35] considered the fundamental solutions of this equation. Chen, Liu and Anh [5] considered the analytical solution for such equation using the method of separating variables. Momani [33] studied Adomian decomposition methods and obtained the analytic and approximate solutions for the fractional telegraph equation. Jiang and Lin [18] derived the exact solution in the form of series by using the reproducing kernel theorem. Biazar et al. [3] discussed the variational iteration method to solve the fractional telegraph equation. Momani [33] and Yıldırım [45] have applied Adomian decomposition method and Homotopy perturbation method to solve the fractional telegraph equations, respectively.

The discontinuous Galerkin method, which has many good features of a finite element and a finite volume method, is a very attractive method to solve partial differential equations due to its flexibility in terms of mesh and shape functions, and can achieve a high order of convergence. In this paper, we first present a finite difference scheme to approximate the time fractional derivatives, and give a semidiscrete scheme in time. Then a fully-discrete method based on the semidiscrete scheme for the fractional telegraph equation in which the spatial direction is approximated by a LDG method is presented and analyzed.

The paper is organized as follows. In Section 2 we will introduce some basic notations and theoretic results. Then we present our finite difference/local discontinuous Galerkin method for the time-fractional telegraph equation, and also discuss the stability and give an error estimate in Section 3. In Section 4, numerical results are also given to illustrate the accuracy of convergence and capability of the method, and the concluding remarks is included in the final section.

2. Notation and theoretic results

Let $\Omega = \bigcup_j I_j$ be the partition of $\Omega = [a, b]$, and $I_j = [x_{j-1/2}, x_{j+1/2}]$, for $j = 1, \ldots, N$. The cell lengths $\Delta x_j = x_{j+1/2} - x_{j-1/2}$, $1 \le j \le N$, and $h = \max_{1\le j\le N} \Delta x_j$.

Denote by $u^+_{j+1/2}$ and $u^-_{j+1/2}$ the traces from the right cell I_{j+1} and the left cell I_j, respectively. The associated discontinuous Galerkin element space V_h^k is defined as the space of piecewise polynomials of the degree up to k,

$$V_h^k = \{v : v \in P^k(I_j), j = 1, 2, \ldots, N\}.$$

We will use the projections $\mathcal{P}$ and $\mathcal{P}^{\pm}$, for $j = 1, \ldots, N$,

$$\int_{I_j} (\mathcal{P}\mu(x) - \mu(x))\omega(x) = 0, \quad \forall\, \omega \in P^k(I_j), \tag{2.1}$$

$$\begin{aligned} &\int_{I_j} (\mathcal{P}^+\mu(x) - \mu(x))\omega(x) = 0, \quad \forall\, \omega \in P^{k-1}(I_j), \\ &\mathcal{P}^+\mu(x^+_{j-1/2}) = \mu(x_{j-1/2}), \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} &\int_{I_j} (\mathcal{P}^-\mu(x) - \mu(x))\omega(x) = 0, \quad \forall\, \omega \in P^{k-1}(I_j), \\ &\mathcal{P}^-\mu(x^-_{j+1/2}) = \mu(x_{j+1/2}). \end{aligned} \tag{2.3}$$

The above projections $\mathcal{P}$ and $\mathcal{P}^{\pm}$ satisfy the following inequality [42, 43, 49]

$$\|\mu^e\| + h\|\mu^e\|_\infty + h^{1/2}\|\mu^e\|_{\tau_h} \le Ch^{k+1}, \tag{2.4}$$

here $\mu^e = \mathcal{P}\mu - \mu$ or $\mu^e = \mathcal{P}^{\pm}\mu - \mu$. The positive constant C, which is independent of h and solely depends on μ. τ_h denotes the set of boundary points of all elements I_j, and an unmarked norm $\|\cdot\|$ is the usual L^2-norm defined on the whole domain Ω, and

$$\begin{aligned} \|\mu^e\|_\infty &= \max_{x\in\Omega} |\mu^e|, \\ \|\mu^e\|_{\tau_h} &= \left(\frac{1}{2}\sum_{i=1}^{N}[((\mu^e)^+)^2_{i-1/2} + ((\mu^e)^-)^2_{i+1/2}]\right)^{1/2}. \end{aligned}$$

Below C will be used as a positive constant which may have a different value in different occurrence.

3. The schemes

In this section, we first present a finite difference method to approximate the time fractional derivatives, and then give the fully implicit discrete scheme with space discretized by the LDG method. Stability and convergence are analysed in detail.

3.1. The semidiscrete scheme

Assume the following mesh to cover the temporal domain $[0,T]$

$$0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T,$$

and denote $\Delta t = T/M$, $M \in \mathbb{N}$, $t_n = n\Delta t$, $n = 0, 1, \ldots, M$ be the mesh points. We first discretize the time fractional derivatives with order α, and then $\alpha - 1$.

Let $v(x,t) = \partial u(x,t)/\partial t$, and from the fact

$$v(x,t_i) = \frac{\partial u(x,t_i)}{\partial t} = \frac{3u(x,t_i) - 4u(x,t_{i-1}) + u(x,t_{i-2})}{2\Delta t} + \gamma_1^n,$$

where the truncation error $|\gamma_1^n| \le C(\Delta t)^2$, we can obtain

$$\begin{aligned}
&\frac{\partial^\alpha u(x,t_n)}{\partial t^\alpha} \\
&= \frac{1}{\Gamma(2-\alpha)} \int_0^{t_n} \frac{\partial v(x,\tau)}{\partial \tau} \frac{d\tau}{(t_n - \tau)^{\alpha-1}} \\
&= \frac{1}{\Gamma(2-\alpha)} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \frac{\partial v(x,\tau)}{\partial \tau} \frac{d\tau}{(t_n - \tau)^{\alpha-1}} \\
&= \frac{1}{\Gamma(2-\alpha)} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \frac{v(x,t_{i+1}) - v(x,t_i)}{\Delta t} \frac{d\tau}{(t_n - \tau)^{\alpha-1}} + \gamma_2^n \\
&= \frac{(\Delta t)^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{i=0}^{n-1} b_{n-i-1} \frac{v(x,t_{i+1}) - v(x,t_i)}{\Delta t} + \gamma_2^n \\
&= \frac{(\Delta t)^{1-\alpha}}{\Gamma(3-\alpha)} \left[v(x,t_n) + \sum_{i=1}^{n-1} (b_{n-i} - b_{n-i-1}) v(x,t_i) - b_{n-1} v(x,t_0) \right] + \gamma_2^n \\
&= \frac{(\Delta t)^{1-\alpha}}{\Gamma(3-\alpha)} \left[\frac{3u(x,t_n) - 4u(x,t_{n-1}) + u(x,t_{n-2})}{2\Delta t} \right. \\
&\qquad \left. + \sum_{i=1}^{n-1} (b_{n-i} - b_{n-i-1}) \frac{3u(x,t_i) - 4u(x,t_{i-1}) + u(x,t_{i-2})}{2\Delta t} - b_{n-1} v(x,t_0) \right] + \gamma_3^n.
\end{aligned} \tag{3.1}$$

As for the time-fractional derivative $\partial^{\alpha-1} u(x,t_n)/\partial t^{\alpha-1}$, after some manual calculation we have [29]

$$\frac{\partial^{\alpha-1} u(x,t_n)}{\partial t^{\alpha-1}} = \frac{(\Delta t)^{1-\alpha}}{\Gamma(3-\alpha)} \sum_{i=0}^{n-1} b_i (u(x,t_{n-i}) - u(x,t_{n-i-1})) + \gamma_4^n, \tag{3.2}$$

where

$$b_0 = 1, \quad b_i = (i+1)^{2-\alpha} - i^{2-\alpha}, \quad i = 1, 2, 3, \ldots$$

when $i = 1$, we take $u(x,-1) = u(x,0) - \Delta t u_1(x) + C(\Delta t)^2$ by using Taylor expansion.

Similar to the proof in [29], the truncation error satisfies

$$|\gamma_3^n| \le C(\Delta t)^{3-\alpha}, \quad |\gamma_4^n| \le C(\Delta t)^{3-\alpha}.$$

We know that

$$\begin{gathered} b_i > 0, \quad i = 1, 2, \ldots, n, \\ 1 = b_0 > b_1 > b_2 > \cdots > b_n, \quad b_n \to 0 \quad (n \to \infty). \end{gathered}$$

Substituting (3.1) into (1.1), we have

$$\begin{aligned} &(3+2\Delta t)u(x,t_n) - \beta\rho\frac{\partial^2 u(x,t_n)}{\partial x^2} \\ =&\sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})(3u(x,t_i)-4u(x,t_{i-1})+u(x,t_{i-2})) \\ &+2\Delta t b_{n-1}v(x,t_0)+4u(x,t_{n-1})-u(x,t_{n-2}) \\ &+2\Delta t\sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})u(x,t_i) \\ &+2\Delta t b_{n-1}u(x,t_0)+\beta f(x,t_n)+\beta(\gamma_3^n+\gamma_4^n), \end{aligned}$$

where $\beta = 2(\Delta t)^\alpha \Gamma(3-\alpha)$.

Let u^k be the numerical approximation to $u(x,t_k)$, $f^n = f(x,t_n)$, the problem (1.1) can be discretized by the following scheme

$$\begin{aligned} (3+2\Delta t)u^n - \beta\rho\frac{\partial^2 u^n}{\partial x^2} =& \sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})(3u^i-4u^{i-1}+u^{i-2}) \\ &+2\Delta t b_{n-1}v^0+4u^{n-1}-u^{n-2} \\ &+2\Delta t\sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})u^i+2\Delta t b_{n-1}u^0+\beta f^n, \end{aligned}$$

where $u^{-1} = u^0 - \Delta t u_1(x)$.

3.2. Fully discrete schemes

For (1.1) we first consider the equivalent first-order system

$$p = u_x, \quad \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \frac{\partial^{\alpha-1}u(x,t)}{\partial t^{\alpha-1}} - \rho p_x = f(x,t).$$

We seek the approximation solutions $u_h^n, p_h^n \in V_h^k$, such that for test functions $\phi, w \in V_h^k$,

$$\begin{aligned}
&\int_\Omega (3+2\Delta t)u_h^n\phi\,dx + \beta\rho\left(\int_\Omega p_h^n\phi_x\,dx - \sum_{j=1}^N \left((\widehat{p_h^n}\phi^-)_{j+1/2} - (\widehat{p_h^n}\phi^+)_{j-1/2}\right)\right)\\
&= \sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})\int_\Omega (3u_h^i - 4u_h^{i-1} + u_h^{i-2})\phi\,dx + 2\Delta t b_{n-1}\int_\Omega v_h^0\phi\,dx\\
&+ 4\int_\Omega u_h^{n-1}\phi\,dx - \int_\Omega u_h^{n-2}\phi\,dx + 2\Delta t\sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})\int_\Omega u_h^i\phi\,dx\\
&+ 2\Delta t b_{n-1}\int_\Omega u_h^0\phi\,dx + \beta\int_\Omega f^n\phi\,dx,\\
&\int_\Omega p_h^n w\,dx + \int_\Omega u_h^n w_x\,dx - \sum_{j=1}^N\left((\widehat{u_h^n}w^-)_{j+1/2} - (\widehat{u_h^n}w^+)_{j-1/2}\right) = 0.
\end{aligned} \tag{3.3}$$

For the term u_h^0, u_h^{-1}, we take the L^2 projection of $u(\cdot,0)$, $u(\cdot,-1)$, respectively,

$$\begin{aligned}
&\int_{I_j} u_h^0\phi\,dx = \int_{I_j}\mathbb{P}u(x,0)\phi\,dx = \int_{I_j} u_0(x)\phi\,dx,\\
&\int_{I_j} u_h^{-1}\phi\,dx = \int_{I_j}\mathbb{P}u(x,-1)\phi\,dx = \int_{I_j} u(x,-1)\phi\,dx, \quad \forall\,\phi\in V_h^k,\ j=1,2,\ldots,N.
\end{aligned}$$

The “hat” terms in (3.3) at the cell boundary points are the so-called “numerical fluxes”. In this paper we could take the following purely alternating numerical fluxes,

$$\widehat{u_h^n} = (u_h^n)^-, \quad \widehat{p_h^n} = (p_h^n)^+, \tag{3.4}$$

or

$$\widehat{u_h^n} = (u_h^n)^+, \quad \widehat{p_h^n} = (p_h^n)^-.$$

Next, without loss of generality we take $f=0$ for simplicity in the theoretic analysis.

Theorem 3.1. *The fully-discrete LDG scheme* (3.3) *is unconditionally stable, and there exist a positive constant C depending on u, T, α, such that*

$$\|u_h^n\| \le C(\|u_h^0\| + \Delta t\|u_1(x)\|), \quad n=1,2,\ldots,M.$$

Proof. We take the test functions $\phi = u_h^n$, $w = \beta\rho p_h^n$ in scheme (3.3), and with the fluxes choice (3.4) we obtain

$$\begin{aligned}
&(3+2\Delta t)\|u_h^n\|^2 + \beta\rho\|p_h^n\|^2 + \beta\rho\sum_{j=1}^N\left(\Upsilon(u_h^n,p_h^n)_{j+1/2} - \Upsilon(u_h^n,p_h^n)_{j-1/2} + \Lambda(u_h^n,p_h^n)_{j-1/2}\right)\\
&= \sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})\int_\Omega(3u_h^i - 4u_h^{i-1} + u_h^{i-2})u_h^n\,dx + 2\Delta t b_{n-1}\int_\Omega v_h^0u_h^n\,dx + 4\int_\Omega u_h^{n-1}u_h^n\,dx\\
&- \int_\Omega u_h^{n-2}u_h^n\,dx + 2\Delta t\sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})\int_\Omega u_h^iu_h^n\,dx + 2\Delta t b_{n-1}\int_\Omega u_h^0u_h^n\,dx,
\end{aligned} \tag{3.5}$$

where

$$
\begin{aligned}
\Upsilon(u_h^n, p_h^n) &= (p_h^n)^-(u_h^n)^- - (p_h^n)^+(u_h^n)^- - (u_h^n)^-(p_h^n)^-,\\
\Lambda(u_h^n, p_h^n) &= (p_h^n)^-(u_h^n)^- - (p_h^n)^+(u_h^n)^+ - (p_h^n)^+(u_h^n)^- + (p_h^n)^+(u_h^n)^+ - (u_h^n)^-(p_h^n)^-\\
&\quad + (u_h^n)^-(p_h^n)^+\\
&= 0.
\end{aligned}
$$

Based on the equation (3.5), we have

$$
\begin{aligned}
(3+2\Delta t)\|u_h^n\|^2 \le & \sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})(3\|u_h^i\| + 4\|u_h^{i-1}\| + \|u_h^{i-2}\|)\|u_h^n\|\\
&+ 2\Delta t b_{n-1}\|v_h^0\|\|u_h^n\| + 4\|u_h^{n-1}\|\|u_h^n\| + \|u_h^{n-2}\|\|u_h^n\|\\
&+ 2\Delta t\sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})\|u_h^i\|\|u_h^n\| + 2\Delta t b_{n-1}\|u_h^0\|\|u_h^n\|,
\end{aligned}
$$

that is

$$
\begin{aligned}
\|u_h^n\| \le & \sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})(3\|u_h^i\| + 4\|u_h^{i-1}\| + \|u_h^{i-2}\|) + 2\Delta t b_{n-1}\|v_h^0\|\\
&+ \frac{4}{3}\|u_h^{n-1}\| + \|u_h^{n-2}\| + \sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})\|u_h^i\| + b_{n-1}\|u_h^0\|.
\end{aligned} \tag{3.6}
$$

Theorem 3.1 will be proved by mathematical induction. Let $n=1$ in (3.6), we have

$$
\|u_h^1\| \le 2\Delta t\|v_h^0\| + \frac{7}{3}\|u_h^0\| + \|u_h^{-1}\|.
$$

Notice that

$$
\begin{aligned}
\int_{I_j} u_h^{-1}\phi\,dx &= \int_{I_j}\mathbb{P}(u(x,0)-\Delta t u_1(x))\phi\,dx\\
&= \int_{I_j} u_h^0\phi\,dx - \Delta t\int_{I_j} u_1(x)\phi\,dx, \quad \phi\in V_h^k.
\end{aligned}
$$

Let $\phi = u_h^{-1}$, we could get

$$
\begin{aligned}
\|u_h^{-1}\|_{I_j}^2 &= \int_{I_j} u_h^0 u_h^{-1}\,dx - \Delta t\int_{I_j} u_1(x)u_h^{-1}\,dx\\
&\le \|u_h^0\|_{I_j}^2 + \frac{1}{4}\|u_h^{-1}\|_{I_j}^2 + (\Delta t)^2\|u_1(x)\|_{I_j}^2 + \frac{1}{4}\|u_h^{-1}\|_{I_j}^2.
\end{aligned}
$$

Let us sum the above inequality over j from 1 to N, we have

$$
\|u_h^{-1}\| \le C(\|u_h^0\| + \Delta t\|u_1(x)\|). \tag{3.7}
$$

Analogue to the proof of (3.7), we can obtain

$$\|v_h^0\| \le \|u_1(x)\|. \tag{3.8}$$

By using (3.7), (3.8) and (3.2), we know that

$$\|u_h^1\| \le C(\|u_h^0\| + \Delta t\|u_1(x)\|).$$

We suppose

$$\|u_h^l\| \le C(\|u_h^0\| + \Delta t\|u_1(x)\|), \quad l = 2, 3, \ldots, K. \tag{3.9}$$

Taking $n = K+1$ in the inequality (3.6), we get

$$\begin{aligned}\|u_h^{K+1}\| \le &\sum_{i=1}^{K}(b_{K-i} - b_{K+1-i})(3\|u_h^i\| + 4\|u_h^{i-1}\| + \|u_h^{i-2}\|) + 2\Delta t b_K\|v_h^0\| \\ &+ \frac{4}{3}\|u_h^K\| + \|u_h^{K-1}\| + \sum_{i=1}^{K}(b_{K-i} - b_{K+1-i})\|u_h^i\| + b_K\|u_h^0\|.\end{aligned}$$

Using (3.9), and the fact

$$\sum_{i=1}^{K}(b_{i-1} - b_i) + b_K = 1,$$

then we could get the following inequality immediately

$$\|u_h^{K+1}\| \le C(\|u_h^0\| + \Delta t\|u_1(x)\|).$$ □

Theorem 3.2. *Assume that $u(x, t_n)$ is the exact solution of the problem* (1.1)*, which is sufficiently smooth with bounded derivatives, and u_h^n is a solution of the scheme* (3.3)*, then there exists a positive constant C, such that*

$$\|u(x, t_n) - u_h^n\| \le C(h^{k+1} + (\Delta t)^{3-\alpha}),$$

where C is independent of Δt, h.

Proof. By virtue of Taylor expansion, we can get

$$|u(x, t_{-1}) - u(x, 0) + \Delta t u_1(x)| \le C(\Delta t)^{3-\alpha}.$$

From the property (2.4), we know,

$$\|u(x, t_{-1}) - u_h^{-1}\| \le C(h^{k+1} + (\Delta t)^2).$$

The exact solution $u(x,t_n)$ of (1.1) satisfies

$$
\begin{aligned}
&(3+2\Delta t)\int_\Omega u(x,t_n)\phi\,dx \\
&+\beta\rho\left(\int_\Omega p(x,t_n)\phi_x\,dx-\sum_{j=1}^{N}\big((p(x,t_n)\phi^-)_{j+1/2}-(p(x,t_n)\phi^+)_{j-1/2}\big)\right) \\
&+\beta\int_\Omega \gamma^n\phi\,dx \\
&=\sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})\int_\Omega(3u(x,t_i)-4u(x,t_{i-1})+u(x,t_{i-2}))\phi\,dx \\
&+2\Delta t b_{n-1}\int_\Omega v(x,t_0)\phi\,dx+4\int_\Omega u(x,t_{n-1})v\,dx \\
&-\int_\Omega u(x,t_{n-2})\phi\,dx+2\Delta t\sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})\int_\Omega u(x,t_i)\phi\,dx \\
&+2\Delta t b_{n-1}\int_\Omega u(x,t_0)\phi\,dx+\beta\int_\Omega f(x,t_n)\phi\,dx,
\end{aligned}
\tag{3.10}
$$

$$
\int_\Omega p(x,t_n)w\,dx+\int_\Omega u(x,t_n)w_x\,dx-\sum_{j=1}^{N}\big((u(x,t_n)w^-)_{j+1/2}-(u(x,t_n)w^+)_{j-1/2}\big)=0,
$$

where $\gamma^n=\gamma_3^n+\gamma_4^n$.

We denote

$$
\begin{aligned}
e_u^n&=u(x,t_n)-u_h^n=\mathcal{P}^-e_u^n-(\mathcal{P}^-u(x,t_n)-u(x,t_n)),\\
e_p^n&=p(x,t_n)-p_h^n=\mathcal{P}^+e_p^n-(\mathcal{P}^+p(x,t_n)-p(x,t_n)).
\end{aligned}
\tag{3.11}
$$

Taking the fluxes (3.4), and subtracting (3.3) from (3.10), we could have the following error equation

$$
\begin{aligned}
&(3+2\Delta t)\int_\Omega e_u^n\phi\,dx+\beta\rho\left(\int_\Omega e_p^n\phi_x\,dx-\sum_{j=1}^{N}\big(((e_p^n)^+\phi^-)_{j+1/2}+((e_p^n)^+\phi^+)_{j-1/2}\big)\right) \\
&-\sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})\int_\Omega(3e_u^i-4e_u^{i-1}+e_u^{i-2})\phi\,dx-2\Delta t b_{n-1}\int_\Omega e_v^0\phi\,dx \\
&-4\int_\Omega e_u^{n-1}\phi\,dx+\int_\Omega e_u^{n-2}\phi\,dx-2\Delta t\sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})\int_\Omega e_u^i\phi\,dx \\
&-2\Delta t b_{n-1}\int_\Omega e_u^0\phi\,dx+\beta\int_\Omega\gamma^n\phi\,dx+\int_\Omega e_p^n w\,dx+\int_\Omega e_u^n w_x\,dx \\
&-\sum_{j=1}^{N}\big(((e_u^n)^-w^-)_{j+1/2}+((e_u^n)^-w^+)_{j-1/2}\big)=0.
\end{aligned}
\tag{3.12}
$$

Using (3.11), the error equation (3.12) could be written as

(3.13)
$$
\begin{aligned}
&(3+2\Delta t)\int_\Omega \mathcal{P}^- e_u^n \phi\, dx + \beta\rho\left(\int_\Omega \mathcal{P}^+ e_p^n \phi_x\, dx - \sum_{j=1}^{N}\big(((\mathcal{P}^+ e_p^n)^+\phi^-)_{j+1/2} + ((\mathcal{P}^+ e_p^n)^+\phi^+)_{j-1/2}\big)\right)\\
&+\int_\Omega \mathcal{P}^+ e_p^n w\, dx + \int_\Omega \mathcal{P}^- e_u^n w_x\, dx - \sum_{j=1}^{N}\big(((\mathcal{P}^- e_u^n)^- w^-)_{j+1/2} + ((\mathcal{P}^- e_u^n)^- w^+)_{j-1/2}\big)\\
=&\sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})\int_\Omega (3\mathcal{P}^- e_u^i - 4\mathcal{P}^- e_u^{i-1} + \mathcal{P}^- e_u^{i-2})\phi\, dx - 2\Delta t b_{n-1}\int_\Omega e_v^0 \phi\, dx\\
&+4\int_\Omega \mathcal{P}^- e_u^{n-1}\phi\, dx - \int_\Omega \mathcal{P}^- e_u^{n-2}\phi\, dx - \beta\int_\Omega \gamma^n \phi\, dx + 2\Delta t b_{n-1}\int_\Omega \mathcal{P}^- e_u^0 \phi\, dx\\
&+2\Delta t\sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})\int_\Omega \mathcal{P}^- e_u^i\phi\, dx - \sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})\\
&\quad\times\int_\Omega \big(3(\mathcal{P}^- u(x,t_i)-u(x,t_i)) - 4(\mathcal{P}^- u(x,t_{i-1})-u(x,t_{i-1})) + (\mathcal{P}^- u(x,t_{i-2})-u(x,t_{i-2}))\big)\phi\, dx\\
&-4\int_\Omega (\mathcal{P}^- u(x,t_{n-1})-u(x,t_{n-1}))\phi\, dx + \int_\Omega (\mathcal{P}^- u(x,t_{n-2})-u(x,t_{n-2}))\phi\, dx\\
&-2\Delta t\sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})\int_\Omega (\mathcal{P}^- u(x,t_i)-u(x,t_i))\phi\, dx - 2\Delta t b_{n-1}\int_\Omega (\mathcal{P}^- u(x,t_0)-u(x,t_0))\phi\, dx\\
&+(3+2\Delta t)\int_\Omega (\mathcal{P}^- u(x,t_n)-u(x,t_n))\phi\, dx + \beta\rho\bigg(\int_\Omega (\mathcal{P}^+ p(x,t_n)-p(x,t_n))\phi_x\, dx\\
&\quad-\sum_{j=1}^{N}\big(((\mathcal{P}^+ p(x,t_n)-p(x,t_n))^+\phi^-)_{j+1/2} + ((\mathcal{P}^+ p(x,t_n)-p(x,t_n))^+\phi^+)_{j-1/2}\big)\bigg)\\
&+\int_\Omega (\mathcal{P}^+ p(x,t_n)-p(x,t_n))w\, dx + \int_\Omega (\mathcal{P}^- u(x,t_n)-u(x,t_n))w_x\, dx\\
&-\sum_{j=1}^{N}\big(((\mathcal{P}^- u(x,t_n)-u(x,t_n))^- w^-)_{j+1/2} + ((\mathcal{P}^- u(x,t_n)-u(x,t_n))^- w^+)_{j-1/2}\big).
\end{aligned}
$$

Taking the test functions $\phi = \mathcal{P}^- e_u^n$, $w = \beta\rho\mathcal{P}^+ e_p^n$ in (3.13), and by virtue of the properties (2.1)–(2.3), then the following inequality holds

$$
\begin{aligned}
&(3+2\Delta t)\|\mathcal{P}^- e_u^n\|^2 + \beta\rho\|\mathcal{P}^+ e_p^n\|^2\\
\le&\sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})\left(3\|\mathcal{P}^- e_u^i\| + 4\|\mathcal{P}^- e_u^{i-1}\| + \|\mathcal{P}^- e_u^{i-2}\|\right)\|\mathcal{P}^- e_u^n\|\\
&+2\Delta t b_{n-1}\|e_v^0\|\|\mathcal{P}^- e_u^n\| + 4\|\mathcal{P}^- e_u^{n-1}\|\|\mathcal{P}^- e_u^n\| + \|\mathcal{P}^- e_u^{n-2}\|\|\mathcal{P}^- e_u^n\| + \beta\|\gamma^n\|\|\mathcal{P}^- e_u^n\|\\
&+2\Delta t\sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})\|\mathcal{P}^- e_u^i\|\|\mathcal{P}^- e_u^n\| + 2\Delta t b_{n-1}\|\mathcal{P}^- e_u^0\|\|\mathcal{P}^- e_u^n\|\\
&+\sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})\big(3\|\mathcal{P}^- u(x,t_i)-u(x,t_i)\| + 4\|\mathcal{P}^- u(x,t_{i-1})-u(x,t_{i-1})\|
\end{aligned}
$$

$$
\begin{aligned}
&\qquad + \|\mathcal{P}^- u(x,t_{i-2}) - u(x,t_{i-2})\|\big)\|\mathcal{P}^- e_u^n\| \\
&+ 4\|\mathcal{P}^- u(x,t_{n-1}) - u(x,t_{n-1})\|\|\mathcal{P}^- e_u^n\| + \|\mathcal{P}^- u(x,t_{n-2}) - u(x,t_{n-2})\|\|\mathcal{P}^- e_u^n\| \\
&+ 2\Delta t b_{n-1}\|\mathcal{P}^- u(x,t_0) - u(x,t_0)\|\|\mathcal{P}^- e_u^n\| \\
&+ 2\Delta t \sum_{i=1}^{n-1}(b_{n-i-1} - b_{n-i})\|\mathcal{P}^- u(x,t_i) - u(x,t_i)\|\|\mathcal{P}^- e_u^n\| \\
&+ (3+2\Delta t)\|\mathcal{P}^- u(x,t_n) - u(x,t_n)\|\|\mathcal{P}^- e_u^n\| + \beta\rho\|\mathcal{P}^+ p(x,t_n) - p(x,t_n))\|\|\mathcal{P}^- e_p^n\|.
\end{aligned}
$$

Based on the fact that

$$
ab \le \frac{1}{4\varepsilon}a^2 + \varepsilon b^2, \quad a^2 + b^2 \le (a+b)^2,
$$

we can get the following inequality immediately

$$
\begin{aligned}
&\|\mathcal{P}^- e_u^n\| \\
&\le \sum_{i=1}^{n-1}(b_{n-i-1} - b_{n-i})\left(3\|\mathcal{P}^- e_u^i\| + 4\|\mathcal{P}^- e_u^{i-1}\| + \|\mathcal{P}^- e_u^{i-2}\|\right) + 2\Delta t b_{n-1}\|e_v^0\| \\
&\quad + 4\|\mathcal{P}^- e_u^{n-1}\| + \|\mathcal{P}^- e_u^{n-2}\| + \|\gamma^n\| + 2\Delta t \sum_{i=1}^{n-1}(b_{n-i-1} - b_{n-i})\|\mathcal{P}^- e_u^i\| + 2\Delta t b_{n-1}\|\mathcal{P}^- e_u^0\| \\
&\quad + \sum_{i=1}^{n-1}(b_{n-i-1} - b_{n-i})\big(3\|\mathcal{P}^- u(x,t_i) - u(x,t_i)\| \\
&\qquad + 4\|\mathcal{P}^- u(x,t_{i-1}) - u(x,t_{i-1})\| + \|\mathcal{P}^- u(x,t_{i-2}) - u(x,t_{i-2})\|\big) \\
&\quad + 4\|\mathcal{P}^- u(x,t_{n-1}) - u(x,t_{n-1})\| + \|\mathcal{P}^- u(x,t_{n-2}) - u(x,t_{n-2})\| \\
&\quad + 2\Delta t \sum_{i=1}^{n-1}(b_{n-i-1} - b_{n-i})\|\mathcal{P}^- u(x,t_i) - u(x,t_i)\| + 2\Delta t b_{n-1}\|\mathcal{P}^- u(x,t_0) - u(x,t_0)\| \\
&\quad + (3+2\Delta t)\|\mathcal{P}^- u(x,t_n) - u(x,t_n)\| + \sqrt{\beta\rho}\|\mathcal{P}^+ p(x,t_n) - p(x,t_n))\|.
\end{aligned} \tag{3.14}
$$

The convergence of the scheme (3.3) will be proved by mathematical induction. Let $n = 1$ in the above inequality (3.14), we can obtain

$$
\begin{aligned}
\|\mathcal{P}^- e_u^1\| \le\ & 2\Delta t\|e_v^0\| + 4\|\mathcal{P}^- e_u^0\| + \|\mathcal{P}^- e_u^{-1}\| + \|\gamma^1\| + 2\Delta t\|\mathcal{P}^- e_u^0\| \\
& + 4\|\mathcal{P}^- u(x,t_0) - u(x,t_0)\| + \|\mathcal{P}^- u(x,t_{-1}) - u(x,t_{-1})\| \\
& + 2\Delta t\|\mathcal{P}^- u(x,t_0) - u(x,t_0)\| + (3+2\Delta t)\|\mathcal{P}^- u(x,t_1) - u(x,t_1)\| \\
& + \sqrt{\beta\rho}\|\mathcal{P}^+ p(x,t_1) - p(x,t_1))\|.
\end{aligned}
$$

Notice the fact that

$$
\|e_u^0\| \le Ch^{k+1}, \quad \|e_v^0\| \le Ch^{k+1}, \quad \|e_u^{-1}\| \le C(h^{k+1} + (\Delta t)^2), \quad \|\gamma^1\| \le C(\Delta t)^{3-\alpha},
$$

we can obtain

$$\|\mathcal{P}^- e_u^1\| \le C(h^{k+1} + (\Delta t)^{3-\alpha}).$$

We suppose

$$\|\mathcal{P}^- e_u^l\| \le C(h^{k+1} + (\Delta t)^{3-\alpha}), \quad l = 1, 2, \ldots, K. \tag{3.15}$$

Take $n = K+1$ in (3.14), we get

$$\begin{aligned}
&\|\mathcal{P}^- e_u^{K+1}\| \\
\le& \sum_{i=1}^{K} (b_{K-i} - b_{K+1-i}) \left(3\|\mathcal{P}^- e_u^i\| + 4\|\mathcal{P}^- e_u^{i-1}\| + \|\mathcal{P}^- e_u^{i-2}\|\right) + 2\Delta t b_K \|e_v^0\| \\
&+ 4\|\mathcal{P}^- e_u^K\| + \|\mathcal{P}^- e_u^{K-1}\| + \|\gamma^{K+1}\| + 2\Delta t \sum_{i=1}^{K} (b_{K-i} - b_{K+1-i}) \|\mathcal{P}^- e_u^i\| + 2\Delta t b_K \|\mathcal{P}^- e_u^0\| \\
&+ \sum_{i=1}^{K} (b_{K-i} - b_{K+1-i}) \big(3\|\mathcal{P}^- u(x,t_i) - u(x,t_i)\| \\
&\qquad + 4\|\mathcal{P}^- u(x,t_{i-1}) - u(x,t_{i-1})\| + \|\mathcal{P}^- u(x,t_{i-2}) - u(x,t_{i-2})\|\big) \\
&+ 4\|\mathcal{P}^- u(x,t_K) - u(x,t_K)\| + \|\mathcal{P}^- u(x,t_{K-1}) - u(x,t_{K-1})\| \\
&+ 2\Delta t \sum_{i=1}^{K} (b_{K-i} - b_{K+1-i}) \|\mathcal{P}^- u(x,t_i) - u(x,t_i)\| \\
&+ 2\Delta t b_K \|\mathcal{P}^- u(x,t_0) - u(x,t_0)\| + \sqrt{\beta\rho} \|\mathcal{P}^+ p(x,t_{K+1}) - p(x,t_{K+1}))\| \\
&+ (3 + 2\Delta t) \|\mathcal{P}^- u(x,t_{K+1}) - u(x,t_{K+1})\|.
\end{aligned}$$

Using the assumption (3.15) and the property (2.4), we have

$$\|\mathcal{P}^- e_u^{K+1}\| \le C(h^{k+1} + (\Delta t)^{3-\alpha}).$$

Then Theorem 3.2 holds by the triangle inequality and the interpolating property (2.4) immediately. □

4. Numerical experiments

In this section, the validity and accuracy of convergence of the presented finite difference/local discontinuous Galerkin method are demonstrated by a test example.

Example 4.1. Let

$$f(x,t) = \frac{6t^{3-\alpha}}{\Gamma(4-\alpha)} \sin(2\pi x) + \frac{6t^{4-\alpha}}{\Gamma(5-\alpha)} \sin(2\pi x) + 4\pi^2 t^3 \sin(2\pi x).$$

Compute the time-fractional telegraph equation (1.1) with $\rho = 1$ at $T = 1$ numerically. The exact solution is

$$u(x,t) = t^3 \sin(2\pi x).$$

The initial condition and the boundary values are obtained directly. The experimental convergence rate κ is given by

$$\kappa = \frac{\log(\|u - u_{h_1}\|/\|u - u_{h_2}\|)}{\log(h_1/h_2)},$$

where u and u_{h_i} are exact and numerical solutions with the space step h_i, respectively.

The L^2 and L^∞ errors and the numerical orders of accuracy in space for piecewise P^k polynomials for several α are listed in Tables 4.1–4.3. It can be concluded that the order of convergence of the present method is in good agreement with the results by the theoretical analysis.

	N	L^2-error	order	L^∞-error	order
P^0	5	0.264326949540113	-	0.621561373185215	-
	10	0.129217214344293	1.03	0.313359291007195	0.99
	20	6.424721062992099E-002	1.01	0.156980797588175	1.00
	40	3.207865052457315E-002	1.00	7.852745266437230E-002	1.00
P^1	5	6.731070096298464E-002	-	0.249586596018240	-
	10	1.695175223856232E-002	1.99	6.466637138748532E-002	1.95
	20	4.244932733871986E-003	2.00	1.630903615444002E-002	1.99
	40	1.061659704993896E-003	2.00	4.101466164679879E-003	1.99
P^2	5	6.673082225915985E-003	-	3.169297582744446E-002	-
	10	8.503922299704529E-004	2.97	3.969530568838230E-003	3.00
	20	1.068093589154052E-004	2.99	5.115680438094785E-004	2.96
	40	1.336700752607358E-005	3.00	6.442691300446294E-005	2.99
P^3	5	5.180718129025192E-004	-	2.662069613503348E-003	-
	10	3.289042835987682E-005	3.98	1.844251227173244E-004	3.85
	20	2.067264781375035E-006	3.99	1.165406407133140E-005	3.98
	40	1.710535105672870E-007	3.60	8.912161295615562E-007	3.71

Table 4.1: Spatial accuracy test for the time-fractional telegraph equation using piecewise P^k polynomials, $\alpha = 1.1$, $\Delta t = 1/1000$, $T = 1$.

Then the convergence order of the scheme in time is tested. All the computations were performed in double precision. With the fixed and sufficiently small step sizes $h = 1/200$ and the varying $\Delta t = 0.08, 0.04, 0.02, 0.01$, respectively. From Table 4.4, one can conclude that the convergence order of the scheme in time is $3 - \alpha$ in L^2-norm and L^∞-norm.

	N	L^2-error	order	L^∞-error	order
P^0	5	0.263772539661747	-	0.620136302765829	-
	10	0.129150891409391	1.03	0.313183577920604	0.99
	20	6.423754896418950E-002	1.01	0.156954961792935	1.00
	40	3.207668202660861E-002	1.00	7.852213681766884E-002	1.00
P^1	5	6.727500011224508E-002	-	0.249359282860865	-
	10	1.694965935784688E-002	1.99	6.462004769896668E-002	1.95
	20	4.244838046759258E-003	2.00	1.627212755394636E-002	1.99
	40	1.061972803523747E-003	2.00	4.065578259994984E-003	2.00
P^2	5	6.664743310941978E-003	-	3.162805983636763E-002	-
	10	8.504933312440419E-004	2.97	3.968008312843709E-003	2.99
	20	1.096992329019471E-004	2.95	5.115101434150278E-004	2.96
	40	1.539724846989132E-005	2.83	7.170126363342799E-005	2.83
P^3	5	5.185970601446759E-004	-	2.685710565843769E-003	-
	10	3.238836964305419E-005	4.00	1.891130668112162E-004	3.82
	20	2.014143199241720E-006	4.01	1.322340700746841E-005	3.84
	40	1.535535584291874E-007	3.71	8.824432540660916E-007	3.91

Table 4.2: Spatial accuracy test for the time-fractional telegraph equation using piecewise P^k polynomials, $\alpha = 1.5$, $\Delta t = 1/1000$, $T = 1$.

5. Conclusion

In this work, we have presented a new finite difference/local discontinuous Galerkin method to solve the time fractional telegraph equation. We first propose a new finite difference method to approximate the time fractional derivatives when $1 < \alpha < 2$, and then give a fully discrete scheme, and prove that the method is unconditionally stable and convergent. To date we are not aware of the same scheme in the published papers. In future we will develop the method discussed in this paper to other kinds of fractional equations and higher-dimensional problems.

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	N	L^2-error	order	L^∞-error	order
P^0	5	0.263596619763754	-	0.619678669709020	-
	10	0.129118405374179	1.03	0.313096311598061	0.99
	20	6.422878978148322E-002	1.01	0.156931036435848	1.00
	40	3.207336315447732E-002	1.00	7.851269909520331E-002	1.00
P^1	5	6.726057884101144E-002	-	0.249167181129090	-
	10	1.694853118433292E-002	1.99	6.450530002514832E-002	1.95
	20	4.245791889524003E-003	2.00	1.616039418034410E-002	2.00
	40	1.066701009253817E-003	1.99	3.955375210516143E-003	2.03
P^2	5	6.660380245045741E-003	-	3.154569653495143E-002	-
	10	8.539452815388900E-004	2.96	3.967031894675847E-003	2.99
	20	1.049541660337010E-004	3.02	5.114594628408880E-004	2.96
	40	1.558824291902685E-005	2.75	7.199456294184820E-005	2.83
P^3	5	5.250217141273616E-004	-	2.745121197051392E-003	-
	10	3.392756788959759E-005	3.95	1.999707874018487E-004	3.78
	20	2.154156320733310E-006	3.98	1.393794504025891E-005	3.84
	40	1.451303017336259E-007	3.89	8.877277347282901E-007	3.97

Table 4.3: Spatial accuracy test for the time-fractional telegraph equation using piecewise P^k polynomials, $\alpha = 1.7$, $\Delta t = 1/1000$, $T = 1$.

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	Δt	L^2-error	order	L^∞-error	order
$\alpha = 1.2$	0.08	3.814082004142095E-004	-	5.393867006779907E-004	-
	0.04	1.059433850536936E-004	1.84	1.498212286663581E-004	1.85
	0.02	3.031054272196768E-005	1.81	4.286002375386566E-005	1.81
	0.01	9.234809901325062E-006	1.71	1.306269682344041E-005	1.71
$\alpha = 1.5$	0.08	1.168752125423230E-003	-	1.652859246798588E-003	-
	0.04	4.014237926405224E-004	1.54	5.676937460528109E-004	1.54
	0.02	1.401552962754790E-004	1.52	1.982042254247141E-004	1.52
	0.01	4.863930743644567E-005	1.53	6.878087961470492E-005	1.53
$\alpha = 1.8$	0.08	3.733664761753826E-003	-	5.280193579009884E-003	-
	0.04	1.612515495557272E-003	1.21	2.280436132674701E-003	1.21
	0.02	7.115090305684545E-004	1.18	1.006220463059870E-003	1.18
	0.01	3.132591173068265E-004	1.18	4.430096972589670E-004	1.18

Table 4.4: Temporal accuracy test using piecewise P^2 polynomials for different α when $N = 200$, $T = 1$.

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