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Classification and Evolution of Bifurcation Curves for a Dirichlet-Neumann Boundary Value Problem and its Application

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In memory of Professor Hwai-Chiuan Wang

Abstract. We study the classification and evolution of bifurcation curves of positive solutions for the one-dimensional Dirichlet-Neumann boundary value problem

$$\begin{cases} u''(x) + \lambda f(u) = 0, & 0 < x < 1, \\ u(0) = 0, \quad u'(1) = -c < 0, \end{cases}$$

where $\lambda > 0$ is a bifurcation parameter and $c > 0$ is an evolution parameter. We mainly prove that, under some suitable assumptions on f, there exists $c_1 > 0$, such that, on the $(\lambda, \|u\|_\infty)$-plane, (i) when $0 < c < c_1$, the bifurcation curve is S-shaped; (ii) when $c \geq c_1$, the bifurcation curve is $\subset$-shaped. Our results can be applied to the one-dimensional perturbed Gelfand equation with $f(u) = \exp\left(\frac{au}{a+u}\right)$ for $a \geq 4.37$.

1. Introduction

In this paper, we study the classification and evolution of bifurcation curves of positive solutions for the one-dimensional Dirichlet-Neumann boundary value problem

$$\begin{cases} u''(x) + \lambda f(u) = 0, & 0 < x < 1, \\ u(0) = 0, \quad u'(1) = -c < 0, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a bifurcation parameter and the value $c > 0$, with which $-c$ is the boundary slope of $u(x)$ at $x = 1$, is treated as an evolution parameter. We assume that nonlinearity $f \in C^2[0, \infty)$ satisfies the following hypotheses (H1)–(H6):

(H1) $f(u), f'(u) > 0$ on $[0, \infty)$.

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(H2) f is convex-concave on $(0,\infty)$; that is, there exists a number $\gamma > 0$ such that

$$f''(u)\begin{cases} > 0 & \text{when } u \in [0,\gamma), \\ = 0 & \text{when } u = \gamma, \\ < 0 & \text{when } u \in (\gamma,\infty). \end{cases}$$

In addition, $\gamma^2 f(\gamma) \geq 3\int_0^\gamma t f(t)\,dt$.

(H3) f is asymptotic sublinear at infinity; that is, $\lim_{u\to\infty} f(u)/u = 0$.

(H4) There exists a number $\tau > 0$ such that

$$[f'(u) + uf''(u)]f(u) - u[f'(u)]^2 \begin{cases} > 0 & \text{when } u \in [0,\tau), \\ = 0 & \text{when } u = \tau, \\ < 0 & \text{when } u \in (\tau,\infty). \end{cases}$$

(H5) $[f'(u)]^2 - f''(u)f(u) > 0$ on $[0,\infty)$.

(H6) Define

$$F(u) = \int_0^u f(t)\,dt \quad \text{for } u \geq 0, \tag{1.2}$$

$$M_1(u) = \frac{uf(u)}{F(u)} \quad \text{for } u > 0, \tag{1.3}$$

$$M_2(u) = \frac{uf'(u)}{f(u)} \quad \text{for } u \geq 0, \tag{1.4}$$

$$N_1(u) = \frac{2f'(0)}{[f(0)]^2}uf(u) + M_1(u) - 2M_2(u) \quad \text{for } u > 0, \tag{1.5}$$

$$N_2(u) = N_1(u) - 3M_2(u) - 2, \quad N_3(u) = -2N_1(u) + 3M_2(u) + 3 \quad \text{for } u > 0, \tag{1.6}$$

$$P_1(u,s) = \frac{uf(u) - sf(s)}{F(u) - F(s)}, \quad P_2(u,s) = \frac{u^2 f'(u) - s^2 f'(s)}{uf(u) - sf(s)} \quad \text{for } 0 \leq s < u, \tag{1.7}$$

$$W(u,s) = P_1(u,s)[N_2(u) + 2P_2(u,s)] \quad \text{for } 0 \leq s < u, \tag{1.8}$$

$$\widetilde{W}_0(u) = \left[\frac{\partial}{\partial s} W(u,s)\right]_{s=0} \quad \text{for } u > 0. \tag{1.9}$$

Then

$$f(\tau) \geq 4f(0) \tag{1.10}$$

and the following three inequalities related to $W(u,s)$ hold:

$$\widetilde{W}_0(u) > 0 \quad \text{for } 0 < u \leq \bar{\rho}, \tag{1.11}$$

$$W(u,s) \geq W(u,0) + s\widetilde{W}_0(u) \quad \text{for } 0 \leq s < u \leq \bar{\rho}, \tag{1.12}$$

$$3\sqrt{f(u)}[W(u,0) + N_3(u)] + 2u\sqrt{f(0)}\widetilde{W}_0(u) > 0 \quad \text{for } 0 < u \leq \bar{\rho}, \tag{1.13}$$

where $\overline{\rho}$ is the unique positive zero of

$$2f(0)[2 - M_1(u)] - f(u)$$

on $(0, \tau)$.

Remark 1.1. The existence and uniqueness of the number $\overline{\rho}$ in $(0, \tau)$ in (H6) is proved in Lemma 3.1(ix) stated behind.

The motivation of this paper arises from the recent work of Liang and Wang [11]. In [11], the authors considered the classification and evolution of bifurcation curves of positive solutions for the one-dimensional perturbed Gelfand equation with Dirichlet-Neumann boundary conditions, i.e., (1.1) with $f(u) = \exp\left(\frac{au}{a+u}\right)$, $a > 0$. In this paper, we generalize their main results to general nonlinearities $f(u)$ under hypotheses (H1)–(H6).

For one-dimensional *zero Dirichlet* boundary value problem with general nonlinearity:

$$\begin{cases} u''(x) + \lambda f(u) = 0, & 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{1.14}$$

the shapes of the bifurcation curve of positive solutions for (1.14) on the $(\lambda, \|u\|_\infty)$-plane have been studied exuberantly; see, e.g., [1, 5, 6, 8, 10] and references therein. While the shapes of the bifurcation curves with the mixed boundary conditions such as (1.1) are much less studied; see [2–4, 7, 11]. We define the bifurcation curve of positive solutions of (1.1) by

$$\widetilde{S}_c = \{(\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of (1.1)}\},$$

while that of (1.14) is defined by

$$S = \{(\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of (1.14)}\}.$$

Moreover, we say that, on the $(\lambda, \|u\|_\infty)$-plane, the shape of a bifurcation curve $\widetilde{S}_c$ (same for S) is S-shaped or $\subset$-shaped if it satisfies the following conditions, respectively.

S-shaped. The bifurcation curve $\widetilde{S}_c$ on the $(\lambda, \|u\|_\infty)$-plane is said to be *S-shaped* if $\widetilde{S}_c$ has *at least two* turning points, say $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ and $(\lambda_*, \|u_{\lambda_*}\|_\infty)$, satisfying $\lambda_* < \lambda^*$ and $\|u_{\lambda^*}\|_\infty < \|u_{\lambda_*}\|_\infty$, and

(i) $\widetilde{S}_c$ starts at some point $(\lambda_0, \|u_{\lambda_0}\|_\infty)$ and initially continues to the *right*,

(ii) at $(\lambda^*, \|u_{\lambda^*}\|_\infty)$, $\widetilde{S}_c$ turns to the *left*,

(iii) at $(\lambda_*, \|u_{\lambda_*}\|_\infty)$, $\widetilde{S}_c$ turns to the *right*,

(iv) $\widetilde{S}_c$ tends to infinity as $\lambda \to \infty$. That is, $\lim_{\lambda\to\infty} \|u_\lambda\|_\infty = \infty$.

Exactly S-shaped. The bifurcation curve $\widetilde{S}_c$ on the $(\lambda, \|u\|_\infty)$-plane is said to be *exactly S-shaped* if $\widetilde{S}_c$ is S-shaped and it has *exactly two* turning points; see Figure 1.1.

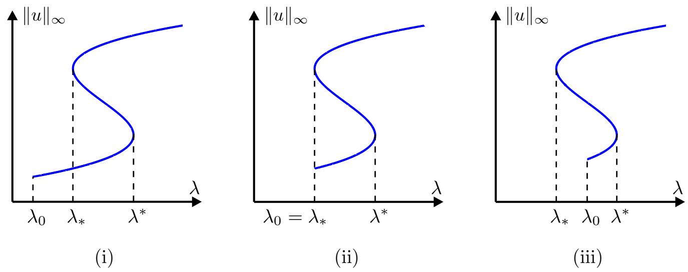

Figure 1.1: Three different types of exactly S-shaped bifurcation curves $\widetilde{S}_c$ with $\lambda_0 > 0$ and $\|u_{\lambda_0}\|_\infty > 0$. (i) Type 1 with $\lambda_0 < \lambda_*$. (ii) Type 2 with $\lambda_0 = \lambda_*$. (iii) Type 3 with $\lambda_0 > \lambda_*$.

Type 1/2/3 S-shaped. Assume that the bifurcation curve $\widetilde{S}_c$ is S-shaped on the $(\lambda, \|u\|_\infty)$-plane. Let $(\lambda_0, \|u_{\lambda_0}\|_\infty)$ be the starting point of $\widetilde{S}_c$, and

$$\overline{\lambda}_{\min} \equiv \min\{\lambda : (\lambda, \|u_\lambda\|_\infty) \text{ is a turning point of } \widetilde{S}_c\}.$$

Then $\widetilde{S}_c$ is said to be type 1 (resp., type 2 and type 3) S-shaped if $\lambda_0 < \overline{\lambda}_{\min}$ (resp., $\lambda_0 = \overline{\lambda}_{\min}$ and $\lambda_0 > \overline{\lambda}_{\min}$); see Figure 1.1(i) (resp., Figures 1.1(ii) and 1.1(iii)).

$\subset$-shaped. The bifurcation curve $\widetilde{S}_c$ on the $(\lambda, \|u\|_\infty)$-plane is said to be *$\subset$-shaped* if $\widetilde{S}_c$ has *at least one* turning point $(\lambda_*, \|u_{\lambda_*}\|_\infty)$, and

(i) $\widetilde{S}_c$ starts at some point $(\lambda_0, \|u_{\lambda_0}\|_\infty)$ and initially continues to the *left*,

(ii) at $(\lambda_*, \|u_{\lambda_*}\|_\infty)$, $\widetilde{S}_c$ turns to the *right*,

(iii) $\lambda_* < \lambda_0$ and $\|u_{\lambda_0}\|_\infty < \|u_{\lambda_*}\|_\infty$,

(iv) $\widetilde{S}_c$ tends to infinity as $\lambda \to \infty$. That is, $\lim_{\lambda\to\infty} \|u_\lambda\|_\infty = \infty$.

Exactly $\subset$-shaped. The bifurcation curve $\widetilde{S}_c$ on the $(\lambda, \|u\|_\infty)$-plane is said to be *exactly $\subset$-shaped* if $\widetilde{S}_c$ is $\subset$-shaped and it has *exactly one* turning point; see Figure 1.2.

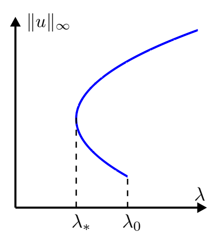

Figure 1.2: Exactly $\subset$-shaped bifurcation curve $\widetilde{S}_c$ with $\lambda_0 > 0$ and $\|u_{\lambda_0}\|_\infty > 0$.

For one-dimensional zero Dirichlet boundary value problem (1.14), under (H1)–(H3) on f, Hung and Wang [6] proved that the bifurcation curve of positive solutions is exactly type 1 S-shaped on the $(\lambda, \|u\|_\infty)$-plane. They gave an application to the one-dimensional perturbed Gelfand problem.

Theorem 1.2. [6, Theorems 2.1(i) and 2.2(i)] *Consider* (1.14) *and suppose that nonlinearity $f \in C^2[0,\infty)$ satisfies* (H1)–(H3). *Then, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve S is a continuous curve which starts at the origin and tends to infinity as $\lambda \to \infty$. Moreover, it is exactly type 1 S-shaped. In particular, $f(u) = \exp\left(\frac{au}{a+u}\right)$ satisfies* (H1)–(H3) *for $a \geq a^* \approx 4.166$ for some a^* defined in* [6, Eq. (3.22)].

For one-dimensional Dirichlet-Neumann boundary value problem (1.1) with $f(u) = \exp\left(\frac{au}{a+u}\right)$, Goddard II, Shivaji and Lee [3, Section 3.4] started to consider with $c = 1$. Their computational results suggested that there exists a positive critical bifurcation value $a^{**} < 4$ such that $\widetilde{S}_1$ is strictly increasing for $0 < a \leq a^{**}$ and is exactly S-shaped for $a > a^{**}$. Hung, Wang and Yu [7] gave rigorous proofs for some of these computational results. Recently, Liang and Wang [11] generalized these analytic results to general $c > 0$. The main result in [11] is stated in the next theorem.

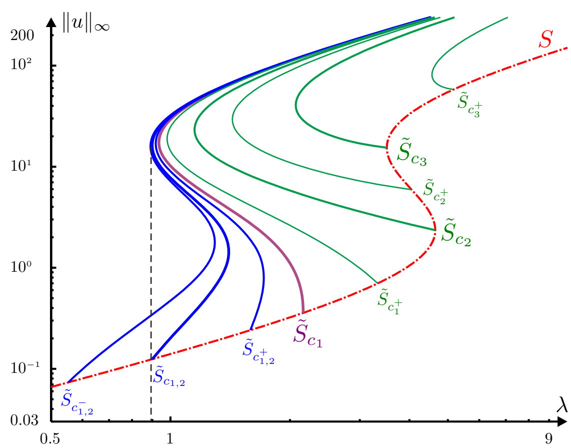

Figure 1.3: (cf. [11, Figure 4]) Numerical simulations of bifurcation curves S and $\widetilde{S}_c$ for $f(u) = \exp\left(\frac{au}{a+u}\right)$ with $a = 5$ and varying $c > 0$ on the $(\lambda, \|u_\lambda\|_\infty)$-plane of the bi-logarithm coordinates. Here $0 < c_{1,2}^- < c_{1,2}$ $(\approx 0.49) < c_{1,2}^+ < c_1$ $(\approx 1.36) < c_1^+ < c_2$ $(\approx 7.72) < c_2^+ < c_3$ $(\approx 47.71) < c_3^+$.

Theorem 1.3. (cf. [11, Theorem 2.3], see Figure 1.3 with $a = 5$) *Consider* (1.1) *with $c > 0$ and $f(u) = \exp\left(\frac{au}{a+u}\right)$ for any fixed $a > 0$. Then, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve $\widetilde{S}_c$ is a continuous curve which starts at some point $(\lambda_0, \|u_{\lambda_0}\|_\infty)$ with $\lambda_0 > 0$ and $\|u_{\lambda_0}\|_\infty > 0$ and it tends to infinity as $\lambda \to \infty$. Moreover, when $a \geq a_1$ (≈ 4.107), where*

a_1 is defined in [6, Eq. (1.4)], *there exists c_1 ($= c_1(a)$) such that the following assertions* (a)–(b) *hold:*

(a) *For $0 < c < c_1$, the bifurcation curve $\widetilde{S}_c$ is S-shaped on the $(\lambda, \|u\|_\infty)$-plane. More precisely, there exist three positive $c_{1,1} \le c_{1,2} \le c_{1,3}$ on $(0, c_1)$, all depending on a, such that the S-shaped belongs to type* 1*, type* 2 *and type* 3 *when $0 < c < c_{1,1}$, $c = c_{1,2}$ and $c_{1,3} < c < c_1$, respectively.*

(b) *For $c \ge c_1$, the bifurcation curve $\widetilde{S}_c$ is $\subset$-shaped on the $(\lambda, \|u\|_\infty)$-plane.*

The paper is organized as follows. Section 2 contains statements of the main results (Theorems 2.1, 2.3–2.5; in particular, Theorems 2.4 and 2.5). Section 3 contains several lemmas needed to prove the main results and their proofs except those of Lemma 3.1(i)–(ii), (ix)–(x) and Lemma 3.7. Section 4 contains proofs of the main results except assertions (a)–(d) stated in the proof of Theorem 2.5 for the function $f(\rho) = \exp\left(\frac{a\rho}{a+\rho}\right)$ with $a \ge 4.37$. (The proofs of Lemma 3.1(i)–(ii), (ix)–(x) and Lemma 3.7 and assertions (a)–(d) stated in the proof of Theorem 2.5 are put in [9] due to their lengthiness.)

2. Main results

The main results are next Theorems 2.1, 2.3–2.5; in particular, Theorems 2.4 and 2.5.

Theorem 2.1. (cf. Figure 1.3 for $f(u) = \exp\big(au/(a+u)\big)$ with $a = 5$) *Consider* (1.1) *with $c > 0$ and suppose that f satisfies* (H1) *and* (H3)–(H5). *Then, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve $\widetilde{S}_c$ is a continuous curve which starts at some point $(\lambda_0, \|u_{\lambda_0}\|_\infty)$ with λ_0 ($= \lambda_0(c)$) > 0 and $\|u_{\lambda_0}\|_\infty > 0$. More precisely, the following assertions* (i)–(iv) *hold:*

(i) *$S \cap \widetilde{S}_c = \{(\lambda_0, \|u_{\lambda_0}\|_\infty)\}$. Moreover, if u_λ is a positive solution of* (1.1) *with $u_\lambda \ne u_{\lambda_0}$, then $\|u_\lambda\|_\infty > \|u_{\lambda_0}\|_\infty$.*

(ii) *$\widetilde{S}_c$ tends to infinity as $\lambda \to \infty$. That is, $\lim_{\lambda\to\infty} \|u_\lambda\|_\infty = \infty$.*

(iii) *For any $\rho > \rho_0(c) \equiv \|u_{\lambda_0(c)}\|_\infty > 0$, there exist exactly two positive $\widetilde{\lambda}(\rho) < \lambda(\rho)$ such that $(\widetilde{\lambda}(\rho), \rho) \in \widetilde{S}_c$ and $(\lambda(\rho), \rho) \in S$.*

(iv) *$\rho_0(c) \in C(0, \infty)$ is a strictly increasing function of c on $(0, \infty)$, $\lim_{c\to 0^+} \rho_0(c) = 0$ and $\lim_{c\to\infty} \rho_0(c) = \infty$.*

Remark 2.2. (cf. Figure 1.3 for $f(u) = \exp\left(\frac{au}{a+u}\right)$ with $a = 5$) Theorem 1.2 together with Theorem 2.1(iv) imply that, if f satisfies (H1)–(H5), then there exist two positive numbers $c_2 < c_3$ such that bifurcation curves S and $\widetilde{S}_c$ intersect at the lower (resp., middle and upper) branch of exactly type 1 S-shaped bifurcation curve S when $c \in (0, c_2)$ (resp., $c \in (c_2, c_3)$ and $c \in (c_3, \infty)$).

Theorem 2.3. (cf. Figure 1.3 for $f(u) = \exp\left(\frac{au}{a+u}\right)$ with $a = 5$) *Consider* (1.1) *with* $c > 0$ *and suppose that* f *satisfies* (H1) *and* (H3)–(H5). *Then the following assertions* (i) *and* (ii) *hold:*

(i) *For any two positive numbers* $\widetilde{c}_1 < \widetilde{c}_2$, $\widetilde{S}_{\widetilde{c}_1}$ *lies at the left-hand side of* $\widetilde{S}_{\widetilde{c}_2}$ *on the* $(\lambda, \|u\|_\infty)$*-plane. That is, for any two positive numbers* $\widetilde{c}_1 < \widetilde{c}_2$ *and* $\rho > \rho_0(\widetilde{c}_2)$, *let* $(\lambda_{\widetilde{c}_i}(\rho), \rho) \in \widetilde{S}_{\widetilde{c}_i}$, $i = 1, 2$. *Then* $\lambda_{\widetilde{c}_1}(\rho) < \lambda_{\widetilde{c}_2}(\rho)$.

(ii) *Let* $\lambda_{\min}(c) \equiv \min\{\lambda : (\lambda, \|u_\lambda\|_\infty) \in \widetilde{S}_c\}$. *Then* $\lambda_{\min}(c)$ *is strictly increasing on* $(0, \infty)$, $\lim_{c\to 0^+} \lambda_{\min}(c) = 0$ *and* $\lim_{c\to\infty} \lambda_{\min}(c) = \infty$.

Theorem 2.4. (cf. Figure 1.3 for $f(u) = \exp\left(\frac{au}{a+u}\right)$ with $a = 5$) *Consider* (1.1) *with* $c > 0$ *and suppose that* f *satisfies* (H1)–(H6). *Then there exists a unique positive* c_1 *such that the following assertions* (i)–(ii) *hold:*

(i) *For* $0 < c < c_1$, *the bifurcation curve* $\widetilde{S}_c$ *is S-shaped on the* $(\lambda, \|u\|_\infty)$*-plane. More precisely, there exist three positive* $c_{1,1} \le c_{1,2} \le c_{1,3}$ *on* $(0, c_1)$ *such that the following assertions* (a)–(c) *hold:*

   (a) *(cf. Figure* 1.1(i)*) If* $0 < c < c_{1,1}$, *then the bifurcation curve* $\widetilde{S}_c$ *is* type 1 *S-shaped on the* $(\lambda, \|u\|_\infty)$*-plane. Moreover, there exist three positive* $\lambda_0 < \lambda_* < \lambda^*$ *which are all strictly increasing functions of* c *on* $(0, c_{1,1})$ *such that* (1.1) *has no positive solution for* $0 < \lambda < \lambda_0$, *at least one positive solution for* $\lambda_0 \le \lambda < \lambda_*$ *and* $\lambda > \lambda^*$, *at least two positive solutions for* $\lambda = \lambda_*$ *and* $\lambda = \lambda^*$, *and at least three positive solutions for* $\lambda_* < \lambda < \lambda^*$.

   (b) *(cf. Figure* 1.1(ii)*) If* $c = c_{1,2}$, *then the bifurcation curve* $\widetilde{S}_c$ *is* type 2 *S-shaped on the* $(\lambda, \|u\|_\infty)$*-plane. Moreover, there exist three positive* $\lambda_0 = \lambda_* < \lambda^*$ *such that* (1.1) *has no positive solution for* $0 < \lambda < \lambda_0$, *at least one positive solution for* $\lambda > \lambda^*$, *at least two positive solutions for* $\lambda = \lambda_*$ *and* $\lambda = \lambda^*$, *and at least three positive solutions for* $\lambda_* < \lambda < \lambda^*$.

   (c) *(cf. Figure* 1.1(iii)*) If* $c_{1,3} < c < c_1$, *then the bifurcation curve* $\widetilde{S}_c$ *is* type 3 *S-shaped on the* $(\lambda, \|u\|_\infty)$*-plane. Moreover, there exist three positive* $\lambda_* < \lambda_0 < \lambda^*$ *which are all strictly increasing functions of* c *on* $(c_{1,3}, c_1)$ *such that* (1.1) *has no positive solution for* $0 < \lambda < \lambda_*$, *at least one positive solution for* $\lambda = \lambda_*$ *and* $\lambda > \lambda^*$, *at least two positive solutions for* $\lambda_* < \lambda < \lambda_0$ *and* $\lambda = \lambda^*$, *and at least three positive solutions for* $\lambda_0 \le \lambda < \lambda^*$.

(ii) *(cf. Figure* 1.2*) For* $c \ge c_1$, *the bifurcation curve* $\widetilde{S}_c$ *is* $\subset$*-shaped on the* $(\lambda, \|u\|_\infty)$*-plane. Moreover, there exist two positive* $\lambda_* < \lambda_0$ *such that* (1.1) *has no positive solution for* $0 < \lambda < \lambda_*$, *at least one positive solution for* $\lambda = \lambda_*$ *and* $\lambda > \lambda_0$, *and at least two positive solutions for* $\lambda_* < \lambda \le \lambda_0$.

Theorem 2.5. (cf. Figure 1.3 with $a = 5$) *Consider* (1.1) *for* $f(u) = \exp\left(\frac{au}{a+u}\right)$ *with* $a \geq 4.37$. *Then* f *satisfies* (H1)–(H6) *with* $\gamma = a(a-2)/2 > \tau = a$ *and hence all results in Theorems* 2.1, 2.3 *and* 2.4 *hold.*

3. Lemmas

To prove our main results for problem (1.1), in this paper, we modify the quadrature method (time-map technique) which was used in [3, 4, 7, 11]. First, the time map formula which we apply to study for zero Dirichlet problem (1.14) takes the form as follows:

$$G(\rho) \equiv \sqrt{2} \int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} \quad \text{for } \rho > 0,$$

where $F(s) = \int_0^s f(t)\,dt$ is defined in (1.2). Note that positive solutions u of (1.14) correspond to

$$\|u\|_\infty = \rho \quad \text{and} \quad G(\rho) = \sqrt{\lambda}. \tag{3.1}$$

Thus, studying the exact number of positive solutions of (1.14) is equivalent to studying the shape of the time map $G(\rho)$ on $(0, \infty)$. We compute that

$$G'(\rho) = \frac{\sqrt{2}}{2} \int_0^\rho \frac{\theta(\rho) - \theta(s)}{\rho[F(\rho) - F(s)]^{3/2}}\,ds, \tag{3.2}$$

where $\theta(s) \equiv 2F(s) - sf(s)$. Moreover,

$$G''(\rho) = \frac{\sqrt{2}}{2} \int_0^\rho \frac{\frac{3}{2}[P_1(\rho, s)]^2 - [P_2(\rho, s) + 2]P_1(\rho, s)}{\rho^2[F(\rho) - F(s)]^{1/2}}\,ds, \tag{3.3}$$

where $P_1(\rho, s)$ and $P_2(\rho, s)$ are defined in (1.7); see [1, Eq. (2.7)]. Note that

$$\lim_{\rho \to 0^+} G(\rho) = 0 \quad \text{and} \quad \lim_{\rho \to \infty} G(\rho) = \infty \tag{3.4}$$

if $f \in C^2[0, \infty)$ satisfies (H1) and (H3); see, e.g., [6, Lemma 3.1].

On the other hand, considering Dirichlet-Neumann problem (1.1), we define

$$\begin{aligned} \widetilde{H}_c(\rho, q) &= 2 \int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} - \int_0^q \frac{ds}{\sqrt{F(\rho) - F(s)}} - \frac{c}{\sqrt{F(\rho) - F(q)}} \\ &= \sqrt{2}G(\rho) - \int_0^q \frac{ds}{\sqrt{F(\rho) - F(s)}} - \frac{c}{\sqrt{F(\rho) - F(q)}} \end{aligned} \tag{3.5}$$

for $0 \leq q < \rho < \infty$; see [3, Eq. (3.29)] for $f(\rho) = \exp\left(\frac{a\rho}{a+\rho}\right)$ and $c = 1$. Then to study the number of positive solutions of (1.1), we need to analyze function $\widetilde{H}_c(\rho, q)$ in the first step. Beforehand, under (H1) and (H3)–(H5), we derive some basic properties related to functions $f(\rho)$, $f'(\rho)$ and $F(\rho)$ in Lemma 3.1 to ease the proofs of the other lemmas in this section.

Lemma 3.1. *Suppose that* $f \in C^2[0,\infty)$ *satisfies* (H1) *and* (H3)–(H5), *and let* $M_1(\rho)$, $M_2(\rho)$, $P_1(\rho,s)$ *and* $P_2(\rho,s)$ *be defined in* (1.3), (1.4) *and* (1.7). *Then the following assertions* (i)–(x) *hold:*

(i) $M_2(\rho) < M_1(\rho)$ *for* $\rho > 0$.

(ii) *Let* $\tau > 0$ *be defined in* (H4). *Then*

$$M_2'(\rho) \begin{cases} > 0 & \text{when } \rho \in [0,\tau), \\ = 0 & \text{when } \rho = \tau, \\ < 0 & \text{when } \rho \in (\tau,\infty). \end{cases}$$

(iii) $P_1(\rho,0) = M_1(\rho)$, $\lim_{s\to\rho^-} P_1(\rho,s) = M_2(\rho)+1$ *for* $\rho > 0$, *and* $\lim_{\rho\to s^+} P_1(\rho,s) = M_2(s)+1$ *for* $s \geq 0$.

(iv) $P_1(\rho,s) > 1$ *for* $0 \leq s < \rho$.

(v) *For* $0 < \rho \leq \tau$, $P_1(\rho,s)$ *is a strictly increasing function of* s *on* $[0,\rho)$. *Moreover,* $M_1(\rho) \leq P_1(\rho,s) < M_2(\rho)+1$ *for* $0 \leq s < \rho \leq \tau$, *where the equality holds if and only if* $s = 0$.

(vi) $M_2(\rho) < P_2(\rho,s)$ *for* $0 < s < \rho \leq \tau$.

(vii) *The function* $Q(\rho,s) \equiv \theta(\rho) - \theta(s) + M_1(\rho)[F(\rho)-F(s)] > 0$ *for* $0 \leq s < \rho$.

(viii) $\left[\frac{f(\rho)}{\sqrt{F(\rho)}}\right]' + \frac{M_2(\rho)+1}{2\rho}\frac{f(\rho)}{\sqrt{F(\rho)}} > 0$ *for* $0 < \rho \leq \tau$.

(ix) *If* f *satisfies* (1.10), *then there exists a unique* $\overline{\rho}$ *on* $(0,\tau)$ *such that*

$$2f(0)[2-M_1(\rho)] - f(\rho) \begin{cases} > 0 & \text{when } \rho \in (0,\overline{\rho}), \\ = 0 & \text{when } \rho = \overline{\rho}, \\ < 0 & \text{when } \rho \in (\overline{\rho},\tau]. \end{cases}$$

(x) *If* f *satisfies* (1.10), *then the function*

$$R(\rho,s) \equiv 2f(0)[\theta(\rho)-\theta(s)] - \frac{f(\rho)}{\sqrt{F(\rho)}}\sqrt{\rho}\frac{[F(\rho)-F(s)]^{3/2}}{\sqrt{\rho-s}} \quad \text{for } 0 \leq s < \rho \tag{3.6}$$

satisfies $R(\rho,s) < 0$ *for* $0 \leq s < \rho$ *and* $\rho > \overline{\rho}$.

The proofs of Lemma 3.1(iii)–(viii) are very similar to those of [11, Lemma 3.1], and hence we omit them here. While the proofs of the remaining parts (i)–(ii) and (ix)–(x) are easy but tedious, and hence we put them in [9].

Lemma 3.2. *Suppose that $f \in C^2[0,\infty)$ satisfies (H1) and (H3)–(H5). Then, for $\widetilde{H}_c(\rho,q)$ with $0 \le q < \rho < \infty$ and $c > 0$, the following assertions (i)–(viii) hold:*

(i) *For $c > 0$, $\lim_{\rho\to 0^+} \widetilde{H}_c(\rho,0) = -\infty$ and $\lim_{\rho\to\infty} \widetilde{H}_c(\rho,0) = \infty$.*

(ii) *For $c > 0$, there exists a unique positive ρ_0 $(= \rho_0(c))$ such that*

$$\widetilde{H}_c(\rho,0) \begin{cases} < 0 & \text{when } \rho \in (0,\rho_0), \\ = 0 & \text{when } \rho = \rho_0, \\ > 0 & \text{when } \rho \in (\rho_0,\infty). \end{cases}$$

(iii) *For fixed $c, \rho > 0$, $\widetilde{H}_c(\rho,q)$ is a strictly decreasing function of q on $[0,\rho)$ and $\lim_{q\to\rho^-} \widetilde{H}_c(\rho,q) = -\infty$.*

(iv) *For $c > 0$, if $0 < \rho < \rho_0(c)$, then $\widetilde{H}_c(\rho,q)$ has no zero q $(= q(\rho,c))$ on $[0,\rho)$, while if $\rho \ge \rho_0(c)$, then $\widetilde{H}_c(\rho,q)$ has a unique zero $q(\rho,c)$ on $[0,\rho)$, i.e.,*

$$\widetilde{H}_c(\rho, q(\rho,c)) = 0. \tag{3.7}$$

Moreover, $q(\rho,c) = 0$ if and only if $\rho = \rho_0(c)$.

(v) *For $c > 0$ and $\rho > \rho_0$, $q(\rho,c) \in C[\rho_0,\infty) \cap C^1(\rho_0,\infty)$ satisfies*

$$\frac{\partial}{\partial\rho} q(\rho,c) = \frac{[F(\rho) - F(q(\rho,c))]^{3/2}\left[2\sqrt{2}G'(\rho) + \int_0^{q(\rho,c)} \frac{f(\rho)}{[F(\rho)-F(s)]^{3/2}}\,ds\right] + cf(\rho)}{2[F(\rho) - F(q(\rho,c))] + cf(q(\rho,c))}. \tag{3.8}$$

Moreover,

$$\lim_{\rho\to\rho_0^+} \frac{\partial}{\partial\rho} q(\rho,c) = \frac{2\sqrt{2}[F(\rho_0)]^{3/2}G'(\rho_0) + cf(\rho_0)}{2F(\rho_0) + cf(0)}. \tag{3.9}$$

(vi) *For $c > 0$ and $\rho \ge \rho_0$,*

$$0 < \rho - q(\rho,c) \le \frac{c^2 f(\rho)}{4f(0)\rho}.$$

Moreover, $\lim_{\rho\to\infty}[\rho - q(\rho,c)] = 0$.

(vii) *$\rho_0(c)$ is a continuous, strictly increasing function of c on $(0,\infty)$, $\lim_{c\to 0^+} \rho_0(c) = 0$ and $\lim_{c\to\infty} \rho_0(c) = \infty$. Moreover, for $c > 0$, $\rho_0(c)$ is the unique positive root of*

$$c = \sqrt{2F(\rho)}G(\rho).$$

(viii) *For $\rho > 0$, $q(\rho,c) \in C(0,\widehat{c}] \cap C^1(0,\widehat{c})$ is a strictly decreasing function of c on $(0,\widehat{c}]$, $\lim_{c\to 0^+} q(\rho,c) = \rho$ and $q(\rho,\widehat{c}) = 0$. Here $\widehat{c}$ $(= \widehat{c}(\rho)) = \sqrt{2F(\rho)}G(\rho)$.*

Proof. Lemma 3.2(iii) and (iv) are slight generalizations of [3, Lemma B and Theorem 3.4(a)], and Lemma 3.2(ii), (vii) and (viii) are from [11, Lemma 3.2(ii), (vii) and (viii)], respectively. Hence these proofs are omitted. We then prove Lemma 3.2(i), (v) and (vi) as follows.

(i) Since $f \in C^2[0,\infty)$ satisfies (H1) and (H3), the two limits in (3.4) hold by [6, Lemma 3.1]. Then, by (3.5), we have that

$$\lim_{\rho\to 0^+} \widetilde{H}_c(\rho,0) = \lim_{\rho\to 0^+}\left[\sqrt{2}G(\rho) - \frac{c}{\sqrt{F(\rho)}}\right] = -\infty,$$
$$\lim_{\rho\to \infty} \widetilde{H}_c(\rho,0) = \lim_{\rho\to \infty}\left[\sqrt{2}G(\rho) - \frac{c}{\sqrt{F(\rho)}}\right] = \infty.$$

So Lemma 3.2(i) holds.

(v) Since $\widetilde{H}_c(\rho, q(\rho,c)) = 0$ by (3.7) and applying the Implicit Function Theorem to (3.5), we have that

$$\begin{aligned}\frac{\partial}{\partial\rho}q(\rho,c) &= -\frac{\frac{\partial}{\partial\rho}\widetilde{H}_c(\rho,q(\rho,c))}{\frac{\partial}{\partial q}\widetilde{H}_c(\rho,q(\rho,c))}\\
&= \frac{\sqrt{2}G'(\rho) + \int_0^{q(\rho,c)}\frac{f(\rho)}{2[F(\rho)-F(s)]^{3/2}}\,ds + \frac{cf(\rho)}{2[F(\rho)-F(s)]^{3/2}}}{\frac{1}{\sqrt{F(\rho)-F(q(\rho,c))}} + \frac{cf(q(\rho,c))}{2[F(\rho)-F(q(\rho,c))]^{3/2}}}\\
&= \frac{[F(\rho)-F(q(\rho,c))]^{3/2}\left[2\sqrt{2}G'(\rho) + \int_0^{q(\rho,c)}\frac{f(\rho)}{[F(\rho)-F(s)]^{3/2}}\,ds\right] + cf(\rho)}{2[F(\rho)-F(q(\rho,c))] + cf(q(\rho,c))}.\end{aligned}$$

Moreover,

$$\lim_{\rho\to\rho_0^+}\frac{\partial}{\partial\rho}q(\rho,c) = \frac{[F(\rho_0)]^{3/2}[2\sqrt{2}G'(\rho_0)] + cf(\rho_0)}{2F(\rho_0) + cf(0)}$$

since $q(\rho_0(c),c) = 0$ by Lemma 3.2(iv). The proof of the assertion $q(\rho,c) \in C[\rho_0,\infty) \cap C^1(\rho_0,\infty)$ is very similar to that of [7, Lemma 3.1(iv)], and hence we omit it here. So Lemma 3.2(v) holds.

(vi) By the definition of $\widetilde{H}_c$ in (3.5), we compute that

$$\begin{aligned}\widetilde{H}_c(\rho,q(\rho,c)) &= 2\int_0^{\rho}\frac{ds}{\sqrt{F(\rho)-F(s)}} - \int_0^{q(\rho,c)}\frac{ds}{\sqrt{F(\rho)-F(s)}} - \frac{c}{\sqrt{F(\rho)-F(q(\rho,c))}}\\
&\geq \int_0^{\rho}\frac{ds}{\sqrt{F(\rho)-F(s)}} - \frac{c}{\sqrt{F(\rho)-F(q(\rho,c))}}\\
&\geq \frac{1}{\sqrt{f(\rho)}}\int_0^{\rho}\frac{ds}{\sqrt{\rho-s}} - \frac{1}{\sqrt{f(q(\rho,c))}}\frac{c}{\sqrt{\rho-q(\rho,c)}}\\
&\qquad\text{(by the Mean Value Theorem and (H1))}\end{aligned}$$

$$= \frac{1}{\sqrt{f(q(\rho,c))}}\left[2\sqrt{\frac{\rho}{f(\rho)}f(q(\rho,c))} - \frac{c}{\sqrt{\rho - q(\rho,c)}}\right]$$
$$\geq \frac{1}{\sqrt{f(q(\rho,c))}}\left[2\sqrt{\frac{\rho}{f(\rho)}f(0)} - \frac{c}{\sqrt{\rho - q(\rho,c)}}\right]$$

by (H1). Then, since $\widetilde{H}_c(\rho, q(\rho,c)) = 0$ by (3.7), we conclude that $0 < \rho - q(\rho,c) \leq \frac{c^2}{4f(0)}\frac{f(\rho)}{\rho}$, and hence $\lim_{\rho\to\infty}[\rho - q(\rho,c)] = 0$ by (H3). So Lemma 3.2(vi) holds.

The proof of Lemma 3.2 is complete. □

The time map formula which we apply to study Dirichlet-Neumann problem (1.1) takes the form as follows:

$$H_c(\rho, q(\rho,c)) \equiv \frac{c^2}{2[F(\rho) - F(q(\rho,c))]} \quad \text{for } \rho \geq \rho_0(c), \tag{3.10}$$

where $q(\rho,c)$ is defined in (3.7). Note that positive solutions u of (1.1) correspond to

$$\|u\|_\infty = \rho \quad \text{and} \quad H_c(\rho, q(\rho,c)) = \lambda, \tag{3.11}$$

see, e.g., [3, Theorem 3.3] and [11, Eq. (3.27)] for $f(\rho) = \exp\left(\frac{a\rho}{a+\rho}\right)$. Thus, studying the number of positive solutions of (1.1) is equivalent to studying the shape of the time map $H_c(\rho, q(\rho,c))$ for $\rho \geq \rho_0(c)$.

The next lemma is from [11, Lemma 3.3].

Lemma 3.3. *Suppose that $f \in C^2[0,\infty)$ satisfies* (H1) *and* (H3)–(H5)*. Then, for $G(\rho)$ with $\rho > 0$ and for $H_c(\rho, q(\rho,c))$ with $\rho \geq \rho_0$ and $c > 0$, the following assertions* (i)–(iii) *hold:*

(i) *$H_c(\rho, q(\rho,c)) \leq [G(\rho)]^2$ for $c > 0$ and $\rho \geq \rho_0$. The equality holds if and only if $\rho = \rho_0$.*

(ii) *$\lim_{\rho\to\infty} H_c(\rho, q(\rho,c)) = \infty$ for $c > 0$.*

(iii) *$H_c(\rho, q(\rho,c)) > \frac{1}{4}[G(\rho)]^2$ for $c > 0$ and $\rho \geq \rho_0$. In addition, $\lim_{c\to 0^+} H_c(\rho, q(\rho,c)) = \frac{1}{4}[G(\rho)]^2$ for $\rho > 0$.*

Lemma 3.4. *Suppose that $f \in C^2[0,\infty)$ satisfies* (H1) *and* (H3)–(H5)*. Then, for $H_c(\rho, q(\rho,c))$ with $\rho \geq \rho_0$ and $c > 0$, the following assertions* (i)–(iii) *hold:*

(i) *For any two positive numbers $\widetilde{c}_1 < \widetilde{c}_2$, $H_{\widetilde{c}_1}(\rho, q(\rho,\widetilde{c}_1)) < H_{\widetilde{c}_2}(\rho, q(\rho,\widetilde{c}_2))$ for $\rho \geq \rho_0(\widetilde{c}_2)$.*

(ii) *Let $\lambda_{\min}(c) \equiv \min_{\rho\geq\rho_0(c)} H_c(\rho, q(\rho,c))$ for $c > 0$. Then $\lambda_{\min}(c)$ is a strictly increasing function of c on $(0,\infty)$, $\lim_{c\to 0^+}\lambda_{\min}(c) = 0$ and $\lim_{c\to\infty}\lambda_{\min}(c) = \infty$.*

(iii) *If* $\lim_{\rho\to\rho_0(\widetilde{c})^+}\frac{d}{d\rho}H_{\widetilde{c}}(\rho,q(\rho,\widetilde{c}))\geq 0$, *then* $\frac{d}{d\rho}H_c(\rho_0(\widetilde{c}),q(\rho_0(\widetilde{c}),c))>0$ *for* $0<c<\widetilde{c}$.

Proof. Lemma 3.4(i)–(ii) are from [11, Lemma 3.4(i)–(ii)]. Hence these proofs are omitted. We next prove Lemma 3.4(iii).

We compute that, by (3.8) and (3.10),

$$\frac{d}{d\rho}H_c(\rho,q(\rho,c))$$

$$= -\frac{c^2\left[f(\rho)-f(q(\rho,c))\frac{\partial}{\partial\rho}q(\rho,c)\right]}{2[F(\rho)-F(q(\rho,c))]^2} \tag{3.12}$$

$$= \frac{c^2 f(q(\rho,c))}{2[F(\rho)-F(q(\rho,c))]^{1/2}\left\{2[F(\rho)-F(q(\rho,c))]+cf(q(\rho,c))\right\}}\Psi(\rho,q(\rho,c)), \tag{3.13}$$

where

$$\Psi(\rho,q)\equiv 2\sqrt{2}G'(\rho)-\left[2\frac{f(\rho)}{f(q)\sqrt{F(\rho)-F(q)}}-\int_0^q\frac{f(\rho)}{[F(\rho)-F(s)]^{3/2}}\,ds\right]. \tag{3.14}$$

Moreover, by (3.9) and the fact that $q(\rho_0(c),c)=0$ by Lemma 3.2(iv), we have that

$$\lim_{\rho\to\rho_0(c)^+}\frac{d}{d\rho}H_c(\rho,q(\rho,c))=\frac{c^2}{\sqrt{F(\rho_0)}[2F(\rho_0)+cf(0)]}\Phi(\rho_0), \tag{3.15}$$

where

$$\Phi(\rho)\equiv\sqrt{2}f(0)G'(\rho)-\frac{f(\rho)}{\sqrt{F(\rho)}}. \tag{3.16}$$

Hence, if $\lim_{\rho\to\rho_0(\widetilde{c})^+}\frac{d}{d\rho}H_{\widetilde{c}}(\rho,q(\rho,\widetilde{c}))\geq 0$ for some $\widetilde{c}>0$, then $\Phi(\rho_0(\widetilde{c}))\geq 0$. It follows that

$$\begin{aligned}\Psi(\rho_0(\widetilde{c}),q(\rho_0(\widetilde{c}),\widetilde{c}))=\Psi(\rho_0(\widetilde{c}),0)&=2\left[\sqrt{2}G'(\rho_0(\widetilde{c}))-\frac{f(\rho_0(\widetilde{c}))}{f(0)\sqrt{F(\rho_0(\widetilde{c}))}}\right]\\&=\frac{2}{f(0)}\Phi(\rho_0(\widetilde{c}))\geq 0\end{aligned}$$

by (3.14) and (3.16). Moreover, we compute that

$$\frac{\partial}{\partial q}\Psi(\rho,q)=2\frac{f'(q)f(\rho)}{[f(q)]^2\sqrt{F(\rho)-F(q)}}>0$$

by (H1), which implies that $\Psi(\rho_0(\widetilde{c}),q(\rho_0(\widetilde{c}),c))>\Psi(\rho_0(\widetilde{c}),q(\rho_0(\widetilde{c}),\widetilde{c}))\geq 0$ for $0<c<\widetilde{c}$ since $q(\rho_0(\widetilde{c}),c)>q(\rho_0(\widetilde{c}),\widetilde{c})$ for $0<c<\widetilde{c}$ by Lemma 3.2(viii). Therefore $\frac{d}{d\rho}H_c(\rho_0(\widetilde{c}),q(\rho_0(\widetilde{c}),c))>0$ by (3.13). So Lemma 3.4(iii) holds.

The proof of Lemma 3.4 is now complete. □

Lemma 3.5. *Suppose that* $f \in C^2[0,\infty)$ *satisfies* (H1) *and* (H3)–(H5). *Then, for* $H_c(\rho, q(\rho, c))$ *with* $\rho \geq \rho_0$ *and* $c > 0$, *the following assertions* (i)–(iii) *hold:*

(i) *There exists a unique positive* c_1 *such that*

$$\lim_{\rho\to\rho_0(c)^+} \frac{d}{d\rho} H_c(\rho, q(\rho,c)) \begin{cases} > 0 & \text{when } c \in (0, c_1), \\ = 0 & \text{when } c = c_1, \\ < 0 & \text{when } c \in (c_1, \infty). \end{cases} \tag{3.17}$$

(ii) *If* f *satisfies* (H6), *then for* $c = c_1$, *there exists* $\widetilde{\rho} > \rho_0(c_1)$ *such that* $\frac{d}{d\rho} H_{c_1}(\rho, q(\rho, c_1)) < 0$ *for* $\rho_0(c_1) < \rho < \widetilde{\rho}$.

(iii) $\frac{d}{d\rho} H_c(\rho, q(\rho, c)) > 0$ *for* $0 < c < c_1$ *and* $\rho_0(c) < \rho < \rho_0(c_1)$.

Proof. Lemma 3.5(iii) is from [11, Lemma 3.5(iii)]. We then prove Lemma 3.5(i) and (ii) as follows.

(i) Notice that, by (3.15), studying the sign of $\lim_{\rho\to\rho_0(c)^+} \frac{d}{d\rho} H_c(\rho, q(\rho,c))$ is equivalent to studying that of $\Phi(\rho)$. Then we have that, for $\rho > 0$,

$$\begin{aligned} \Phi(\rho) &= \sqrt{2} G'(\rho) f(0) - \frac{f(\rho)}{\sqrt{F(\rho)}} \quad \text{(by (3.16))} \\ &= \int_0^\rho \frac{f(0)[\theta(\rho) - \theta(s)]}{\rho[F(\rho) - F(s)]^{3/2}} \, ds - \frac{f(\rho)}{\sqrt{F(\rho)}} \int_0^\rho \frac{ds}{2\sqrt{\rho(\rho - s)}} \\ &\qquad \text{(by (3.2) and since } \int_0^\rho \frac{ds}{2\sqrt{\rho(\rho-s)}} = 1) \\ &= \int_0^\rho \frac{R(\rho, s)}{2\rho[F(\rho) - F(s)]^{3/2}} \, ds, \end{aligned} \tag{3.18}$$

where the function $R(\rho, s)$ is defined in (3.6). Then, by Lemma 3.1(ix)–(x), we have that

$$\Phi(\rho) < 0 \quad \text{for } \rho > \overline{\rho}. \tag{3.19}$$

Next, we show that $\lim_{\rho\to 0^+} \Phi(\rho) = \infty$. Note that, by integration by parts, we have that

$$\sqrt{2} G'(\rho) = 2 \frac{f(\rho)}{f(0)\sqrt{F(\rho)}} - 2 \int_0^\rho \frac{f'(s) f(\rho)}{[f(s)]^2 \sqrt{F(\rho) - F(s)}} \, ds,$$

see [11, Proof of Lemma 3.6 on p. 8375]. So we can represent, by (3.16), $\Phi(\rho)$ as follows:

$$\Phi(\rho) = \frac{f(\rho)}{\sqrt{F(\rho)}} - 2f(0) \int_0^\rho \frac{f'(s) f(\rho)}{[f(s)]^2 \sqrt{F(\rho) - F(s)}} \, ds.$$

Consequently, let $\breve{\rho} > 0$ be an arbitrary number such that, for $0 < \rho \leq \breve{\rho}$, $0 < f(0) < f(\rho) \leq 2f(0)$ and $f'(\rho) \leq 2f'(0)$, where the existence of $\breve{\rho} > 0$ follows directly by (H1).

Then, for $0 < \rho \le \breve{\rho}$,

$$\begin{aligned}\Phi(\rho) &\ge \frac{f(\rho)}{\sqrt{F(\rho)}} - 2f(0)\int_0^\rho \frac{4f'(0)f(0)}{[f(0)]^2\sqrt{F(\rho)-F(s)}}\,ds \\ &\ge \frac{f(\rho)}{\sqrt{F(\rho)}} - 8f'(0)\int_0^\rho \frac{1}{\sqrt{F(\rho)-F(s)}}\,ds \\ &= \frac{f(\rho)}{\sqrt{F(\rho)}} - 4\sqrt{2}f'(0)G(\rho).\end{aligned}$$

Hence $\lim_{\rho\to 0^+}\Phi(\rho) = \infty$ since $\lim_{\rho\to 0^+} G(\rho) = 0$ by (3.4), $F(0) = 0$ and $f(0) > 0$.

Finally, for $0 < \rho < \tau$, we compute that

$$\begin{aligned}&\Phi'(\rho) + \frac{M_2(\rho)+1}{2\rho}\Phi(\rho) \\ &= \sqrt{2}f(0)\left[G''(\rho) + \frac{M_2(\rho)+1}{2\rho}G'(\rho)\right] - \left\{\left[\frac{f(\rho)}{\sqrt{F(\rho)}}\right]' + \frac{M_2(\rho)+1}{2\rho}\frac{f(\rho)}{\sqrt{F(\rho)}}\right\} \\ &< \sqrt{2}f(0)\left[G''(\rho) + \frac{M_2(\rho)+1}{2\rho}G'(\rho)\right] \quad \text{(by Lemma 3.1(viii))} \\ &= f(0)\int_0^\rho \frac{\frac{3}{2}[P_1(\rho,s)]^2 - \left[P_2(\rho,s) + 2 + \frac{M_2(\rho)+1}{2}\right]P_1(\rho,s) + [M_2(\rho)+1]}{\rho^2\sqrt{F(\rho)-F(s)}}\,ds \\ &\qquad \text{(by (3.2) and (3.3))} \\ &< f(0)\int_0^\rho \frac{\frac{3}{2}[P_1(\rho,s)]^2 - \frac{3}{2}\left[M_2(\rho)+\frac{5}{3}\right]P_1(\rho,s) + [M_2(\rho)+1]}{\rho^2\sqrt{F(\rho)-F(s)}}\,ds \quad \text{(by Lemma 3.1(vi))} \\ &= f(0)\int_0^\rho \frac{\frac{3}{2}\left[P_1(\rho,s)-\frac{2}{3}\right]\{P_1(\rho,s)-[M_2(\rho)+1]\}}{\rho^2\sqrt{F(\rho)-F(s)}}\,ds < 0\end{aligned}$$

by Lemma 3.1(iv)–(v), which implies that $\Phi'(\rho) < 0$ whenever $\Phi(\rho) = 0$ on $(0,\tau)$. Combining the facts that $\lim_{\rho\to 0^+}\Phi(\rho) = \infty$ and $\Phi(\overline{\rho}) \le 0$ by (3.19), we conclude that there exists a unique ρ_0^* on $(0,\overline{\rho}]$ such that

$$\Phi(\rho)\begin{cases} > 0 & \text{when } \rho \in (0,\rho_0^*), \\ = 0 & \text{when } \rho = \rho_0^*, \\ < 0 & \text{when } \rho \in (\rho_0^*,\infty). \end{cases} \tag{3.20}$$

By Lemma 3.2(vii), there exists $c_1 > 0$ such that $\rho_0(c_1) = \rho_0^*$. Moreover, $\rho_0(c) \in (0,\rho_0^*)$ (resp., (ρ_0^*,∞)) if and only if $c \in (0,c_1)$ (resp., $c \in (c_1,\infty)$). Hence we have that

$$\Phi(\rho_0(c))\begin{cases} > 0 & \text{when } c \in (0,c_1), \\ = 0 & \text{when } c = c_1, \\ < 0 & \text{when } c \in (c_1,\infty). \end{cases}$$

Then, by (3.15), Lemma 3.5(i) holds.

(ii) Let ρ_0^* $(= \rho_0(c_1))$ be defined in (3.20). Then $\Psi(\rho_0^*, q(\rho_0^*, c_1)) = \Psi(\rho_0^*, 0) = \frac{2}{f(0)}\Phi(\rho_0^*) = 0$ by Lemma 3.2(iv), (3.14), (3.16) and (3.20). So if we can show that $\lim_{\rho\to(\rho_0^*)^+} \frac{d}{d\rho}\Psi(\rho, q(\rho, c_1)) < 0$, then there exists $\widetilde{\rho} > \rho_0^*$ such that $\Psi(\rho, q(\rho, c_1)) < 0$ for $\rho_0^* < \rho < \widetilde{\rho}$, and hence $\frac{d}{d\rho}H_{c_1}(\rho, q(\rho, c_1)) < 0$ for $\rho_0^* < \rho < \widetilde{\rho}$ by (3.13), which completes the proof. Indeed, we compute that, by (3.8) and (3.14),

$$\begin{aligned}\frac{d}{d\rho}\Psi(\rho, q(\rho, c)) = {} & 2\sqrt{2}G''(\rho) - 2\frac{f'(\rho)f(q(\rho,c)) - f'(q(\rho,c))\frac{\partial}{\partial\rho}q(\rho,c)f(\rho)}{[f(q(\rho,c))]^2\sqrt{F(\rho) - F(q(\rho,c))}} \\ & + \frac{[f(\rho)]^2}{f(q(\rho,c))[F(\rho) - F(q(\rho,c))]^{3/2}} \\ & + \int_0^{q(\rho,c)} \frac{f'(\rho)[F(\rho) - F(s)] - \frac{3}{2}[f(\rho)]^2}{[F(\rho) - F(s)]^{5/2}}\, ds.\end{aligned}$$

It follows that

$$\begin{aligned}
& \lim_{\rho\to(\rho_0^*)^+} \frac{d}{d\rho}\Psi(\rho, q(\rho, c_1)) \\
& = 2\sqrt{2}G''(\rho_0^*) + \frac{-2\left\{f(0)f'(\rho_0^*) - \frac{f'(0)}{f(0)}[f(\rho_0^*)]^2\right\}F(\rho_0^*) + f(0)[f(\rho_0^*)]^2}{[f(0)]^2[F(\rho_0^*)]^{3/2}} \\
& \qquad \left(\text{since } q(\rho_0^*, c_1) = 0 \text{ and } \lim_{\rho\to(\rho_0^*)^+}\frac{\partial}{\partial\rho}q(\rho, c_1) = \frac{f(\rho_0^*)}{f(0)} \text{ by (3.12) and (3.17)}\right) \\
& = 2\sqrt{2}G''(\rho_0^*) + \frac{N_1(\rho_0^*)}{\rho_0^*}\frac{f(\rho_0^*)}{f(0)\sqrt{F(\rho_0^*)}} \quad (N_1(\rho) \text{ is defined in (1.5)}) \\
& = 2\sqrt{2}G''(\rho_0^*) + \frac{N_1(\rho_0^*)}{\rho_0^*}[\sqrt{2}G'(\rho_0^*)] \quad (\text{by (3.18) and (3.20)}) \\
& = \int_0^{\rho_0^*} \frac{3[P_1(\rho_0^*, s)]^2 - 2[P_2(\rho_0^*, s) + 2]P_1(\rho_0^*, s) + N_1(\rho_0^*)[2 - P_1(\rho_0^*, s)]}{\rho_0^{*2}\sqrt{F(\rho_0^*) - F(s)}}\, ds \\
& = \int_0^{\rho_0^*} \frac{3[P_1(\rho_0^*, s) - 1]\left\{P_1(\rho_0^*, s) - [M_2(\rho_0^*) + 1]\right\} - P_1(\rho_0^*, s)[N_2(\rho_0^*) + 2P_2(\rho_0^*, s)] - N_3(\rho_0^*)}{\rho_0^{*2}\sqrt{F(\rho_0^*) - F(s)}}\, ds \\
& \qquad (N_2(\rho) \text{ and } N_3(\rho) \text{ are defined in (1.6)}) \\
& \le \int_0^{\rho_0^*} \frac{-P_1(\rho_0^*, s)[N_2(\rho_0^*) + 2P_2(\rho_0^*, s)] - N_3(\rho_0^*)}{\rho_0^{*2}\sqrt{F(\rho_0^*) - F(s)}}\, ds \quad (\text{by Lemma 3.1(iv) and (v)}) \\
& = -\int_0^{\rho_0^*} \frac{W(\rho_0^*, s) + N_3(\rho_0^*)}{\rho_0^{*2}\sqrt{F(\rho_0^*) - F(s)}}\, ds,
\end{aligned}$$

where $W(\rho, s)$ is defined in (1.8). Then to prove $\lim_{\rho\to(\rho_0^*)^+} \frac{d}{d\rho}\Psi(\rho, q(\rho, c_1)) < 0$, it suffices to prove that

$$\int_0^{\rho} \frac{W(\rho, s) + N_3(\rho)}{\sqrt{F(\rho) - F(s)}}\, ds > 0 \quad \text{for } 0 < \rho \le \overline{\rho} \tag{3.21}$$

since $\rho_0^* \le \overline{\rho}$. We shall prove (3.21) for each $\rho \in (0, \overline{\rho}]$ in the following two cases.

Case 1*:* $W(\rho,0)+N_3(\rho)\geq 0$ for $\rho\in(0,\overline{\rho}]$. We have that $W(\rho,s)+N_3(\rho)\geq W(\rho,0)+s\widetilde{W}_0(\rho)+N_3(\rho)>0$ for $s\in(0,\rho]$ by (1.11) and (1.12) in (H6). Hence (3.21) holds.

Case 2*:* $W(\rho,0)+N_3(\rho)<0$ for $\rho\in(0,\overline{\rho}]$. We have that

$$\begin{aligned}
&\int_0^\rho \frac{W(\rho,s)+N_3(\rho)}{\sqrt{F(\rho)-F(s)}}\,ds\\
&\geq \int_0^\rho \frac{W(\rho,0)+s\widetilde{W}_0(\rho)+N_3(\rho)}{\sqrt{F(\rho)-F(s)}}\,ds \quad \text{(by (1.12) in (H6))}\\
&= \int_0^\rho \frac{W(\rho,0)+N_3(\rho)}{\sqrt{F(\rho)-F(s)}}\,ds+\int_0^\rho \frac{s\widetilde{W}_0(\rho)}{\sqrt{F(\rho)-F(s)}}\,ds\\
&\geq \int_0^\rho \frac{W(\rho,0)+N_3(\rho)}{\sqrt{f(0)}\sqrt{\rho-s}}\,ds+\int_0^\rho \frac{s\widetilde{W}_0(\rho)}{\sqrt{f(\rho)}\sqrt{\rho-s}}\,ds\\
&\qquad \text{(by (1.11), (H1) and the Mean Value Theorem)}\\
&= \frac{2\sqrt{\rho}}{3\sqrt{f(0)}\sqrt{f(\rho)}}\left\{3\sqrt{f(\rho)}[W(\rho,0)+N_3(\rho)]+2\rho\sqrt{f(0)}\widetilde{W}_0(\rho)\right\}>0
\end{aligned}$$

by (1.13) in (H6).

By Cases 1 and 2, (3.21) holds. So Lemma 3.5(ii) holds.

The proof of Lemma 3.5 is now complete. □

The next lemma is from [11, Lemma 3.6].

Lemma 3.6. *Suppose that* $f\in C^2[0,\infty)$ *satisfies* (H1) *and* (H3)–(H5)*. Then, for* $G(\rho)$ *with* $\rho>0$ *and for* $H_c(\rho,q(\rho,c))$ *with* $\rho\geq\rho_0$ *and* $c>0$*, if* $G'(\rho)\leq 0$ *for some* $\rho>0$*, then* $\frac{d}{d\rho}H_c(\rho,q(\rho,c))<0$ *for* $0<c\leq\widehat{c}$*. Here* $\widehat{c}=\sqrt{2F(\rho)}G(\rho)$ *is defined in Lemma* 3.2(viii)*.*

In the next Lemma 3.7, we show some properties of the nonlinearity $f(\rho)$ $(=f(\rho,a))=\exp\left(\frac{a\rho}{a+\rho}\right)$ with $a\geq 4$. When there is no confusion arising, throughout the paper, we would omit the parameter a in representing functions resulting from $f(\rho)$ $(=f(\rho,a))=\exp\left(\frac{a\rho}{a+\rho}\right)$ for the reason of clarity such as the functions defined in (1.2)–(1.9).

Lemma 3.7. *Consider* $f(\rho)=\exp\left(\frac{a\rho}{a+\rho}\right)$ *with* $a\geq 4.37$*, and let* $N_2(\rho)$ $(=N_2(\rho,a))$ *be defined in* (1.6)*. Then the following assertions* (i)–(iii) *hold:*

(i) *There exists a unique positive zero* $\overline{\rho}$ *of* $2f(0)[2-M_1(\rho)]-f(\rho)$ *and* $\overline{\rho}<1/2$*.*

(ii) *For* $0<\rho\leq\overline{\rho}$*,* $N_2(\rho,a)$ *is a strictly decreasing function of* a *on* $[4.37,\infty)$ *and* $-2<N_2(\rho)<-1$*.*

(iii) *For* $0<\rho\leq\overline{\rho}$*,* $N_4(\rho)$ $(=N_4(\rho,a))\equiv F(\rho)\widetilde{W}_0(\rho)$ *is a positive, strictly increasing function of* a *on* $[4.37,\infty)$*.*

The proof of Lemma 3.7 is lengthy, and hence it is given in [9].

4. Proofs of the main results

In this section, we prove our main results (Theorems 2.1, 2.3–2.5). We note that, from the relationship between bifurcation curve S and time map G as in (3.1), and that between bifurcation curves $\widetilde{S}_c$ and the time map H_c as in (3.11), the assertions in Theorems 2.1, 2.3 and 2.4 can be concluded directly from Lemmas 3.1–3.6. Hence we shall just give the main framework of these proofs; cf. proofs of [11, Theorems 2.1–2.3] in [11, Section 4].

Proof of Theorem 2.1. First, by (3.10), (3.11) and Lemma 3.2(iv), for any $c > 0$, the bifurcation curve $\widetilde{S}_c$ is a continuous curve which starts at some point $(\lambda_0, \|u_{\lambda_0}\|_\infty)$ on the $(\lambda, \|u_\lambda\|_\infty)$-plane with λ_0 $(= H(\rho_0(c), q(\rho_0(c), c))) = H(\rho_0(c), 0) > 0$ and $\|u_\lambda\|_\infty = \rho_0 > 0$. In addition, Theorem 2.1(i) and (iii) follow from Lemma 3.3(i), while Theorem 2.1(ii) and (iv) follow from Lemma 3.3(ii) and Lemma 3.2(vii), respectively. Hence the proof of Theorem 2.1 is complete. □

Proof of Theorem 2.3. Theorem 2.3(i) and (ii) follow from Lemma 3.4(i) and (ii), respectively. □

Proof of Theorem 2.4. We first note that, from the relationship between the bifurcation curve S and time map G as in (3.1) and since the bifurcation curve S of (1.14) is exactly type 1 S-shaped on the $(\lambda, \|u_\lambda\|_\infty)$-plane by Theorem 1.2, there exist two positive $\overline{\rho}_1 < \overline{\rho}_2$ such that

$$G'(\rho) \begin{cases} \geq 0 & \text{when } \rho \in (0, \overline{\rho}_1) \cup (\overline{\rho}_2, \infty), \\ = 0 & \text{when } \rho = \overline{\rho}_1 \text{ or } \overline{\rho}_2, \\ \leq 0 & \text{when } \rho \in (\overline{\rho}_1, \overline{\rho}_2). \end{cases}$$

Let c_1 and ρ_0^* $(= \rho_0(c_1))$ be defined as in (3.17) and (3.20), respectively. Then we have that

$$\rho_0^* \ (= \rho_0(c_1)) < \overline{\rho}_1 \tag{4.1}$$

since $H_c(\rho, q(\rho, c)) \leq [G(\rho)]^2$ for any $c > 0$ and $\rho \geq \rho_0$ by Lemma 3.3(i) and since

$$\lim_{\rho \to \rho_0(c)^+} \frac{d}{d\rho} H_{c_1}(\rho, q(\rho, c)) \geq 0$$

for $c \leq c_1$ by (3.17).

(I) We prove Theorem 2.4(i). Let $0 < c < c_1$. Then $H_c(\rho, q(\rho, c))$ is defined on the interval $[\rho_0(c), \infty)$ with $0 < \rho_0(c) < \rho_0^* < \overline{\rho}_1$ by Lemma 3.2(vii) and (4.1). Consequently, since (a) $\lim_{\rho \to \rho_0(c)^+} \frac{d}{d\rho} H_c(\rho, q(\rho, c)) > 0$ by Lemma 3.5(i), (b) $\frac{d}{d\rho} H_c(\rho, q(\rho, c)) < 0$ for $\overline{\rho}_1 \leq \rho \leq \overline{\rho}_2$ by Lemma 3.6, and (c) $\lim_{\rho \to \infty} H_c(\rho, q(\rho, c)) = \infty$ by Lemma 3.3(ii), we conclude that the bifurcation curve $\widetilde{S}_c$ is S-shaped on the $(\lambda, \|u\|_\infty)$-plane. We next show that the S-shaped bifurcation curve $\widetilde{S}_c$ can be of either type 1, type 2 or type 3 for each different value of c on $(0, c_1)$. In fact, since $\lim_{c\to 0^+} \lambda_{\min}(c) = 0$ by Lemma 3.4(ii) and since $H_c(\rho, q(\rho, c)) > \frac{1}{4}[G(\rho)]^2$ by Lemma 3.3(iii), there exists $c_{1,1} \in (0, c_1)$ such that, for $0 < c < c_{1,1}$, the S-shaped bifurcation curve $\widetilde{S}_c$ is of type 1 on the $(\lambda, \|u\|_\infty)$-plane. In addition, since (d) there exists $\widetilde{\rho} > \rho_0(c_1)$ such that $\frac{d}{d\rho} H_{c_1}(\rho, q(\rho, c_1)) < 0$ for $\rho_0(c_1) < \rho < \widetilde{\rho}$ by Lemma 3.5(ii), (e) $H_c(\rho, q(\rho, c)) < H_{c_1}(\rho, q(\rho, c_1))$ for $0 < c < c_1$ by Lemma 3.4(i), and (f) $\lim_{\rho\to\rho_0(c)^+} \frac{d}{d\rho} H_c(\rho, q(\rho, c)) > 0$ for $0 < c < c_1$ by Lemma 3.5(i), there exists $c_{1,3} \in [c_{1,1}, c_1)$ such that, for $c_{1,3} < c < c_1$, the S-shaped bifurcation curve $\widetilde{S}_c$ is of type 3 on the $(\lambda, \|u\|_\infty)$-plane. Moreover, by the continuity of evolution of bifurcation curves $\widetilde{S}_c$ from 0^+ to c_1^-, there exists $c_{1,2} \in [c_{1,1}, c_{1,3}]$ such that the S-shaped bifurcation curve $\widetilde{S}_{c_{1,2}}$ is of type 2 on the $(\lambda, \|u\|_\infty)$-plane.

(II) We prove Theorem 2.4(ii). Let $c \geq c_1$. Since $\frac{d}{d\rho} H_c(\rho, q(\rho, c)) < 0$ for ρ is sufficient close to $(\rho_0)^+$ by Lemma 3.5(i)–(ii) and since $\lim_{\rho\to\infty} H_c(\rho, q(\rho, c)) = \infty$, we have that the bifurcation curve $\widetilde{S}_c$ is $\subset$-shaped on the $(\lambda, \|u\|_\infty)$-plane for $c \geq c_1$.

Finally, the multiplicity results of positive solutions for (1.1) with $c > 0$ follow immediately from the shape of bifurcation curve $\widetilde{S}_c$. Hence the proof of Theorem 2.4 is now complete. □

Proof of Theorem 2.5. Let $f(\rho)$ $(= f(\rho, a)) = \exp\left(\frac{a\rho}{a+\rho}\right)$. Then, for $a > 0$, (H1) and (H3) hold as provided in [6, Theorem 2.2(i)]. While, for $a \geq 4.37$, (H2) holds with $\gamma = a(a-2)/2 > 0$ as provided in [6, Theorem 2.2(i)]. We then verify hypotheses (H4)–(H5) for $a > 0$ and (H6) for $a \geq 4.37$ as follows.

(I) We verify (H4) for $a > 0$ with choosing

$$\tau = a. \tag{4.2}$$

We compute that

$$[f'(u) + uf''(u)]f(u) - u[f'(u)]^2 = \frac{a^2(a-u)}{(a+u)^3} \exp\left(\frac{2au}{a+u}\right) \begin{cases} > 0 & \text{when } u \in [0, a), \\ = 0 & \text{when } u = \tau = a, \\ < 0 & \text{when } u \in (a, \infty). \end{cases}$$

So (H4) holds for $a > 0$ with $\tau = a$.

(II) We verify (H5) for $a > 0$. By direct computation, we have that

$$[f'(u)]^2 - f''(u)f(u) = \frac{2a^2}{(a+u)^3} \exp\left(\frac{2au}{a+u}\right) > 0$$

for $u \geq 0$. So (H5) holds for $a > 0$.

(III) We verify (H6) for $a \geq 4.37$. It suffices to prove (1.10)–(1.13) for $a \geq 4.37$. We first note that the assertions given in Lemma 3.1(i)–(viii) hold since (H1) and (H3)–(H5) hold for $a > 0$ as claimed above.

(i) We verify (1.10) for $a \geq 4.37$. We compute that

$$f(\tau) - 4f(0) = \exp\left(\frac{a}{2}\right) - 4 \geq \exp\left(\frac{4.37}{2}\right) - 4 > \exp(2) - 4 > 0.$$

So (1.10) holds for $a \geq 4.37$.

Then, by Lemma 3.1(ix), Lemma 3.7(i) and (4.2), we have that $0 < \overline{\rho}$ $(< 1/2) < \tau$ $(= a)$ for $a \geq 4.37$.

(ii) We verify (1.11) for $a \geq 4.37$. We compute that, for $0 < \rho < \overline{\rho}$ $(< 1/2)$,

$$\begin{aligned}\widetilde{W_0}(\rho) &= \frac{f(0)}{F(\rho)}\left\{[M_1(\rho) - 1]N_2(\rho) + 2M_1(\rho)M_2(\rho)\right\} \\ &\geq \frac{f(0)}{F(\rho)}\left\{-2[M_1(\rho) - 1] + 2M_1(\rho)M_2(\rho)\right\} \\ &\qquad (\text{since } M_1(\rho) = P_1(\rho, 0) > 1 \text{ by Lemma 3.1(iv) and } N_2(\rho) > -2 \text{ by Lemma 3.7(ii)}) \\ &= \frac{2f(0)}{F(\rho)}\left\{[M_2(\rho) - 1]M_1(\rho) + 1\right\}.\end{aligned}$$

Moreover, for $0 < \rho < \overline{\rho}$ $(< 1/2)$, since $M_2(\rho) - 1 < M_2(1/2) - 1 = -(2a^2 + 4a + 1)/(2a + 1)^2 < 0$ by the fact that $M_2(\rho)$ is a strictly increasing function of ρ on $[0, a]$ as claimed in Lemma 3.1(ii) and since $M_1(\rho) < M_2(\rho) + 1$ by Lemma 3.1(v), we conclude that

$$\widetilde{W_0}(\rho) \geq \frac{2f(0)}{F(\rho)}\left\{[M_2(\rho) - 1][M_2(\rho) + 1] + 1\right\} = \frac{2f(0)}{F(\rho)}[M_2(\rho)]^2 > 0.$$

So (1.11) holds $a \geq 4.37$.

(iii) We verify (1.12) for $a \geq 4.37$. Define $W_1(\rho, s) = W(\rho, s) - [W(\rho, 0) + s\widetilde{W_0}(\rho)]$ for $0 \leq s < \rho$. Then showing (1.12) is equivalent to showing that $W_1(\rho, s) \geq 0$ for $0 \leq s < \rho \leq \overline{\rho}$. It is easy to see that $W_1(\rho, 0) = 0$ and $\left[\frac{\partial}{\partial s}W_1(\rho, s)\right]_{s=0} = 0$. Moreover, by direct computation, we have that

$$\begin{aligned}\frac{\partial}{\partial s}W_1(\rho, s) &= \frac{\partial}{\partial s}W(\rho, s) - \widetilde{W_0}(\rho) = \frac{f(s)}{F(\rho) - F(s)}[W(\rho, s) - U(\rho, s)] - \widetilde{W_0}(\rho), \\ \frac{\partial^2}{\partial s^2}W_1(\rho, s) &= \frac{f(s)}{F(\rho) - F(s)}\left[V(\rho, s)\frac{\partial}{\partial s}W(\rho, s) - \frac{\partial}{\partial s}U(\rho, s)\right],\end{aligned}$$

where

$$U(\rho, s) \equiv \left[1 + \frac{sf'(s)}{f(s)}\right]N_2(\rho) + \frac{4sf'(s) + 2s^2 f''(s)}{f(s)}, \quad V(\rho, s) \equiv \frac{f'(s)}{[f(s)]^2}[F(\rho) - F(s)] + 2.$$

Hence, for any fixed $\rho \in (0, \overline{\rho}]$, if $0 \leq s < \rho$ satisfies $\frac{\partial}{\partial s}W_1(\rho, s) = 0$, then

$$\frac{\partial^2}{\partial s^2}W_1(\rho, s) = \frac{f(s)}{F(\rho) - F(s)}W_2(\rho, s), \tag{4.3}$$

where

$$W_2(\rho, s) \equiv V(\rho, s)\widetilde{W_0}(\rho) - \frac{\partial}{\partial s}U(\rho, s).$$

Applying the relationship between $W_1(\rho,s)$ and $W_2(\rho,s)$ in (4.3), we claim that, for any fixed $\rho\in(0,\overline{\rho}]$, $W_1(\rho,s)$ is an increasing function of s on $[0,\rho)$ if the following assertions (a)–(d) hold:

(a) $\frac{\partial^2}{\partial s^2}W_2(\rho,s)>0$ for $0\le s<\rho\le\overline{\rho}$ $(<1/2)$.

(b) $W_2(\rho,0)>0$ for $0<\rho\le\overline{\rho}$ $(<1/2)$.

(c) $\lim_{s\to\rho^-}W_2(\rho,s)<0$ for $0<\rho\le\overline{\rho}$ $(<1/2)$.

(d) $\lim_{s\to\rho^-}W_2(\rho,s)=-2\lim_{s\to\rho^-}\left[\frac{\partial}{\partial s}W_1(\rho,s)\right]$ for $0<\rho\le\overline{\rho}$ $(<1/2)$.

Indeed, suppose that $W_1(\rho,s)$ is not an increasing function of s on $[0,\rho)$ for some fixed $\rho\in(0,\overline{\rho}]$. Then there exists some $\overline{s}\in(0,\rho)$ such that $\frac{\partial}{\partial s}W_1(\rho,\overline{s})<0$. It follows that there exists some $s_*\in(0,\overline{s})$, which is a local maximum of $W_1(\rho,s)$ on $(0,\overline{s})$ since $\left[\frac{\partial}{\partial s}W_1(\rho,s)\right]_{s=0}=0$ and $\left[\frac{\partial^2}{\partial s^2}W_1(\rho,s)\right]_{s=0}>0$ by (4.3) and assertion (b). So $\frac{\partial}{\partial s}W_1(\rho,s_*)=0$ and $\frac{\partial^2}{\partial s^2}W_1(\rho,s_*)\le 0$. Following by (4.3), we have that $W_2(\rho,s_*)\le 0$. Consequently, $W_2(\rho,s)<0$ for $s_*<s<\rho$ by assertions (a)–(c). However, since $\frac{\partial}{\partial s}W_1(\rho,\overline{s})<0$ and $\lim_{s\to\rho^-}\left[\frac{\partial}{\partial s}W_1(\rho,s)\right]>0$ by assertions (c)–(d), there exists some $s^*\in(\overline{s},\rho)$, which is a local minimum of $W_1(\rho,s)$ on $(\overline{s},\rho)$. So $\frac{\partial}{\partial s}W_1(\rho,s^*)=0$ and $\frac{\partial^2}{\partial s^2}W_1(\rho,s^*)\ge 0$. Again, by (4.3), $W_2(\rho,s^*)\ge 0$. So we get a contradiction to the fact that $W_2(\rho,s)<0$ for $s_*<s<\rho$, and hence $W_1(\rho,s)$ is an increasing function of s on $[0,\rho)$. Thus $W_1(\rho,s)\ge W_1(\rho,0)=0$ for $0\le s<\rho\le\overline{\rho}$. So (1.12) holds for $a\ge 4.37$.

The proofs of assertions (a)–(d) for $f(\rho)=\exp\left(\frac{a\rho}{a+\rho}\right)$ with $a\ge 4.37$ are lengthy, and hence they are given in [9].

(iv) We verify (1.13) for $a\ge 4.37$. We compute that

$$
\begin{aligned}
&3\sqrt{f(\rho)}[W(\rho,0)+N_3(\rho)]+2\rho\sqrt{f(0)}\widetilde{W}_0(\rho)\\
&=F(\rho)\widetilde{W}_0(\rho)\left[3\frac{\sqrt{f(\rho)}}{f(0)}+\frac{2\sqrt{f(0)}\rho}{F(\rho)}\right]\\
&\quad+3\sqrt{f(\rho)}\left[1-2\frac{f'(0)}{[f(0)]^2}\rho f(\rho)-M_1(\rho)+2M_2(\rho)\right]\\
&=N_4(\rho,a)\left[3\sqrt{f(\rho)}+\frac{2\rho}{F(\rho)}\right]+3\sqrt{f(\rho)}[1-N_1(\rho,a)]\quad(\text{since } f(0)=f'(0)=1)\\
&\equiv N_4(\rho,a)N_5(\rho,a)+N_6(\rho,a)\equiv N_7(\rho,a).
\end{aligned}
\tag{4.4}
$$

Here $N_1(\rho,a)$ and $N_4(\rho,a)$ are defined in (1.5) and Lemma 3.7(iii), respectively. Now, we show that

$$
N_5(\rho,a)=3\sqrt{f(\rho)}+\frac{2\rho}{F(\rho)}\quad\text{and}\quad N_6(\rho,a)=3\sqrt{f(\rho)}[1-N_1(\rho,a)]
$$

are both strictly increasing functions of a on $[4,\infty)$ for any fixed $\rho\in(0,\overline{\rho}]$.

We first show that $N_5(\rho,a)$ a strictly increasing function of a on $[4,\infty)$ for any fixed $\rho\in(0,\overline{\rho}]$. Let $a\geq 4$ and $0<\rho\leq\overline{\rho}$ $(<1/2)$. We compute that

$$\begin{aligned}
\frac{\partial}{\partial a}N_5(\rho,a) &= \frac{2\rho}{[F(\rho)]^2}\left[\frac{3}{4}\frac{\rho[F(\rho)]^2}{(a+\rho)^2}\exp\left(\frac{a\rho}{2(a+\rho)}\right)-\int_0^\rho\frac{s^2}{(a+s)^2}\exp\left(\frac{as}{a+s}\right)ds\right]\\
&\geq \frac{2\rho}{[F(\rho)]^2}\left[\frac{3}{4}\frac{\rho^3}{(a+\rho)^2}\exp\left(\frac{a\rho}{2(a+\rho)}\right)-\int_0^\rho\frac{s^2}{(a+s)^2}\exp\left(\frac{as}{a+s}\right)ds\right]\\
&\qquad \left(\text{since } F(\rho)=\int_0^\rho f(s)\,ds=\int_0^\rho\exp\left(\frac{as}{a+s}\right)ds\geq\int_0^\rho 1\,ds=\rho\right)\\
&\equiv \frac{2\rho}{[F(\rho)]^2}N_8(\rho,a).
\end{aligned}\tag{4.5}$$

We have that $N_8(0,a)=0$ and, for $0<\rho\leq\overline{\rho}$ $(<1/2)$,

$$\begin{aligned}
\frac{\partial}{\partial\rho}N_8(\rho,a) &= \frac{\rho^2\exp\left(\frac{a\rho}{2(a+\rho)}\right)}{(a+\rho)^2}\left[\frac{3}{8}\frac{(\rho+6)a^2+8\rho a+2\rho^2}{(a+\rho)^2}-\exp\left(\frac{a\rho}{2(a+\rho)}\right)\right]\\
&\geq \frac{\rho^2\exp\left(\frac{a\rho}{2(a+\rho)}\right)}{(a+\rho)^2}\left[\frac{3}{8}\left[\frac{(\rho+6)a^2+8\rho a+2\rho^2}{(a+\rho)^2}\right]_{a=4}-\exp\left(\frac{\rho}{2}\right)\right]\\
&\qquad \left(\text{since } \frac{\partial}{\partial a}\frac{(\rho+6)a^2+8\rho a+2\rho^2}{(a+\rho)^2}=\frac{2\rho[(2+\rho)a+2\rho]}{(a+\rho)^3}>0\right)\\
&= \frac{\rho^2\exp\left(\frac{a\rho}{2(a+\rho)}\right)}{(a+\rho)^2}\left[\frac{3}{8}\frac{2\rho^2+48\rho+96}{(4+\rho)^2}-\exp\left(\frac{\rho}{2}\right)\right]\\
&\geq \frac{\rho^2\exp\left(\frac{a\rho}{2(a+\rho)}\right)}{(a+\rho)^2}\left[\frac{3}{8}\frac{2\rho^2+48\rho+96}{(4+\rho)^2}-\exp\left(\frac{\rho}{2}\right)\right]_{\rho=1/2}\\
&\qquad \left(\text{since } \frac{\partial}{\partial\rho}\left[\frac{3}{8}\frac{2\rho^2+48\rho+96}{(4+\rho)^2}-\exp\left(\frac{\rho}{2}\right)\right]=-\frac{12\rho}{(4+\rho)^3}-\frac{1}{2}\exp\left(\frac{\rho}{2}\right)<0\right)\\
&= \frac{\rho^2\exp\left(\frac{a\rho}{2(a+\rho)}\right)}{(a+\rho)^2}\left[\frac{241}{108}-\exp\left(\frac{1}{4}\right)\right]>0,
\end{aligned}$$

since $241/108-\exp(1/4)$ $(\approx 0.947)>0$. Hence $N_8(\rho,a)>0$ for $a\geq 4$ and $0<\rho\leq\overline{\rho}$. It implies that, by (4.5), $N_5(\rho,a)$ is a strictly increasing function of a on $[4,\infty)$ for any fixed $\rho\in(0,\overline{\rho}]$.

We next show that $N_6(\rho,a)$ a strictly increasing function of a on $[4,\infty)$ for any fixed $\rho\in(0,\overline{\rho}]$. Let $a\geq 4$ and $0<\rho\leq\overline{\rho}$ $(<1/2)$. We compute that

$$\begin{aligned}
\frac{\partial}{\partial a}N_6(\rho,a) &= \frac{3}{2}\frac{\rho^2}{(a+\rho)^2}\sqrt{f(\rho)}[2M_2(\rho)-M_1(\rho)-2\rho f(\rho)+1]\\
&\quad +3\sqrt{f(\rho)}\left[\frac{4a\rho^2}{(a+\rho)^3}-\frac{\partial}{\partial a}M_1(\rho)-\frac{2\rho^3 f(\rho)}{(a+\rho)^2}\right].
\end{aligned}$$

Then, since $M_1(\rho) < M_2(\rho) + 1$ by Lemma 3.1(v) and since

$$\frac{\partial}{\partial a} M_1(\rho, a) = \frac{M_1(\rho)}{(a+\rho)^2} \left[\rho^2 - \frac{\frac{\partial}{\partial a} F(\rho)}{F(\rho)} (a+\rho)^2 \right] < \frac{\rho^2}{(a+\rho)^2} M_1(\rho) < \frac{\rho^2}{(a+\rho)^2} [M_2(\rho) + 1],$$

we have that

$$\begin{aligned} \frac{\partial}{\partial a} N_6(\rho, a) &> \frac{3}{2} \frac{\rho^2}{(a+\rho)^2} \sqrt{f(\rho)} \{2M_2(\rho) - [M_2(\rho) + 1] - 2\rho f(\rho) + 1\} \\ &\quad + 3\sqrt{f(\rho)} \left\{ \frac{4a\rho^2}{(a+\rho)^3} - \left[\frac{\rho^2}{(a+\rho)^2} [M_2(\rho) + 1] \right] - \frac{2\rho^3 f(\rho)}{(a+\rho)^2} \right\} \\ &= \frac{3\sqrt{f(\rho)}}{2(a+\rho)^4} N_9(\rho, a), \end{aligned} \tag{4.6}$$

where

$$N_9(\rho, a) \equiv [-6\rho^3 f(\rho) - \rho^3 + 6\rho^2] a^2 + [-12\rho^4 f(\rho) + 4\rho^3] a - [6\rho^5 f(\rho) + 2\rho^4].$$

We compute that, for $a \geq 4$ and $0 < \rho \leq \overline{\rho}$ $(< 1/2)$,

$$\begin{aligned} \frac{\partial}{\partial a} N_9(\rho, a) &= -2\rho^2(3\rho^3 + 6\rho^2 + 6a\rho) f(\rho) - 2\rho^2(-2\rho + a\rho - 6a) \\ &> -2\rho^2(3\rho^3 + 6\rho^2 + 6a\rho) \exp(1/2) - 2\rho^2(-2\rho + a\rho - 6a) \\ &= 2a\rho^2[-6\rho \exp(1/2) - (\rho - 6)] - 6\rho^2(\rho^3 + 2\rho^2) \exp(1/2) + 4\rho^3 \\ &\geq 8\rho^2[-6\rho \exp(1/2) - (\rho - 6)] - 6\rho^2(\rho^3 + 2\rho^2) \exp(1/2) + 4\rho^3 \\ &\qquad (\text{since } -6\rho \exp(1/2) - (\rho - 6) > -3\exp(1/2) - (1/2 - 6) \ (\approx 0.554) > 0) \\ &= 2\rho^2 \left\{ -\exp(1/2) \left[3\rho^3 + 6\rho^2 + \left(24 + \frac{2}{\exp(1/2)} \right) \rho \right] + 24 \right\} \\ &\geq 2\rho^2 \left\{ -\exp(1/2) \left[3\rho^3 + 6\rho^2 + \left(24 + \frac{2}{\exp(1/2)} \right) \rho \right]_{\rho=1/2} + 24 \right\} \\ &= 2\rho^2 \left[23 - \frac{111}{8} \exp(1/2) \right] \ (\approx 0.248\rho^2) > 0 \end{aligned}$$

and

$$\begin{aligned} N_9(\rho, 4) &= \left\{ -6\rho[\rho^2 + 8\rho + 16] \exp\left(\frac{4\rho}{4+\rho} \right) - 2\rho^2 + 96 \right\} \rho^2 \\ &> \left\{ -6\rho[\rho^2 + 8\rho + 16] \exp\left(\frac{4\rho}{4+\rho} \right) - 2\rho^2 + 96 \right\}_{\rho=1/2} \rho^2 \\ &= \left[\frac{191}{2} - \frac{243}{4} \exp\left(\frac{4}{9} \right) \right] \rho^2 \ (\approx 0.753\rho^2) > 0. \end{aligned}$$

Hence $N_9(\rho, a) > 0$ for $a \geq 4$ and $0 < \rho \leq \overline{\rho}$. It implies that, by (4.6), $N_6(\rho, a)$ is a strictly increasing function of a on $[4, \infty)$ for any fixed $\rho \in (0, \overline{\rho}]$.

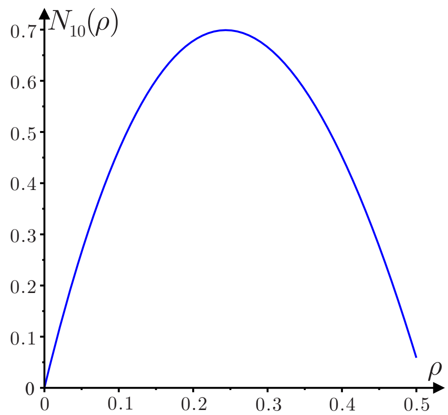

Figure 4.1: Graph of $N_{10}(\rho)$ on $(0, 1/2)$.

Hence, by (4.4) and Lemma 3.7(iii), we conclude that $N_7(\rho, a)$ is a strictly increasing function of a on $[4.37, \infty)$ for any fixed $\rho \in (0, \overline{\rho}]$. It follows that

$$N_7(\rho, a) > N_7(\rho, 4.37) \equiv N_{10}(\rho) > 0$$

for $0 < \rho \le \overline{\rho}$ ($< 1/2$); see Figure 4.1. So (1.13) holds for $a \ge 4.37$.

The proof of Theorem 2.5 is now complete. $\square$

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