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Difference of Weighted Composition Operators on the Space of Cauchy Integral Transforms

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Abstract. In this paper, we provide a complete function theoretic characterizations for boundedness and compactness of difference of weighted composition operators from the space of Cauchy integral transforms to logarithmic weighted-type spaces. Surprisingly, an interesting feature of these characterizations is that they are free from pseudo-hyperbolic distance between $\varphi(z)$ and $\psi(z)$, which is different from the previous characterizations of difference of weighted composition operators acting between different holomorphic function spaces.

1. Introduction

Let $\mathcal{S} = \mathcal{S}(\mathbf{D})$ be the class of all holomorphic self-maps of the unit disk $\mathbf{D}$ of the complex plane $\mathbf{C}$ and $\mathbf{T}$ the boundary of $\mathbf{D}$. Denote by $H(\mathbf{D})$ the space of all holomorphic functions on $\mathbf{D}$. Then, for $\varphi \in \mathbf{S}$ and $u \in H(\mathbf{D})$, the weighted composition operator induced by u and φ is given by

$$uC_\varphi(f) = u \cdot f \circ \varphi, \quad f \in H(\mathbf{D}).$$

We can regard this operator as a generalization of a multiplication operator M_u induced by u and a composition operator C_φ induced by φ, where $M_u f = u \cdot f$ and $C_\varphi(f) = f \circ \varphi$. An extensive study on the theory of (weighted) composition operators has been established during the past four decades in various settings. We refer to standard references [10, 30] for various aspects on the theory of composition operators acting on holomorphic function spaces.

We first recall our function spaces to work on. Let $\mathcal{M}$ be the space of all complex Borel measures on $\mathbf{T}$. The space $\mathcal{F}$ of Cauchy integral transforms consists of all functions $f \in H(\mathbf{D})$ which admits a representation of the form

$$f(z) = \int_{\mathbf{T}} \frac{d\mu(\xi)}{1 - \bar{\xi} z} \quad \text{for some } \mu \in \mathcal{M}.$$

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The space $\mathcal{F}$ is a Banach space with respect to the norm

$$\|f\|_{\mathcal{F}} = \inf\left\{\|\mu\| : f(z) = \int_{\mathbf{T}} \frac{d\mu(\xi)}{1-\bar{\xi}z}\right\},$$

where $\|\mu\|$ denotes the total variation of the measure μ.

The space $\mathcal{F}$ can be viewed in a natural way as the dual space of the disk algebra, or equivalently, as the quotient of the Banach space $\mathcal{M}$ of Borel measures and the space of measures whose Cauchy transforms vanish. It follows from the F. and M. Riesz theorem that the Borel measure μ has a vanishing Cauchy transform if and only if it has the form $d\mu = f\,dm$, where $f \in \overline{H_0^1}$, the subspace of L^1 consisting of functions with mean value 0 whose conjugate belongs to the Hardy space H^1, and m is the normalized Lebesgue measure on $\mathbf{T}$. Thus $\mathcal{F}$ is isometrically isomorphic to $\mathcal{M}/\overline{H_0^1}$.

By the Lebesgue decomposition theorem, the space $\mathcal{M}$ of Borel measures admits a direct sum decomposition $\mathcal{M} = L^1 \oplus \mathcal{M}_s$, where L^1 is identified with absolutely continuous measures $\mathcal{M}_a := \{\mu \in \mathcal{M} : \mu \ll m\}$ and $\mathcal{M}_s := \{\mu \in \mathcal{M} : \mu \perp m\}$. Since $\overline{H_0^1} \subset L^1$, then $\mathcal{F}$ is isometrically isomorphic to $L^1/\overline{H_0^1} \oplus \mathcal{M}_s$.

Similarly, the space of Cauchy integral transforms $\mathcal{F}$ can be decomposed as $\mathcal{F} = \mathcal{F}_a \oplus \mathcal{F}_s$, where $\mathcal{F}_a$ is isometrically isomorphic to $L^1/\overline{H_0^1}$, and $\mathcal{F}_s$ is isometrically isomorphic to $\mathcal{M}_s$.

It is also known that

$$H^1 \subset \mathcal{F} \subsetneq \bigcap_{0<p<1} H^p,$$

where H^p is the standard Hardy space. When $p = \infty$, H^∞ will denote the space of bounded holomorphic functions on $\mathbf{D}$ with the norm $\|f\|_\infty = \sup_{z\in\mathbf{D}} |f(z)|$. For further results about the space of Cauchy integral transforms, we refer to [2,8,14,15] and references therein.

Let ν be a positive continuous function on $\mathbf{D}$ (weight). A weight ν is called typical if it is radial, i.e., $\nu(z) = \nu(|z|)$, $z \in \mathbf{D}$ and $\nu(|z|)$ decreasingly converges to 0 as $|z| \to 1$. A positive continuous function ν on the interval $[0,1)$ is called normal if there are $\delta \in [0,1)$ and τ and t, $0 < \tau < t$ such that

$$\frac{\nu(r)}{(1-r)^\tau} \text{ is decreasing on } [\delta,1) \text{ and } \lim_{r\to 1} \frac{\nu(r)}{(1-r)^\tau} = 0,$$
$$\frac{\nu(r)}{(1-r)^t} \text{ is increasing on } [\delta,1) \text{ and } \lim_{r\to 1} \frac{\nu(r)}{(1-r)^t} = \infty.$$

If we say that a function $\nu\colon \mathbf{D} \to [0,\infty)$ is a normal weighted function, we also assume that it is radial. Now, the logarithmic weighted-type space $LA_{\ln}(\nu)$ is the space of all $f \in H(\mathbf{D})$ such that the norm

$$\|f\|_{LA_{\ln}(\nu)} := \sup_{z\in\mathbf{D}} \nu(|z|)|f(z)| \ln \frac{2}{1-|z|^2}$$

is finite. Likewise we denote by $LA_{\ln,0}(\nu)$ the little logarithmic weighted-type space of holomorphic functions f on $\mathbf{D}$ for which

$$\lim_{|z|\to 1} \nu(|z|)|f(z)| \ln \frac{2}{1-|z|^2} = 0.$$

As is well known, the space $LA_{\ln}(\nu)$ equipped with the norm $\|\cdot\|_{LA_{\ln}(\nu)}$ is a Banach space and $LA_{\ln,0}(\nu)$ is a closed subspace of $LA_{\ln}(\nu)$.

As is well known in the setting of $\mathbf{D}$, every composition operator is bounded on the Hardy spaces or weighted Bergman spaces due to the Littlewood Subordination principle. Much efforts have been expended in the early stage on characterizing those holomorphic maps which induce compact composition operators. It was also known that the composition operator C_φ acts as a bounded operator on the space of Cauchy integral transforms $\mathcal{F}$. Recently, Cima and Matheson [9] considered the problem of characterizing the compactness of C_φ on $\mathcal{F}$ and have established that C_φ is compact on $\mathcal{F}$ if and only if it is compact on H^2. With the basic questions such as boundedness and compactness settled, more attention has been paid to the study of the topological structure of the composition operators in the operator norm topology and this topic is of continuing interests in the theory of composition operators. Berkson [1] first focused attention to the topological structure with his isolation result on the Hardy spaces in 1981 which was refined later by Shapiro and Sundberg [31], and also by MacCluer [22]. In [31], Shapiro and Sundberg posed a question on whether two composition operators belong to the same connected component, when their difference is compact. The above mentioned question of Shapiro and Sundberg initiated another direction of the study of compact differences of composition operators on various settings, which has been a very active topic. MacCluer et al. [23] first used the pseudo-hyperbolic metric to characterize compact difference of two composition operators on H^∞ and then Hosokawa et al. [17] considered compactness of difference of two weighted composition operators on H^∞. Subsequently, the characterization of compact difference of composition operators in the Bergman space setting was obtained by Moorhouse [25] which also involved pseudo-hyperbolic metric. Recently, Wang et al. [37] considered the compact difference of weighted composition operators on the weighted Bergman spaces with the pseudo-hyperbolic metric. For further results on compact differences on various other settings, we refer to [3–7,11,16–21,26,28,29,35,36,38] and references therein. Almost all the characterizations of differences of composition operators acting between any two holomorphic function spaces involve pseudo-hyperbolic metric.

Motivated by these results, in this paper we characterize boundedness and compactness of difference of weighted composition operators from the space of Cauchy integral transforms to logarithmic weighted-type spaces. Surprisingly, our characterizations are free from pseudo-hyperbolic metric, which is a common feature of all the characterizations of difference of weighted composition operators acting between different spaces of holomorphic function spaces, see [16, 18, 20, 35, 38]. Recently, boundedness and compactness of weighted composition operators between the space of Cauchy integral transforms and logarithmic weighted-type spaces were considered by Sharma [32]. For further results about composition operators on the space of Cauchy integral transforms, we refer to [9, 12, 13, 33, 34] and references therein.

In Section 2, we recall some basic results to be used in later sections. In Section 3.1, we characterize boundedness and compactness of difference of weighted composition operators between the space of Cauchy integral transforms and logarithmic weighted-type spaces. The difference of weighted composition operators from the Cauchy integral transform spaces to little logarithmic weighted-type spaces is considered in Section 3.2.

2. Prerequisites

In this section we collect some basic auxiliary facts to be used in later sections. Let X and Y be Banach spaces with respective norms $\|\cdot\|_X$ and $\|\cdot\|_Y$. As usual, we say that a linear operator T from X to Y is bounded if there exists a positive constant C such that $\|T(f)\|_Y \leq C\|f\|_X$ for all f in X. This bounded operator T is said to be compact if the image of every bounded set of X is relatively compact (i.e., has compact closure) in Y. Equivalently, $T\colon X \to Y$ is compact if and only if the image of every bounded sequence in X has a subsequence that converges in Y.

We have the following convenient compactness criterion for the operator $T = u_1C_\varphi - u_2C_\psi$ acting from the space of Cauchy integral transforms to logarithmic weighted-type spaces.

Lemma 2.1. *Assume ν is a normal weighted function on $\mathbf{D}$. Let $\varphi, \psi \in \mathcal{S}$ and $u_1, u_2 \in H(\mathbf{D})$. Suppose the operator $T = u_1C_\varphi - u_2C_\psi\colon \mathcal{F} \to LA_{\ln}(\nu)$ is bounded. Then T is compact if and only if $Tf_n \to 0$ in $LA_{\ln}(\nu)$ for any bounded sequence $\{f_n\}$ in $\mathcal{F}$ such that $\{f_n\} \to 0$ uniformly on compact subsets of $\mathbf{D}$.*

A proof can be found in [10, Proposition 3.11] for composition operators on a Hardy space over the unit disk and it can be modified for composition operators on $\mathcal{F}$.

For $a \in \mathbf{D}$, let the Möbius transformation $\eta_a(z)\colon \mathbf{D} \to \mathbf{D}$ be defined by

$$\eta_a(z) = \frac{a - z}{1 - \overline{a}z}.$$

It is clear that the inverse of η_a under composition is η_a, that is, $(\eta_a \circ \eta_a)(z) = z$ for $z \in \mathbf{D}$ and $|\eta_a'(z)| = (1 - |a|^2)/|1 - \overline{a}z|^2$. Moreover,

$$1 - |\eta_a(z)|^2 = (1 - |z|^2)|\eta_a'(z)| = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \overline{a}z|^2}. \tag{2.1}$$

For convenience in the sequel, we note that

$$d\lambda_a(z) = \frac{(1 - |\eta_a(z)|^2)^2}{(1 - |z|^2)^2}\, dA(z)$$

where dA is the area measure on $\mathbf{D}$ normalized to have total mass 1.

To characterize compactness of difference of weighted composition operators in Section 3.1, we need a equivalent norm for $LA_{\ln}(\nu)$, see [32].

Lemma 2.2. *Let ν be a normal weighted function on $\mathbf{D}$. Then $f \in LA_{\ln}(\nu)$ if and only if*

$$\|f\|_{LA_{\ln}(\nu)} \approx \sup_{a\in\mathbf{D}} \int_{\mathbf{D}} |f(z)|\nu(|z|) \ln \frac{2}{1-|z|^2}\, d\lambda_a(z) < \infty.$$

To consider the compactness of difference of weighted composition operators in Section 3.2, we require the following lemma to characterize the compact subset $F \subset LA_{\ln,0}(\nu)$ as follows. The proof of the following lemma is similar to that of [24, Lemma 1] and the details are omitted here.

Lemma 2.3. *Let $\nu\colon \mathbf{D} \to [0,\infty)$ be a normal weight function. A subset F of $LA_{\ln,0}(\nu)$ is compact if and only if it is closed, bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in F} |f(z)|\nu(z) \ln \frac{2}{1-|z|^2} = 0.$$

In the rest of the paper, we use the same letter C to denote various positive constants which may change the notation at each occurrence. Variables indicating the dependency of C will be often specified in a parenthesis. We use $X \lesssim Y$ or $Y \gtrsim X$ for nonnegative quantities X and Y to mean $X \le CY$ for some constant $C > 0$. Similarly, we use the notation $X \approx Y$ if both $X \lesssim Y$ and $Y \lesssim X$ hold.

3. Main results

3.1. Boundedness and compactness of $T\colon \mathcal{F} \to LA_{\ln}(\nu)$

In this subsection, we characterize boundedness and compactness of $T = u_1C_\varphi - u_2C_\psi$ acting from the space of Cauchy integral transforms to logarithmic weighted-type spaces. Moreover, the exact value of operator norm of T is also computed.

Theorem 3.1. *Let $\varphi, \psi \in \mathcal{S}$, $u_1, u_2 \in H(\mathbf{D})$ and take ν to be a normal weighted function on $\mathbf{D}$. Then the following statements are equivalent:*

(a) $u_1C_\varphi - u_2C_\psi\colon \mathcal{F} \to LA_{\ln}(\nu)$ *is bounded.*

(b) $M_1 := \sup_{\xi\in\mathbf{T}} \sup_{z\in\mathbf{D}} \left| \frac{u_1(z)}{1-\bar{\xi}\varphi(z)} - \frac{u_2(z)}{1-\bar{\xi}\psi(z)} \right| \nu(z) \ln \frac{2}{1-|z|^2} < \infty.$

(c) $M_2 := \sup_{\xi\in\mathbf{T}}\sup_{a\in\mathbf{D}} \int_{\mathbf{D}} \left|\frac{u_1(z)}{1-\overline{\xi}\varphi(z)} - \frac{u_2(z)}{1-\overline{\xi}\psi(z)}\right| \nu(z)\ln\frac{2}{1-|z|^2}\,d\lambda_a(z) < \infty.$

Moreover, if $u_1C_\varphi - u_2C_\psi\colon \mathcal{F}\to LA_{\ln}(\nu)$ *is bounded, we obtain that*

$$M_1 \approx M_2 \tag{3.1}$$

and

$$\|u_1C_\varphi - u_2C_\psi\|_{\mathcal{F}\to LA_{\ln}(\nu)} = M_1. \tag{3.2}$$

Proof. (a) $\Leftrightarrow$ (b). First assume that (a) holds. Consider the family of functions

$$f_\xi(z) = \frac{1}{1-\overline{\xi}z}, \quad \xi\in\mathbf{T}. \tag{3.3}$$

Then $\|f_\xi\|_{\mathcal{F}} = 1$ for every $\xi\in\mathbf{T}$. Thus by the boundedness of $u_1C_\varphi - u_2C_\psi\colon \mathcal{F}\to LA_{\ln}(\nu)$, we obtain that $(u_1C_\varphi - u_2C_\psi)f \in LA_{\ln}(\nu)$ for every $f\in\mathcal{F}$. In particular, $(u_1C_\varphi - u_2C_\psi)f_\xi \in LA_{\ln}(\nu)$ for each $\xi\in\mathbf{T}$. Moreover,

$$\begin{aligned} M_1 &:= \sup_{\xi\in\mathbf{T}}\sup_{z\in\mathbf{D}} \left|\frac{u_1(z)}{1-\overline{\xi}\varphi(z)} - \frac{u_2(z)}{1-\overline{\xi}\psi(z)}\right| \nu(z)\ln\frac{2}{1-|z|^2} \\ &= \sup_{\xi\in\mathbf{T}} \left\|(u_1C_\varphi - u_2C_\psi)\left(\frac{1}{1-\overline{\xi}z}\right)\right\|_{LA_{\ln}(\nu)} \\ &\le \|u_1C_\varphi - u_2C_\psi\|_{\mathcal{F}\to LA_{\ln}(\nu)} \sup_{\xi\in\mathbf{T}}\left\|\frac{1}{1-\overline{\xi}z}\right\|_{\mathcal{F}} \\ &= \|u_1C_\varphi - u_2C_\psi\|_{\mathcal{F}\to LA_{\ln}(\nu)} \end{aligned} \tag{3.4}$$

and so (b) holds, as desired.

Conversely, suppose the condition (b) holds. Let $f\in\mathcal{F}$, then there is a $\mu\in\mathcal{M}$ such that $\|\mu\| = \|f\|_{\mathcal{F}}$ and

$$f(z) = \int_{\mathbf{T}} \frac{d\mu(\xi)}{1-\overline{\xi}z}. \tag{3.5}$$

Let $\tau(z) = \varphi(z)$ or $\psi(z)$. Replacing z in (3.5) by $\tau(z)$ and multiplying such obtained equality by $u_i(z)$ respectively for $i = 1,2$, we have

$$u_1(z)f(\varphi(z)) = \int_{\mathbf{T}} \frac{u_1(z)}{1-\overline{\xi}\varphi(z)}\,d\mu(\xi), \tag{3.6}$$

$$u_2(z)f(\psi(z)) = \int_{\mathbf{T}} \frac{u_2(z)}{1-\overline{\xi}\psi(z)}\,d\mu(\xi). \tag{3.7}$$

Using an elementary inequality, together (3.6) with (3.7), we have

$$
\begin{aligned}
&\nu(z)\ln\frac{2}{1-|z|^2}|u_1(z)f(\varphi(z))-u_2(z)f(\psi(z))| \\
&=\left|\int_{\mathbf{T}}\left(\frac{u_1(z)\nu(z)\ln\frac{2}{1-|z|^2}}{1-\overline{\xi}\varphi(z)}-\frac{u_2(z)\nu(z)\ln\frac{2}{1-|z|^2}}{1-\overline{\xi}\psi(z)}\right)d\mu(\xi)\right| \\
&\le\int_{\mathbf{T}}\left|\frac{u_1(z)\nu(z)\ln\frac{2}{1-|z|^2}}{1-\overline{\xi}\varphi(z)}-\frac{u_2(z)\nu(z)\ln\frac{2}{1-|z|^2}}{1-\overline{\xi}\psi(z)}\right|d|\mu|(\xi) \qquad (3.8)\\
&\le\sup_{\xi\in\mathbf{T}}\sup_{z\in\mathbf{D}}\nu(z)\ln\frac{2}{1-|z|^2}\left|\frac{u_1(z)}{1-\overline{\xi}\varphi(z)}-\frac{u_2(z)}{1-\overline{\xi}\psi(z)}\right|\int_{\mathbf{T}}d|\mu|(\xi) \\
&\le\sup_{\xi\in\mathbf{T}}\sup_{z\in\mathbf{D}}\nu(z)\ln\frac{2}{1-|z|^2}\left|\frac{u_1(z)}{1-\overline{\xi}\varphi(z)}-\frac{u_2(z)}{1-\overline{\xi}\psi(z)}\right|\|f\|_{\mathcal{F}}.
\end{aligned}
$$

Taking the supremum in the last inequality over all $z\in\mathbf{D}$, it follows that

$$
\|(u_1C_\varphi-u_2C_\psi)f\|_{LA_{\ln}(\nu)}\le\sup_{\xi\in\mathbf{T}}\sup_{z\in\mathbf{D}}\nu(z)\ln\frac{2}{1-|z|^2}\left|\frac{u_1(z)}{1-\overline{\xi}\varphi(z)}-\frac{u_2(z)}{1-\overline{\xi}\psi(z)}\right|\|f\|_{\mathcal{F}}. \qquad (3.9)
$$

This shows that $\|(u_1C_\varphi-u_2C_\psi)f\|_{LA_{\ln}(\nu)}\le M_1\|f\|_{\mathcal{F}}$, hence $u_1C_\varphi-u_2C_\psi\colon\mathcal{F}\to LA_{\ln}(\nu)$ is bounded, namely, (a) holds. Moreover, it is easy to see that equality (3.2) holds by (3.4) and (3.9).

(b) $\Leftrightarrow$ (c). Suppose that (c) holds. Note that

$$
D(a,(1-|a|)/2)=\{z\in\mathbf{D}:|z-a|<(1-|a|)/2\}.
$$

Since ν is a normal weighted function on $\mathbf{D}$, then

$$
\nu(|z|)\ln\frac{2}{1-|z|^2}\approx\nu(|a|)\ln\frac{2}{1-|a|^2}
$$

for $z\in D(a,(1-|a|)/2)$. Also it is well known that $|1-\overline{a}z|\approx(1-|a|^2)$ for $z\in D(a,(1-|a|)/2)$. Using these facts, and the subharmonicity of the function

$$
g(z)=\left|\frac{u_1(z)}{1-\overline{\xi}\varphi(z)}-\frac{u_2(z)}{1-\overline{\xi}\psi(z)}\right|,
$$

we have that

$$
\begin{aligned}
M_2&\ge\sup_{\xi\in\mathbf{T}}\sup_{a\in\mathbf{D}}\int_{D(a,(1-|a|)/2)}\left|\frac{u_1(z)}{1-\overline{\xi}\varphi(z)}-\frac{u_2(z)}{1-\overline{\xi}\psi(z)}\right|\nu(z)\ln\frac{2}{1-|z|^2}\,d\lambda_a(z) \qquad (3.10)\\
&\gtrsim\sup_{\xi\in\mathbf{T}}\sup_{a\in\mathbf{D}}\left|\frac{u_1(a)}{1-\overline{\xi}\varphi(a)}-\frac{u_2(a)}{1-\overline{\xi}\psi(a)}\right|\nu(a)\ln\frac{2}{1-|a|^2}=M_1.
\end{aligned}
$$

Conversely, assume that (b) holds. Using the identity (2.1) and Proposition 1.4.10 in [27], we have that

$$
M_2\le M_1\sup_{a\in\mathbf{D}}\int_{\mathbf{D}}\frac{(1-|a|)^2}{|1-\overline{a}z|^4}\,dA(z)\lesssim M_1. \qquad (3.11)
$$

Moreover, the asymptotic relation (3.1) follows from (3.10) and (3.11). The proof is complete. □

Corollary 3.2. *Let ν be a normal weighted function on $\mathbf{D}$ and $\varphi, \psi \in \mathcal{S}$. Then the following statements are equivalent:*

(a) $C_\varphi - C_\psi \colon \mathcal{F} \to LA_{\ln}(\nu)$ *is bounded.*

(b) $M_3 := \sup_{\xi\in\mathbf{T}} \sup_{z\in\mathbf{D}} \left| \frac{1}{1-\bar{\xi}\varphi(z)} - \frac{1}{1-\bar{\xi}\psi(z)} \right| \nu(z) \ln \frac{2}{1-|z|^2} < \infty.$

(c) $M_4 := \sup_{\xi\in\mathbf{T}} \sup_{a\in\mathbf{D}} \int_{\mathbf{D}} \left| \frac{1}{1-\bar{\xi}\varphi(z)} - \frac{1}{1-\bar{\xi}\psi(z)} \right| \nu(z) \ln \frac{2}{1-|z|^2}\, d\lambda_a(z) < \infty.$

Moreover, if $C_\varphi - C_\psi \colon \mathcal{F} \to LA_{\ln}(\nu)$ is bounded, then we obtain that

$$\|C_\varphi - C_\psi\|_{\mathcal{F}\to LA_{\ln}(\nu)} = M_3$$

and $M_3 \approx M_4$.

Using the fact that the family of functions

$$\left\{ f_\xi = \frac{1}{1-\bar{\xi}z} : \xi \in \mathbf{T} \right\}$$

satisfies $\|f_\xi\|_{\mathcal{F}} = 1$, $\xi \in \mathbf{T}$, we can easily obtain the following result.

Corollary 3.3. *Suppose ν is a normal weighted function on $\mathbf{D}$. Let $\varphi, \psi \in \mathcal{S}$ and $u_1, u_2 \in H(\mathbf{D})$. Then $u_1C_\varphi - u_2C_\psi \colon \mathcal{F} \to LA_{\ln}(\nu)$ is bounded if and only if the family of functions*

$$\left\{ \frac{u_1}{1-\bar{\xi}\varphi} - \frac{u_2}{1-\bar{\xi}\psi} : \xi \in \mathbf{T} \right\}$$

is norm-bounded in $LA_{\ln}(\nu)$.

Corollary 3.4. *Let $\varphi, \psi \in \mathcal{S}$. Assume $\nu \colon \mathbf{D} \to [0, \infty)$ is a normal weight function. Then $C_\varphi - C_\psi \colon \mathcal{F} \to LA_{\ln}(\nu)$ is bounded if and only if the family of functions*

$$\left\{ \frac{1}{1-\bar{\xi}\varphi} - \frac{1}{1-\bar{\xi}\psi} : \xi \in \mathbf{T} \right\}$$

is norm-bounded in $LA_{\ln}(\nu)$.

Theorem 3.5. *Let $\varphi, \psi \in \mathcal{S}$ and $u_1, u_2 \in H(\mathbf{D})$. Suppose ν is a normal weight function on $\mathbf{D}$ and $u_1C_\varphi, u_2C_\psi \colon \mathcal{F} \to LA_{\ln}(\nu)$ are bounded. Then the following statements are equivalent.*

(a) $u_1C_\varphi - u_2C_\psi \colon \mathcal{F} \to LA_{\ln}(\nu)$ *is compact.*

(b) $\lim_{r\to 1}\sup_{\xi\in\mathbf{T}}\sup_{a\in\mathbf{D}}\int_{\min(|\varphi(z)|,|\psi(z)|)>r}\left|\frac{u_1(z)}{1-\overline{\xi}\varphi(z)}-\frac{u_2(z)}{1-\overline{\xi}\psi(z)}\right|\nu(z)\ln\frac{2}{1-|z|^2}\,d\lambda_a(z)=0.$

Proof. (b) $\Rightarrow$ (a). Let $\{f_k\}_{k\in\mathbb{N}}$ be a bounded sequence in $\mathcal{F}$, say by L and converging to 0 uniformly on compact subsets of $\mathbf{D}$ as $k\to\infty$. By Lemma 2.1, we need to show that $\|(u_1C_\varphi-u_2C_\psi)f_k\|_{LA_{\ln}(\nu)}\to 0$ as $k\to\infty$. For each $k\in\mathbb{N}$, we can find a $\mu_k\in\mathcal{M}$, with $\|\mu_k\|=\|f_k\|_{\mathcal{F}}$ such that

$$f_k(z)=\int_{\mathbf{T}}\frac{d\mu_k(\xi)}{1-\overline{\xi}z}.$$

Note that

$$g_k(z)=u_1(z)f_k(\varphi(z))-u_2(z)f_k(\psi(z))$$

and

$$M:=\sup_{a\in\mathbf{D}}\left(\int_{\min(|\varphi(z)|,|\psi(z)|)>r}+\int_{\max(|\varphi(z)|,|\psi(z)|)\le r}\right)\nu(z)|g_k(z)|\ln\frac{2}{1-|z|^2}\,d\lambda_a(z).$$

By Lemma 2.2, applied to the function $u_1(f_k\circ\varphi)-u_2(f_k\circ\psi)$ we have

$$\|(u_1C_\varphi-u_2C_\psi)f_k\|_{LA_{\ln}(\nu)}\approx M. \tag{3.12}$$

Since $u_1C_\varphi\colon\mathcal{F}\to LA_{\ln}(\nu)$ and $u_2C_\psi\colon\mathcal{F}\to LA_{\ln}(\nu)$ are bounded, so by taking $f(z)=1$ in $\mathcal{F}$, we have that $u_i\in LA_{\ln}(\nu)$ for $i=1,2$. Also by Lemma 2.2, it is namely that

$$\sup_{a\in\mathbf{D}}\int_{\mathbf{D}}|u_i(z)|\nu(z)\ln\frac{2}{1-|z|^2}\,d\lambda_a(z)<\infty \tag{3.13}$$

for $i=1,2$. By the condition (b), for every $\varepsilon>0$, there is an $r_1\in(0,1)$ such that for $r\in(r_1,1)$, and we have

$$\sup_{\xi\in\mathbf{T}}\sup_{a\in\mathbf{D}}\int_{\min(|\varphi(z)|,|\psi(z)|)>r}\left|\frac{u_1(z)}{1-\overline{\xi}\varphi(z)}-\frac{u_2(z)}{1-\overline{\xi}\psi(z)}\right|\nu(z)\ln\frac{2}{1-|z|^2}\,d\lambda_a(z)<\varepsilon. \tag{3.14}$$

Since the set $\{w:|w|\le r\}$ is compact, we have $\sup_{|w|\le r}|f_k(w)|<\varepsilon$ for sufficiency large k, say $k\ge k_0$. Thus, using (3.12)–(3.14), Fubini's theorem, we have

$$\begin{aligned}
&\|(u_1C_\varphi-u_2C_\psi)f_k\|_{LA_{\ln}(\nu)}\\
&\lesssim\sup_{\max(|\varphi(z)|,|\psi(z)|)\le r}|f_k(\varphi(z))|\sup_{a\in\mathbf{D}}\int_{\max(|\varphi(z)|,|\psi(z)|)\le r}|u_1(z)|\nu(z)\ln\frac{2}{1-|z|^2}\,d\lambda_a(z)\\
&\quad+\sup_{\max(|\varphi(z)|,|\psi(z)|)\le r}|f_k(\psi(z))|\sup_{a\in\mathbf{D}}\int_{\max(|\varphi(z)|,|\psi(z)|)\le r}|u_2(z)|\nu(z)\ln\frac{2}{1-|z|^2}\,d\lambda_a(z)\\
&\quad+\int_{\mathbf{T}}\int_{\min(|\varphi(z)|,|\psi(z)|)>r}\left|\frac{u_1(z)}{1-\overline{\xi}\varphi(z)}-\frac{u_2(z)}{1-\overline{\xi}\psi(z)}\right|\nu(z)\ln\frac{2}{1-|z|^2}\,d\lambda_a(z)\,d|\mu_k|(\xi)\\
&\lesssim\left(1+\int_{\mathbf{T}}d|\mu_k|(\xi)\right)\varepsilon\lesssim(1+\|f_k\|_{\mathcal{F}})\varepsilon\\
&\lesssim(1+L)\varepsilon\lesssim\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, then (b) $\Rightarrow$ (a) holds.

(a) $\Rightarrow$ (b). Since $u_1 C_\varphi \colon \mathcal{F} \to LA_{\ln}(\nu)$ and $u_2 C_\psi \colon \mathcal{F} \to LA_{\ln}(\nu)$ are bounded, so by taking $f(z) = 1$ in $\mathcal{F}$, we have that

$$\sup_{a\in\mathbf{D}} \int_{\mathbf{D}} |u_1(z)|\nu(z) \ln \frac{2}{1-|z|^2}\, d\lambda_a(z) < \infty$$

and

$$\sup_{a\in\mathbf{D}} \int_{\mathbf{D}} |u_2(z)|\nu(z) \ln \frac{2}{1-|z|^2}\, d\lambda_a(z) < \infty.$$

Thus, by the absolute continuity of integral, for every $\varepsilon > 0$, we can choose $r \in (0,1)$ such that

$$\sup_{a\in\mathbf{D}} \int_{\min(|\varphi(z)|,|\psi(z)|)>r} |u_1(z)|\nu(z) \ln \frac{2}{1-|z|^2}\, d\lambda_a(z) < \varepsilon \tag{3.15}$$

and

$$\sup_{a\in\mathbf{D}} \int_{\min(|\psi(z)|,|\psi(z)|)>r} |u_2(z)|\nu(z) \ln \frac{2}{1-|z|^2}\, d\lambda_a(z) < \varepsilon. \tag{3.16}$$

Let $B_{\mathcal{F}} = \{f \in \mathcal{F} : \|f\|_{\mathcal{F}} \leq 1\}$ be the closed unit ball in $\mathcal{F}$. Next, let $f \in B_{\mathcal{F}}$ and $f_t(z) = f(tz)$, $f_t \in \mathcal{F}$, $0 < t < 1$. Then $\sup_{0<t<1} \|f_t\|_{\mathcal{F}} \leq \|f\|_{\mathcal{F}}$, $f_t \in \mathcal{F}$, $t \in (0,1)$ (see [32]) and $f_t \to f$ uniformly on compact subsets of $\mathbf{D}$ as $t \to 1$. The compactness of $u_1 C_\varphi - u_2 C_\psi \colon \mathcal{F} \to LA_{\ln}(\nu)$ implies that

$$\lim_{t\to 1} \|(u_1 C_\varphi - u_2 C_\psi) f_t - (u_1 C_\varphi - u_2 C_\psi) f\|_{LA_{\ln}(\nu)} = 0.$$

Thus for every $\varepsilon > 0$, there is a $t \in (0,1)$ such that

$$\begin{aligned} &\sup_{a\in\mathbf{D}} \int_{\mathbf{D}} |(u_1(z) f_t(\varphi(z)) - u_2(z) f_t(\psi(z))) - (u_1(z) f(\varphi(z)) - u_2(z) f(\psi(z)))| \\ &\qquad \times \nu(z) \ln \frac{2}{1-|z|^2}\, d\lambda_a(z) < \varepsilon. \end{aligned} \tag{3.17}$$

From (3.15)–(3.17), we have

$$\begin{aligned} &\sup_{a\in\mathbf{D}} \int_{\min(|\varphi(z)|,|\psi(z)|)>r} |u_1(z) f(\varphi(z)) - u_2(z) f(\psi(z))|\nu(z) \ln \frac{2}{1-|z|^2}\, d\lambda_a(z) \\ &\leq \sup_{a\in\mathbf{D}} \int_{\mathbf{D}} |(u_1(z) f_t(\varphi(z)) - u_2(z) f_t(\psi(z))) - (u_1(z) f(\varphi(z)) - u_2(z) f(\psi(z)))| \\ &\qquad \times \nu(z) \ln \frac{2}{1-|z|^2}\, d\lambda_a(z) \\ &\quad + \sup_{a\in\mathbf{D}} \int_{\min(|\varphi(z)|,|\psi(z)|)>r} (|f_t(\varphi(z)) u_1(z)| + |f_t(\psi(z)) u_2(z)|)\nu(z) \ln \frac{2}{1-|z|^2}\, d\lambda_a(z) \\ &\leq \varepsilon(1 + 2\|f_t\|_\infty). \end{aligned}$$

Thus we claim that for every $f \in B_{\mathcal{F}}$, there is a $\delta_0 \in (0,1)$, $\delta_0 = \delta_0(f,\varepsilon)$, such that for $r \in (\delta_0, 1)$

$$\sup_{a\in\mathbf{D}} \int_{\min(|\varphi(z)|,|\psi(z)|)>r} |u_1(z)f(\varphi(z)) - u_2(z)f(\psi(z))|\nu(z)\ln\frac{2}{1-|z|^2}\,d\lambda_a(z) < \varepsilon. \tag{3.18}$$

Since $u_1C_\varphi - u_2C_\psi\colon \mathcal{F} \to LA_{\ln}(\nu)$ is compact, the ball $B_{\mathcal{F}}$ is mapped by $u_1C_\varphi - u_2C_\psi$ into a relatively compact subset of $LA_{\ln}(\nu)$. Hence for every $\varepsilon > 0$, there is a finite collection of functions $f_1, f_2, \ldots, f_k \in B_{\mathcal{F}}$, and there is a $j \in \{1,2,\ldots,k\}$ such that

$$\begin{aligned}\sup_{a\in\mathbf{D}} \int_{\mathbf{D}} &|(u_1(z)f(\varphi(z)) - u_2(z)f(\psi(z))) - (u_1(z)f_j(\varphi(z)) - u_2(z)f_j(\psi(z)))| \\ &\times \nu(z)\ln\frac{2}{1-|z|^2}\,d\lambda_a(z) < \varepsilon.\end{aligned} \tag{3.19}$$

Moreover, by (3.18), we have that for $\delta := \max_{1\le j\le k}\delta_j(f_j,\varepsilon)$ and $r \in (\delta, 1)$

$$\sup_{a\in\mathbf{D}} \int_{\min(|\varphi(z)|,|\psi(z)|)>r} |u_1(z)f_j(\varphi(z)) - u_2(z)f_j(\psi(z))|\nu(z)\ln\frac{2}{1-|z|^2}\,d\lambda_a(z) < \varepsilon \tag{3.20}$$

for each $j \in \{1,2,\ldots,k\}$. Together (3.19) with (3.20), we obtain that for $r \in (\delta,1)$ and every $f \in B_{\mathcal{F}}$

$$\sup_{a\in\mathbf{D}} \int_{\min(|\varphi(z)|,|\psi(z)|)>r} |u_1(z)f(\varphi(z)) - u_2(z)f(\psi(z))|\nu(z)\ln\frac{2}{1-|z|^2}\,d\lambda_a(z) \lesssim \varepsilon. \tag{3.21}$$

If we apply (3.21) to the function $f_\xi(z) = 1/(1-\overline{\xi}z)$, we have that

$$\sup_{\xi\in\mathbf{T}}\sup_{a\in\mathbf{D}} \int_{\min(|\varphi(z)|,|\psi(z)|)>r} \left|\frac{u_1(z)}{1-\overline{\xi}\varphi(z)} - \frac{u_2(z)}{1-\overline{\xi}\psi(z)}\right| \nu(z)\ln\frac{2}{1-|z|^2}\,d\lambda_a(z) \lesssim \varepsilon,$$

from which (b) follows as desired. This completes the proof. □

Corollary 3.6. *Let $\varphi, \psi \in \mathcal{S}$ and $\nu\colon \mathbf{D} \to [0,\infty)$ be a normal weight function. If $C_\varphi, C_\psi\colon \mathcal{F} \to LA_{\ln}(\nu)$ are bounded, then $C_\varphi - C_\psi\colon \mathcal{F} \to LA_{\ln}(\nu)$ is compact if and only if*

$$\lim_{r\to 1}\sup_{\xi\in\mathbf{T}}\sup_{a\in\mathbf{D}} \int_{\min(|\varphi(z)|,|\psi(z)|)>r} \left|\frac{1}{1-\overline{\xi}\varphi(z)} - \frac{1}{1-\overline{\xi}\psi(z)}\right| \nu(z)\ln\frac{2}{1-|z|^2}\,d\lambda_a(z) = 0.$$

3.2. Boundedness and compactness of $T\colon \mathcal{F} \to LA_{\ln,0}(\nu)$

In the last section, the boundedness and compactness of $T = u_1C_\varphi - u_2C_\psi$ acting from the Cauchy integral transform spaces to the little logarithmic weighted-type spaces are considered.

Theorem 3.7. *Suppose $\nu\colon \mathbf{D}\to[0,\infty)$ is a normal weight function. Let $\varphi,\psi\in\mathcal{S}$ and $u_1,u_2\in H(\mathbf{D})$. Then $u_1C_\varphi-u_2C_\psi\colon\mathcal{F}\to LA_{\ln,0}(\nu)$ is bounded if and only if $u_1C_\varphi-u_2C_\psi\colon\mathcal{F}\to LA_{\ln}(\nu)$ is bounded and*

$$\lim_{|z|\to 1}\left|\frac{u_1(z)}{1-\overline{\xi}\varphi(z)}-\frac{u_2(z)}{1-\overline{\xi}\psi(z)}\right|\nu(z)\ln\frac{2}{1-|z|^2}=0 \tag{3.22}$$

for every $\xi\in\mathbf{T}$.

Proof. First assume that $u_1C_\varphi-u_2C_\psi\colon\mathcal{F}\to LA_{\ln,0}(\nu)$ is bounded, then for every $f\in\mathcal{F}$, we have that $(u_1C_\varphi-u_2C_\psi)f\in LA_{\ln,0}(\nu)\subset LA_{\ln}(\nu)$. So by the closed graph theorem, we have that $u_1C_\varphi-u_2C_\psi\colon\mathcal{F}\to LA_{\ln}(\nu)$ is bounded. Once again consider the family of test functions in (3.3). Then $\|f_\xi\|_{\mathcal{F}}=1$. Hence by the boundedness of $u_1C_\varphi-u_2C_\psi\colon\mathcal{F}\to LA_{\ln,0}(\nu)$, we have $(u_1C_\varphi-u_2C_\psi)f_\xi\in LA_{\ln,0}(\nu)$ for every $\xi\in\mathbf{T}$, namely,

$$\lim_{|z|\to 1}\left|\frac{u_1(z)}{1-\overline{\xi}\varphi(z)}-\frac{u_2(z)}{1-\overline{\xi}\psi(z)}\right|\nu(z)\ln\frac{2}{1-|z|^2}=0$$

for every $\xi\in\mathbf{T}$.

Conversely, suppose that $u_1C_\varphi-u_2C_\psi\colon\mathcal{F}\to LA_{\ln}(\nu)$ is bounded and (3.22) holds. It follows from (3.22) that the inner expression of (3.8) tends to zero for every $\xi\in\mathbf{T}$ as $|z|\to 1$. Because of the boundedness of $u_1C_\varphi-u_2C_\psi\colon\mathcal{F}\to LA_{\ln}(\nu)$ the condition (b) in Theorem 3.1 holds. Moreover, the integrand in (3.8) is dominated by M_1, where M_1 is in Theorem 3.1. Hence by the bounded convergence theorem, the integral in (3.8) tends to zero as $|z|\to 1$, implying

$$\lim_{|z|\to 1}|(u_1C_\varphi-u_2C_\psi)f|\nu(z)\ln\frac{2}{1-|z|^2}=0.$$

Thus, we conclude that if $f\in\mathcal{F}$, then $(u_1C_\varphi-u_2C_\psi)f\in LA_{\ln,0}(\nu)$. Therefore, the boundedness of $u_1C_\varphi-u_2C_\psi\colon\mathcal{F}\to LA_{\ln,0}(\nu)$ follows by the closed graph theorem. The proof is complete. □

Corollary 3.8. *Let $\varphi,\psi\in\mathcal{S}$. Assume ν is a normal weighted function on $\mathbf{D}$. Then $C_\varphi-C_\psi\colon\mathcal{F}\to LA_{\ln,0}(\nu)$ is bounded if and only if $C_\varphi-C_\psi\colon\mathcal{F}\to LA_{\ln}(\nu)$ is bounded and*

$$\lim_{|z|\to 1}\left|\frac{1}{1-\overline{\xi}\varphi(z)}-\frac{1}{1-\overline{\xi}\psi(z)}\right|\nu(z)\ln\frac{2}{1-|z|^2}=0$$

for every $\xi\in\mathbf{T}$.

Theorem 3.9. *Let ν be a normal weighted function on $\mathbf{D}$ and $\varphi,\psi\in\mathcal{S}$, $u_1,u_2\in H(\mathbf{D})$. Then $u_1C_\varphi-u_2C_\psi\colon\mathcal{F}\to LA_{\ln,0}(\nu)$ is compact if and only if*

$$\lim_{|z|\to 1}\sup_{\xi\in\mathbf{T}}\left|\frac{u_1(z)}{1-\overline{\xi}\varphi(z)}-\frac{u_2(z)}{1-\overline{\xi}\psi(z)}\right|\nu(z)\ln\frac{2}{1-|z|^2}=0. \tag{3.23}$$

Proof. By Lemma 2.3, a closed set E in $LA_{\ln,0}(\nu)$ is compact if and only if it is bounded and satisfies

$$\lim_{|z|\to 1}\sup_{f\in E}\nu(z)|f(z)|\ln\frac{2}{1-|z|^2}=0.$$

Thus the set $\{(u_1C_\varphi-u_2C_\psi)f : f\in B_{\mathcal{F}}\}$ has compact closure in $LA_{\ln,0}(\nu)$ if and only if

$$\lim_{|z|\to 1}\sup\{|(u_1C_\varphi-u_2C_\psi)f(z)|\nu(z)\ln\frac{2}{1-|z|^2} : f\in B_{\mathcal{F}}\}=0. \tag{3.24}$$

Let $f\in\mathcal{F}$, then there is a $\mu\in\mathcal{M}$ such that $\|\mu\|=\|f\|_{\mathcal{F}}$ and

$$f(z)=\int_{\mathrm{T}}\frac{d\mu(\xi)}{1-\overline{\xi}z},\quad \xi\in\mathbf{T}.$$

Then, for each $f\in B_{\mathcal{F}}$, we can easily get that

$$\begin{aligned}
&|(u_1C_\varphi-u_2C_\psi)f(z)|\nu(z)\ln\frac{2}{1-|z|^2}\\
&\le\int_{\mathbf{T}}\left|\frac{u_1(z)}{1-\overline{\xi}\varphi(z)}-\frac{u_2(z)}{1-\overline{\xi}\psi(z)}\right|\nu(z)\ln\frac{2}{1-|z|^2}\,d|\mu|(\xi)\\
&\le\|\mu\|\sup_{\xi\in\mathbf{T}}\left|\frac{u_1(z)}{1-\overline{\xi}\varphi(z)}-\frac{u_2(z)}{1-\overline{\xi}\psi(z)}\right|\nu(z)\ln\frac{2}{1-|z|^2}\\
&\le\|f\|_{\mathcal{F}}\sup_{\xi\in\mathbf{T}}\left|\frac{u_1(z)}{1-\overline{\xi}\varphi(z)}-\frac{u_2(z)}{1-\overline{\xi}\psi(z)}\right|\nu(z)\ln\frac{2}{1-|z|^2}.
\end{aligned}\tag{3.25}$$

Using (3.23) in (3.25), we obtain (3.24). Thus, $u_1C_\varphi-u_2C_\psi\colon\mathcal{F}\to LA_{\ln,0}(\nu)$ is compact.

Conversely, assume that $u_1C_\varphi-u_2C_\psi\colon\mathcal{F}\to LA_{\ln,0}(\nu)$ is compact. Taking the test functions in (3.3) and using the fact that $\|f_\xi\|_{\mathcal{F}}=1$, we can easily get that (3.23) follows from (3.24). This completes the proof. □

Corollary 3.10. *Suppose $\nu\colon\mathbf{D}\to[0,\infty)$ is a normal weight function. Let $\varphi,\psi\in\mathcal{S}$. Then $C_\varphi-C_\psi\colon\mathcal{F}\to LA_{\ln,0}(\nu)$ is compact if and only if*

$$\lim_{|z|\to 1}\sup_{\xi\in\mathbf{T}}\left|\frac{1}{1-\overline{\xi}\varphi(z)}-\frac{1}{1-\overline{\xi}\psi(z)}\right|\nu(z)\ln\frac{2}{1-|z|^2}=0.$$

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References

[1] E. Berkson, *Composition operators isolated in the uniform operator topology*, Proc. Amer. Math. Soc. **81** (1981), no. 2, 230–232.

[2] P. Bourdon and J. A. Cima, *On integrals of Cauchy-Stieltjes type*, Houston J. Math. **14** (1988), no. 4, 465–474.

[3] B. R. Choe, H. Koo and I. Park, *Compact differences of composition operators over polydisks*, Integral Equations Operator Theory **73** (2012), no. 1, 57–91.

[4] ______, *Compact differences of composition operators on the Bergman spaces over the ball*, Potential Anal. **40** (2014), no. 1, 81–102.

[5] B. R. Choe, H. Koo and W. Smith, *Difference of composition operators over the half-plane*, Trans. Amer. Math. Soc. **369** (2017), no. 5, 3173–3205.

[6] B. R. Choe, H. Koo and M. Wang, *Compact double differences of composition operators on the Bergman spaces*, J. Funct. Anal. **272** (2017), no. 6, 2273–2307.

[7] B. R. Choe, H. Koo, M. Wang and J. Yang, *Compact linear combinations of composition operators induced by linear fractional maps*, Math. Z. **280** (2015), no. 3-4, 807–824.

[8] J. A. Cima and T. H. MacGregor, *Cauchy transforms of measures and univalent functions*, in: *Complex Analysis I*, (College Park, Md., 1985–86), 78–88, Lecture Notes in Math. **1275**, Springer, Berlin, 1987.

[9] J. A. Cima and A. Matheson, *Cauchy transforms and composition operators*, Illinois J. Math. **42** (1998), no. 1, 58–69.

[10] C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995.

[11] K. Heller, B. D. MacCluer and R. J. Weir, *Compact differences of composition operators in several variables*, Integral Equations Operator Theory **69** (2011), no. 2, 247–268.

[12] R. A. Hibschweiler, *Composition operators on spaces of Cauchy transforms*, in: *Studies on Composition Operators* (Laramie, WY, 1996), 57–63, Contemp. Math. **213**, Amer. Math. Soc., Providence, RI, 1998.

[13] ______, *Composition operators on spaces of fractional Cauchy transforms*, Complex Anal. Oper. Theory **6** (2012), no. 4, 897–911.

[14] R. A. Hibschweiler and T. H. MacGregor, *Closure properties of families of Cauchy-Stieltjes transforms*, Proc. Amer. Math. Soc. **105** (1989), no. 3, 615–621.

[15] ______, *Multipliers of families of Cauchy-Stieltjes transforms*, Trans. Amer. Math. Soc. **331** (1992), no. 1, 377–394.

[16] T. Hosokawa, *Differences of weighted composition operators on the Bloch spaces*, Complex Anal. Oper. Theory **3** (2009), no. 4, 847–866.

[17] T. Hosokawa, K. Izuchi and S. Ohno, *Topological structure of the space of weighted composition operators on H^∞*, Integral Equations Operator Theory **53** (2005), no. 4, 509–526.

[18] T. Hosokawa and S. Ohno, *Differences of weighted composition operators acting from Bloch space to H^∞*, Trans. Amer. Math. Soc. **363** (2011), no. 10, 5321–5340.

[19] L. Jiang and C. Ouyang, *Compact differences of composition operators on holomorphic function spaces in the unit ball*, Acta Math. Sci. Ser. B Engl. Ed. **31** (2011), no. 5, 1679–1693.

[20] Z. J. Jiang and S. Stević, *Compact differences of weighted composition operators from weighted Bergman spaces to weighted-type spaces*, Appl. Math. Comput. **217** (2010), no. 7, 3522–3530.

[21] H. Koo and M. Wang, *Joint Carleson measure and the difference of composition operators on $A^p_\alpha(\mathbf{B}_n)$*, J. Math. Anal. Appl. **419** (2014), no. 2, 1119–1142.

[22] B. D. MacCluer, *Components in the space of composition operators*, Integral Equations Operator Theory **12** (1989), no. 5, 725–738.

[23] B. D. MacCluer, S. Ohno and R. Zhao, *Topological structure of the space of composition operators on H^∞*, Integral Equations Operator Theory **40** (2001), no. 4, 481–494.

[24] K. Madigan and A. Matheson, *Compact composition operators on the Bloch space*, Trans. Amer. Math. Soc. **347** (1995), no. 7, 2679–2687.

[25] J. Moorhouse, *Compact differences of composition operators*, J. Funct. Anal. **219** (2005), no. 1, 70–92.

[26] P. Nieminen and E. Saksman, *On compactness of the difference of composition operators*, J. Math. Anal. Appl. **298** (2004), no. 2, 501–522.

[27] W. Rudin, *Function Theory in the Unit Ball of* $\mathbf{C}^n$, Fundamental Principles of Mathematical Science **241**, Springer-Verlag, New York, 1980.

[28] E. Saukko, *Difference of composition operators between standard weighted Bergman spaces*, J. Math. Anal. Appl. **381** (2011), no. 2, 789–798.

[29] ______, *An application of atomic decomposition in Bergman spaces to the study of differences of composition operators*, J. Funct. Anal. **262** (2012), no. 9, 3872–3890.

[30] J. H. Shapiro, *Composition Operators and Classical Function Theory*, Universitext: Tracts in Mathematics, Springer-Verlag, New York, 1993.

[31] J. H. Shapiro and C. Sundberg, *Isolation amongst the composition operators*, Pacific J. Math. **145** (1990), no. 1, 117–152.

[32] A. K. Sharma, *Weighted composition operators from Cauchy integral transforms to logarithmic weighted-type spaces*, Ann. Funct. Anal. **4** (2013), no. 1, 163–174.

[33] A. K. Sharma and R. Krishan, *Difference of composition operators from the space of Cauchy integral transforms to the Dirichlet space*, Complex Anal. Oper. Theory **10** (2016), no. 1, 141–152.

[34] A. K. Sharma, R. Krishan and E. Subhadarsini, *Difference of composition operators from the space of Cauchy integral transforms to Bloch-type spaces*, Integral Transforms Spec. Funct. **28** (2017), no. 2, 145–155.

[35] S. Stević, *Essential norm of differences of weighted composition operators between weighted-type spaces on the unit ball*, Appl. Math. Comput. **217** (2010), no. 5, 1811–1824.

[36] M. Wang and C. Pang, *Compact double differences of composition operators over the half-plane*, Complex Anal. Oper. Theory **12** (2018), no. 1, 261–292.

[37] M. Wang, X. Yao and F. Chen, *Compact differences of weighted composition operators on the weighted Bergman spaces*, J. Inequal. Appl. 2017, Paper No. 2, 14 pp.

[38] X. Zhu and W. Yang, *Differences of composition operators from weighted Bergman spaces to Bloch spaces*, Filomat **28** (2014), no. 9, 1935–1941.

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