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Hausdorff Measure of the Escaping Parameter Without Endpoints is Zero for Exponential Family

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Abstract. In this paper, we prove that the escaping parameter $\mathfrak{I}$ omitted endpoints has Hausdorff measure 0 for the gauge function $t/(\log \frac{1}{t})^s$, where $s > 1$. This is a counterpart result on parameter space of exponential family of Theorem 1.1 in paper [5].

1. Introduction

Let f be a transcendental entire function and denote by $f^n(z) = f \circ f^{n-1}(z)$ the n time iteration of f, for $n \geq 1$. Based on the behavior of the point z under iteration of f, the complex plane $\mathbb{C}$ is split into two fundamentally different parts: the Fatou set $F(f)$, where $\{f^n(z)\}$ forms a normal family in the sense of Montel (i.e., equicontinuous), and the Julia set $J(f)$, where it is chaotic. Both the Fatou set and the Julia set are completely invariant, i.e., $z \in F(f)$ if and only if $f(z) \in F(f)$. For more properties of these sets, we refer to the surveys [3, 20] and books [2, 15, 23].

In 1984, Devaney and Krych [7, p. 50] proved that the Julia set of exponential function $J(E_\lambda)$ consists of uncountably many pairwise disjoint curves, which are called the "hairs" for $0 < \lambda < 1/e$. Here E_λ, $\lambda \neq 0$ denotes the exponential family $\lambda \exp(z)$.

The escaping set of f is defined by $I(f) = \{z : f^n(z) \to \infty \text{ as } n \to \infty\}$. The importance of this set arises from Eremenko's result [8] according to the fact $J(f) = \partial I(f)$. In 2003, Schleicher and Zimmer [21, Corollary 6.9] proved that the escaping points of f are organized in the form of differentiable curves called "dynamical rays" and every escaping point of E_λ is either on a ray or the endpoint of a ray.

Let $\mathcal{S}_0$ denote the set consisting of external address $\underline{s}$, which is exponential bounded. As we known, the dynamics rays with a fast address $\underline{s}$ must land on its endpoint. So we denote $X_{\underline{s}}$ by a closed interval $[t_{\underline{s}}, \infty)$ when $\underline{s}$ is a fast address, and denote $X_{\underline{s}} = (t_{\underline{s}}, \infty)$ for otherwise. (See Definition 2.1 for external address; exponential bounded and fast address etc.)

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Theorem 1.1. [21, Corollary 6.9] *If λ is not an escaping parameter, there is a continuous bijection*

$$g_\lambda\colon Y = \{(\underline{s}, t) \in \mathcal{S}_0 \times \mathbb{R} : t \in X_{\underline{s}}\} \to I(E_\lambda).$$

If λ is an escaping parameter, then there exists $(\underline{s}_\lambda, t_\lambda)$ such that $g_\lambda(\underline{s}_\lambda, t_\lambda) = 0$. We have to restrict the continuous injective map $g_\lambda\colon Y \setminus Y_\lambda \to I(E_\lambda)$ where

$$Y_\lambda = Y \setminus \{(\underline{s}, t) : \exists\, n \geq 1 \text{ such that } \sigma^n \underline{s} = \underline{s}_\lambda \text{ and } F^n(t) \leq t_\lambda\}.$$

Moreover, $E_\lambda^k(g_{\underline{s}}(t)) = F^k(t) + 2\pi i s_{k+1} + o(1)$ as $k \to \infty$ and $E_\lambda(g_{\underline{s}}(t)) = g_{\sigma(\underline{s})}(F(t))$.

McMullen [14] proved that the Hausdorff dimension of $J(E_\lambda)$ is 2 for $\lambda \neq 0$. In fact, he proved that the Hausdorff dimension of the escaping set $I(E_\lambda)$ is 2 and $I(E_\lambda) \subset J(E_\lambda)$. Recently, Peter [16] studied the Hausdorff measure respect to some gauge functions (see Definition 1.2) of Julia set of exponential functions.

We denote the endpoints of escaping set of exponential function by $\mathfrak{C}$. Karpińska [12, Theorem 1.1] proved that Julia set of E_λ omitting endpoints, i.e., $J(E_\lambda) \setminus \mathfrak{C}$, has Hausdorff dimension 1, and the Hausdorff dimension of endpoints is 2 for some λ. This phenomenon is called "Karpińska paradox". In [21, Corollary 7.1], Schleicher and Zimmer extended the above result and obtained a similar result for general λ: the union of all dynamic rays has Hausdorff dimension 1, while the set of escaping rays endpoints has Hausdorff dimension 2.

There are also some other important results on Hausdorff measure and Lebsegue measure, see [4, 11, 22].

A gauge function is a monotonically increasing function $h\colon [0, \varepsilon) \to [0, +\infty)$, which is continuous from the right of point 0 and $h(0) = 0$.

Definition 1.2. Let A be a subset of $\mathbb{R}^m$, and let $\delta > 0$ be a constant. Then

$$\mathrm{HM}^h(A) := \liminf_{\delta \to 0} \left\{ \sum_{j=1}^{\infty} h(\mathrm{diam}(A_j)) : A \subset \bigcup_{j=1}^{\infty} A_j \text{ and } \mathrm{diam}(A_j) < \delta \right\}$$

is the Hausdorff measure with respect to the gauge function h, where $\mathrm{diam}(A_j) = \sup_{x,y \in A_j} |x - y|$ is the standard Euclidean diameter of A_j.

The Hausdorff measure is an outer measure of Borel sets which are measurable. In particular, when $h^s(r) = r^s$ $(s > 0)$, $\mathrm{HM}^h(A)$ is called the s-dimension Hausdorff measure of a set A, denoted by $\mathrm{HM}^{h^s}(A)$. If $\mathrm{HM}^{h^s}(A) < \infty$, then $\mathrm{HM}^{h^t}(A) = 0$ for all constants $t > s$; if $\mathrm{HM}^{h^s}(A) > 0$, then $\mathrm{HM}^{h^t}(A) = \infty$ for all $t < s$. From the above argument, there exists a constant s such that $\mathrm{HM}^{h^t}(A) = 0$ for all $t > s$ and $\mathrm{HM}^{h^t}(A) = \infty$ for all $t < s$. The constant s is called the Hausdorff dimension of A, and is denoted by $s = \mathrm{HD}(A)$.

Recently, Bergweiler and Wang [5] proved two results on Hausdorff measure of $J(E_\lambda) \setminus \mathfrak{C}$, we state one of these results as follows.

Theorem 1.3. [5, Theorem 1.1] *Let $s > 1$. Then* $\mathrm{HM}^h(J(E_\lambda) \setminus \mathfrak{C}) = 0$ *for* $h(t) = t/(\log(1/t))^s$.

In order to state conveniently in parameter space, we denote the exponential maps by $E_\kappa = \exp(z) + \kappa$ rather than $\lambda \exp(z)$, but they are conjugate by translation.

Devaney [6] studied the parameter space of E_κ and introduced the bifurcation graph, whose boundary is very complicated. It contains the escaping parameter

$$\mathfrak{I} = \{\kappa \in \mathbb{C} : E_\kappa^n(\kappa) \to \infty \text{ as } n \to \infty\}.$$

Förster and Schleicher [10] proved that the set $\mathfrak{I}$ consists of uncountably many disjoint curves in parameter space which are called "parameter rays". Förster, Rempe and Schleicher [9] also give a classification of $\mathfrak{I}$: every escaping parameter is either on a parameter ray or the endpoint of one such ray.

Theorem 1.4. [9, Proposition 2.2] *Let $\underline{s} \in \mathcal{S}_0$ and $t \in X_{\underline{s}}$. Then there exists a unique parameter $\kappa = H_{\underline{s}}(t)$ with $\kappa = g_{\underline{s}}^\kappa(t)$. Furthermore, $H_{\underline{s}}(t)$ is continuous in t. Conversely, for any escaping parameter κ, there exist unique $\underline{s} \in \mathcal{S}_0$ and $t \in X_{\underline{s}}$ such that $\kappa = H_{\underline{s}}(t)$.*

Furthermore, Qiu [18, Theorem 1] showed that the Hausdorff dimension of $\mathfrak{I}$ is 2. Later, in [1, Theorem 1] the authors proved the "Karpińska paradox" for the escaping parameter $\mathfrak{I}$, i.e., the Hausdorff dimensions of the union of all parameter rays $\mathfrak{I} \setminus \mathfrak{C}_I$ and the set of endpoints $\mathfrak{C}_I$ are 1 and 2, respectively, where $\mathfrak{C}_I$ denotes the endpoints of $\mathfrak{I}$. For the parameter space of exponential map. We have to mention a very important result on the bifurcation graph, which is given by Rempe and Schleicher [19].

Since the parameter and dynamics rays have the same structure (compare Theorem 1.1 with Theorem 1.4), it is natural to ask whether an analogous result to Theorem 1.3 could be obtained for $\mathfrak{I}$? Our main result addresses this question as follows.

Theorem 1.5. *Let $s > 1$. Then* $\mathrm{HM}^h(\mathfrak{I} \setminus \mathfrak{C}_I) = 0$ *with respect to the gauge function* $t/(\log \frac{1}{t})^s$.

2. Notation and preliminaries

In this part, we will give some basic definitions, some of them have been mentioned in Section 1.

Definition 2.1 (External address or itinerary)**.** Let $\mathcal{S} = \{(s_1 s_2 s_3 \cdots) : s_k \in \mathbb{Z} \text{ for all } k\}$ be the space of sequences over the integers, and let σ be the shift map on $\mathcal{S}$. For any $z \in \mathbb{C}$ with $E_\lambda^n(z) \notin \mathbb{R}^-$ for all $n \in \mathbb{N}$, the external address or itinerary $\underline{s} = (s_1 s_2 s_3 \cdots) \in \mathcal{S}$ of z is the sequence of numbers of the strips $\{z : (2s_i - 1)\pi \le \mathrm{Im}(z) < (2s_i + 1)\pi\}$ containing the point $E_\lambda^{(i-1)}(z)$.

We will fix $F(t) = \exp(t) - 1$, $(t > 0)$ as a comparison function. For any sequence $\underline{s} = (s_1 s_2 s_3 \cdots)$, if there are A and $x \geq 0$ such that $|s_k| \leq AF^{(k-1)}(x)$ for all $k \geq 1$, then $\underline{s}$ is called exponentially bounded. An external address $\underline{s}$ is fast if for every A and $x > 0$, all sufficiently large n have a k such that $s_{n+k} > AF^{(k-1)}(x)$. Otherwise, the external address $\underline{s}$ is slow. We denote the exponentially bounded sequences $\underline{s}$ by $\mathcal{S}_0$.

Definition 2.2 (Minimal potential of external address)**.** For any sequence $\underline{s} = (s_1 s_2 s_3 \cdots)$, define its minimal potential $t_{\underline{s}} \in [0, \infty]$ to be

$$t_{\underline{s}} = \inf \left\{ t > 0 : \limsup_{k \geq 1} \frac{|s_k|}{F^{(k-1)}(t)} = 0 \right\}.$$

Moreover, $t_{\sigma(\underline{s})} = F(t_{\underline{s}})$.

Recall that $\underline{s} \in A$ is a exponentially bounded if and only if $t_{\underline{s}} < \infty$. Any orbit $\{z_n\}$ which avoids $\mathbb{R}^-$ has an exponentially bounded external address.

We also need to recall parameter rays $H_{\underline{s}}$ as introduced in [9, 10]. For every $\underline{s} \in \mathcal{S}$, there is a well-defined minimal potential $t_{\underline{s}}$ and an injective curve $H_{\underline{s}} \colon (t_{\underline{s}}, \infty) \to \mathbb{C}$ with the following properties: for any escaping parameter $\kappa \in I$, there exist a unique $\underline{s} \in S_0$ and a unique potential $t > t_{\underline{s}}$ such that $\kappa = H_{\underline{s}}(t)$.

In order to prove Theorem 1.5, we need the following lemmas.

Koebe growth and distortion theorems imply that all univalent functions $f \colon \mathbb{D} \to \mathbb{C}$ such that $f(0) = 0$, $f'(0) = 1$ is normal. The following lemma is a simple application of the Koebe growth and distortion theorems.

Lemma 2.3. *Let $z_0 \in \mathbb{C}$, $r > 0$ and f be a univalent function in $D(z_0, r)$. If $z \in D(z_0, r)$, then*

$$r^2 |f'(z_0)| \frac{r - |z - z_0|}{(r + |z - z_0|)^3} \leq |f'(z)| \leq r^2 |f'(z_0)| \frac{r + |z - z_0|}{(r - |z - z_0|)^3}$$

and

$$r^2 |f'(z_0)| \frac{|z - z_0|}{(r + |z - z_0|)^2} \leq |f(z) - f(z_0)| \leq r^2 |f'(z_0)| \frac{|z - z_0|}{(r - |z - z_0|)^2}.$$

For our use, we need the following version.

Lemma 2.4. *Let Ω be a domain and K be a compact subset of Ω. Then there exists a positive constant C (depending on K) such that for every univalent function f on Ω and every pair of points $\mu, \nu \in K$, we have*

$$\frac{|f'(\mu)|}{|f'(\nu)|} \leq C.$$

Proof. Since $\mu, \nu \in K$, there must exist finitely many points $\mu_1, \mu_2, \dots, \mu_n \in K$ and positive real numbers $r_1, r_2, \dots, r_n$ such that

$$\mu_1 \in D(\nu, r_1) \subset \Omega, \quad \mu_2 \in D(\mu_1, r_2) \subset \Omega, \quad \dots,$$
$$\mu_k \in D(\mu_{k-1}, r_k) \subset \Omega, \quad \dots, \quad \mu \in D(\mu_n, r_{n+1}) \subset \Omega,$$

where $k \in \{1, 2, \dots, n\}$.

Using Lemma 2.3 to the above disks, we have

$$\frac{|f'(\mu_1)|}{|f'(\nu)|} \le r_1^2 \frac{r_1 + |\mu_1 - \nu|}{(r_1 - |\mu_1 - \nu|)^3} = C(r_1),$$
$$\vdots$$
$$\frac{|f'(\mu_k)|}{|f'(\mu_{k-1})|} \le r_k^2 \frac{r_k + |\mu_k - \mu_{k-1}|}{(r_k - |\mu_k - \mu_{k-1}|)^3} = C(r_k),$$
$$\vdots$$
$$\frac{|f'(\mu)|}{|f'(\mu_n)|} \le r_{n+1}^2 \frac{r_{n+1} + |\mu - \mu_n|}{(r_{n+1} - |\mu - \mu_n|)^3} = C(r_n).$$

It follows from these inequalities that $|f'(\mu)|/|f'(\nu)| \le C(r_1) \cdots C(r_k) \cdots C(r_n) = C$, thus the claim follows. □

How escaping points are organized in the complex plane is an interesting question, Schleicher and Zimmer [21] obtained a complete answer on this question.

Lemma 2.5. [21, Proposition 4.5] *For every exponentially bounded external address $\underline{s}$ and every $t > t_{\underline{s}}$, the point $g_{\underline{s}}(t)$ satisfies the asymptotic bound $E_\lambda^k(g_{\underline{s}}(t)) = F^k(t) - \kappa + 2\pi i s_{k+1} + o(1)$ as $k \to \infty$. In particular, it satisfies*

$$\frac{|\operatorname{Im}(E_\lambda^k(g_{\underline{s}}(t)))|^p}{\operatorname{Re}(E_\lambda^k(g_{\underline{s}}(t)))} \to 0$$

as $k \to \infty$, for every $p > 0$.

Here we define a standard square Q to be open square of sidelength $\pi/2$ with sides parallel to the axes. The double square of Q is a square $\widehat{Q}$ of sidelength π with parallel sides and common center.

Let $\Lambda \subset \mathbb{C}$ and let $N \in \mathbb{N}$. Suppose that $E_\kappa^N \colon \Lambda \to Q$; $\kappa \mapsto E_\kappa^N(\kappa)$, thus

$$E_\kappa^{N+1}(\kappa) = E_\kappa(E_\kappa^N(\kappa)) = E_{\kappa(z)}(z),$$

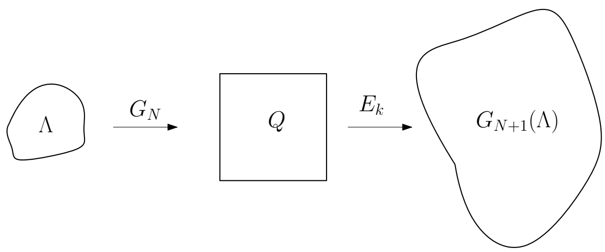
where $E_\kappa^N(\kappa) = z \in Q$.

Then, we can define the n-th iteration of this map on set Q is $E_{\kappa(z)}^n(z) = E_\kappa^{N+n}(\kappa)$, where $z \in Q$ and κ satisfies $E_\kappa^N(\kappa) = z$.

Remark 2.6. In order to distinguish function $E^n_\kappa(\kappa)$ from function $E^n_{\kappa(z)}(z)$ (which also depends on κ), we will use notation $G_n(\kappa) = E^n_\kappa(\kappa)$ when $\kappa \in \Lambda$. In the following, we simplify the index and write $E_{\kappa(z)}(z)$ instead of $E_\kappa(z)$.

Lemma 2.7. [1, Lemma 2] *Suppose that* Λ, Q *are sets such that* $G_N\colon \Lambda \to Q$ *is a conformal isomorphism,* $\mathrm{Re}(Q) > \tau > 1$ *and* $|G'_N(\kappa)| > 2$ *for all* $\kappa \in \Lambda$. *Then* $G_{N+1}\colon \Lambda \to G_{N+1}(\Lambda)$ *is a conformal isomorphism with*

$$|G'_{N+1}(\kappa)| > 2|G'_N(\kappa)| \quad \text{for all } \kappa \in \Lambda.$$



Moreover, maps $E_\kappa\colon Q \to G_{N+1}(\Lambda)$ *are conformal isomorphisms with*

$$|E'_\kappa(z)| > e^\tau - 1 \quad \text{for all } z \in Q.$$

For $p > 1$ and $\tau \geq 0$, we define

$$\chi_{p,\tau} = \{\kappa : \mathrm{Re}(\kappa) > \tau \text{ and } \mathrm{Im}(\kappa) < (\mathrm{Re}(\kappa))^{1/p}\}$$

and

$$\mathfrak{I}_{p,0} = \{\kappa : G_n(\kappa) \in \chi_{p,0} \text{ for all sufficiently large } n\}.$$

Exponential mapping defined on a domain D whose diameter satisfied $\sup_{z,w\in D} |z - w| < 2\pi$ will be conformal. It can also be a conformal mapping for function G_n under some conditions on parameter space, the detail is as follows.

Lemma 2.8. [1, Lemma 3] *Let* $p > 1$, $\tau > 0$ *be constants and* $\Lambda \subset \mathbb{C}$ *be an open set. If* $G_n(\kappa) \in \chi_{p,0}$ *for all but finitely many* n *and* $|G'_n(\kappa)| \to \infty$ *for some* $\kappa \in I \cap \Lambda$, *then there exist an* $N \in \mathbb{N}$, *a neighborhood* $U \subset \Lambda$ *of* κ *and a standard square* $Q \subset \chi_{p,\tau}$ *with center* $G_N(\kappa)$ *such that* $G_N\colon U \to \widehat{Q}$ *is a conformal isomorphism, where* $\widehat{Q}$ *is double square of* Q. *Moreover,* $G_n(\kappa) \in \chi_{p,\tau}$ *for all* $n \geq N$.

Lemma 2.9. [1, Lemma 6] *Parameter rays satisfy* $\frac{d}{d\kappa}G_n(\kappa) \to \infty$ *and* $\frac{d}{dt}G_n(H_{\underline{s}}(t)) \to \infty$ *as* $n \to \infty$.

The following lemma is important to prove the Hausforff measure is zero.

Lemma 2.10. [5, Lemma 2.2] *Let $A \subset \mathbb{R}^m$ and let h be a gauge function. Suppose that for all $x \in A$ and $\delta, \varepsilon > 0$, there exists $\rho(x) \in (0,1)$, $d(x) \in (0,\delta)$ and $N(x) \in \mathbb{N}$ satisfying $N(x)h(d(x)) \leq \varepsilon\rho(x)^m$ such that $D(x,\rho(x)) \cap A$ can be covered by $N(x)$ sets of diameter at most $d(x)$. Then $\mathrm{HM}^h(A) = 0$.*

Under bilipschitz mappings, zero and infinity Hausdorff measure are preserved.

Lemma 2.11. [17, Lemma 2.8] *Let $A \subset \mathbb{C}$ and f be a bilipschitz mapping. Suppose that h is a gauge function. If $\mathrm{HM}^h(A) = 0$ (resp. ∞), then $\mathrm{HM}^h(f(A)) = 0$ (resp. ∞).*

3. Proof of Theorem 1.5

The main ideas of this part come from [1,5]. Since we will estimate the Hausdorff measure in parameter space, it needs more carefully to control the difference of values $E_\kappa(z)$ between different parameter κ.

For every escaping parameter $\kappa \in \mathfrak{I} \setminus \mathfrak{C}_I$, the definition implies that κ is an escaping point for E_κ. It follows from Lemma 2.5 that $\mathrm{Re}(\kappa) > 0$ and $|\mathrm{Im}(G_n(\kappa))| < \mathrm{Re}(G_n(\kappa))^{1/p}$ for all sufficiently large n. That is, $\mathfrak{I} \setminus \mathfrak{C}_I \subset \mathfrak{I}_{p,0}$. In the following, we will show that the theorem holds for the set $\mathfrak{I}_{p,0}$.

Suppose that $\tau > 1$ and $Q \subset \chi_{p,\tau}$ is a standard square. Take $\kappa \in \mathfrak{I}_{p,0}$, from Lemmas 2.7 and 2.8, we know that there exist $N \in \mathbb{N}$ and a neighborhood U of κ such that $G_N \colon U \to \widehat{Q}$ is a conformal mapping. Let Λ be the component of $(G_N)^{-1}(Q)$ contains κ, then $G_N \colon \Lambda \to Q$ is a conformal isomorphism. Furthermore, Λ is bounded and $\mathrm{Re}(z) > \tau > 1$ for all $z \in Q$.

Because of $G'_n(\kappa) \to \infty$, we can choose above N large enough such that $|G'_N(\kappa)| > 2$ for all $\kappa \in \Lambda$. It follows from Lemma 2.7 that $E_\kappa \colon Q \to E_\kappa(Q)$ is also a conformal isomorphism. Let

$$r_0 = \inf_{z\in Q}\{\mathrm{Re}(z)\}, \quad R_0 = \sup_{z\in Q}\{\mathrm{Re}(z)\} = r_0 + \frac{\pi}{2}.$$

Since $\exp(Q)$ is a quarter annulus with inner and outer radius are e^{r_0} and $e^{\pi/2}e^{r_0}$, respectively. Then $E_\kappa(Q)$ is a domain which is similar to a quarter annulus.

We denote $r_1 = \inf_{z\in\Omega}\{\mathrm{Re}(z)\}$, $R_1 = \sup_{z\in\Omega}\{\mathrm{Re}(z)\}$, where $\Omega = E_\kappa(Q) \cap \chi_{p,\tau}$. So

$$\frac{e^{r_0}}{2} \leq r_1 < R_1 \leq 2e^{\pi/2}e^{r_0}.$$

Moreover, we can choose several standard squares Q_{1i} with double squares $\widehat{Q}_{1i}$, such that the union of Q_{1i} covers the set Ω. Denote $(E_\kappa)^{-1}(Q_{1i}) \cap Q$ by U_{1i}. Fix a point $z \in Q$, without loss of generality, we assume Q_{11} is a square contained in $E_\kappa(z)$. To simplify notations, we will use Q_1 and U_1 instead of Q_{11} and U_{11}, respectively. By the same reason we deduce that $E_\kappa(Q_1)$ is a domain similar to a quarter annulus with real part between $r_2 = e^{r_1'}/2$ and $R_2 = 2e^{r_1'+\pi/2}$, where $r_1 \le r_1' \le R_1$. Then we get that $Q_2 \subset E_\kappa^2(U_1)$ which contains the point $E_\kappa^2(z)$. Pull back Q_2 by $(E_\kappa^2)^{-1}$, and denote the component in Q by U_2.

Repeating the above process, we can get a sequence of standard squares $\{Q_n\}$, double squares $\{\widehat{Q}_n\}$, $\{U_n\}$ and $\{\widehat{U}_n\}$.

Set $U = \bigcap_{n=1}^{\infty} \left(\bigcup_{i=1}^{N(i)} U_{ni}\right)$. By the definition of U_{ni}, we have

$$U = \{z \in Q : \; E_\kappa^n(z) \in \chi_{p,\tau} \text{ for all } n\,(\in \mathbb{N}) \ge N\}.$$

Since E_κ is univalent in $\widehat{Q}$, it implies from Lemma 2.4 that

$$\frac{|E'_{\kappa(\mu)}(\mu)|}{|E'_{\kappa(\nu)}(\nu)|} \le C \tag{3.1}$$

for some constant C and every pair $\mu, \nu \in Q$.

Recalling the definitions of R_j and r_j, for all $j \in \mathbb{N}$, we have

$$1 < \alpha \le \frac{R_j}{r_j} \le 4e^{\pi/2}. \tag{3.2}$$

In view of (3.1),

$$\frac{\pi}{2C}|E'_\kappa(z)| \le R_j - r_j \le \frac{C\pi}{2}|E'_\kappa(z)|.$$

Using (3.2) to the inequality above, we deduce that

$$M^{-1}R_j \le |(E_\kappa)'(z)| \le MR_j,$$

and

$$M^{-n}\prod_{j=1}^{n} R_j \le |(E_\kappa^n)'(z)| \le M^n \prod_{j=1}^{n} R_j \tag{3.3}$$

for sufficiently large constant M and every $z \in U$.

In the following, we will use Lemma 2.10 for the set U. Noting that $E_\kappa^n(U_{n-1}) \cap \chi_{p,\tau} = E_\kappa(Q_{n-1}) \cap \chi_{p,\tau}$ can be covered by $N_n(z)$ squares with sidelength which is $2(R_n)^{1/p}$, and center lie on the real axis. Then

$$N_n(z) \le \frac{R_n}{R_n^{1/p}} = R_n^{1-1/p}. \tag{3.4}$$

We will get d_n and ρ_n by pulling back those squares. E_κ^n is a conformal map on Q, from Lemma 2.3, it follows that there exists a constant C such that the diameter of the component of $(E_\kappa^n)^{-1}$ which contained in U is less than $CR_n^{1/p}/|(E_\kappa^n)'(z)|$, i.e.,

$$d_n(z) = \frac{CR_n^{1/p}}{|(E_\kappa^n)'(z)|}. \tag{3.5}$$

By the same argument, we get

$$\rho_n(z) = \frac{c}{|(E_\kappa^{n-1})'(z)|}, \tag{3.6}$$

where c is a constant.

We now develop the following lemma needed for the proof.

Lemma 3.1. *Suppose that $z \in U$, $n \in \mathbb{N}$ and $s > 1$. Then for sufficiently large τ, there exist constants $\eta \leq (s-1)/4$, η_1 such that*

$$R_{n+1} \geq r_{n+1} \geq e^{\eta_1 r_n} = e^{\eta R_n}. \tag{3.7}$$

A similar result for the exponential map is proved by Karpińska and Urbański [13, Lemma 2.3]. We will use their method to the exponential family E_κ here.

Proof of Lemma 3.1. For any $z \in U$. By (3.2), we get that

$$\begin{aligned} r_{n+1} &\geq (4e^{\pi/2})^{-1} R_{n+1} \geq (4e^{\pi/2})^{-1} |E_\kappa(E_\kappa^n(z))| \\ &= \frac{\exp\{\operatorname{Re}(\kappa) - \pi/2\}}{4} \exp\{\operatorname{Re}(E_\kappa^n(z))\} \\ &= \frac{\exp\{\operatorname{Re}(\kappa) - \pi/2\}}{4} e^{\beta}, \end{aligned}$$

where $\beta = \sqrt{|E_\kappa^n(z)|^2 - |\operatorname{Im}(E_\kappa^n(z))|^2}$.

Since $E_\kappa^n(z) \in \chi_{p,\tau}$ and since $x^{1/p} \leq x/(8e^{\pi/2})$ for large $x \in \mathbb{R}^+$ we have

$$\begin{aligned} |E_\kappa^n(z)|^2 - |\operatorname{Im}(E_\kappa^n(z))|^2 &\geq r_n^2 - |\operatorname{Re}(E_\kappa^n(z))|^{2/p} \geq r_n^2 - \left(\frac{|\operatorname{Re}(E_\kappa^n(z))|}{8e^{\pi/2}}\right)^2 \\ &\geq r_n^2 - \left(\frac{R_n}{8e^{\pi/2}}\right)^2 \geq r_n^2 - \left(\frac{r_n}{2}\right)^2 = \frac{\sqrt{3}}{2} r_n. \end{aligned}$$

Then we can get the inequality (3.7) for $\eta_1 = \sqrt{3}/2$ and $\eta = \min\{\eta_1/4e^{\pi/2}, (s-1)/4\}$, thus the proof is completed. □

Recall that both d_n and ρ_n tend to 0 as $n \to \infty$, and for any $\varepsilon > 0$ there exists a positive integer n satisfying $\varepsilon \in (1/(\log R_n)^{2\eta}, 1/(\log R_{n-1})^{2\eta}]$. Combining (3.3), (3.4), (3.5) and (3.6), we deduce that

$$\begin{aligned} N_n(z) h(d_n(z)) &\leq R_n^{1-1/p} \frac{d_n(z)}{(\log(1/d_n(z)))^s} \\ &\leq R_n^{1-1/p} \frac{R_n^{1/p}}{R_n \log R_n} \frac{1}{(\log R_n + \log\log R_n - \log R_n^{1/p})^s} \\ &\leq R_n^{1-1/p} \frac{R_n^{1/p-1}}{\log R_n((1-1/p)\log R_n)^s} \leq \frac{c_0}{(\log R_n)^{1+s}} \\ &\leq \frac{c_0}{(\log R_n)^{2+4\eta}} \leq \frac{c_0 \varepsilon}{(\log R_n)^{2+2\eta}}, \end{aligned}$$

where $c_0 = 1/(1-1/p)^s$.

Furthermore, for any constant γ and $\delta > 0$, it follows from (3.7) that

$$\gamma \prod_{j=1}^{n-1} R_j \leq (\log R_n)^{1+\delta}. \tag{3.8}$$

We can choose $\gamma = c_0 M^n/c$ and $\delta = \eta$. Then (3.8) will be $N_n(z)h(d_n(z)) \leq \varepsilon\rho_n^2(z)$, which implies $\mathrm{HM}^h(U) = 0$ from Lemma 2.10. Recall that $G_N \colon \mathfrak{I}_{p,0} \cap \Lambda \to U$ is a conformal isomorphism, so it is a Lipschitz map on the compact set $\mathfrak{I}_{p,0} \cap \Lambda$. Furthermore, the inverse function is also a Lipschitz map. From Lemma 2.11, it follows that $\mathrm{HM}^h(\mathfrak{I}_{p,0} \cap \Lambda) = 0$.

We can repeat the above process for all standard squares Q and positive integers N, that is: there is a countable set $\{\Lambda_i\}$ which covers $\mathfrak{I}_{p,0}$ and $\mathrm{HM}^h(\mathfrak{I}_{p,0} \cap \Lambda_i) = 0$ for all i.

Therefore, we get $\mathrm{HM}^h(\mathfrak{I}_{p,0}) = 0$, and the proof is completed.

Moreover, the authors in [5] also proved that the Hausdorff measure with respect to some gauge function of the dynamics rays is ∞.

Theorem 3.2. *Let $s > 1$. Then $\mathrm{HM}^h(J(E_\lambda) \setminus \mathfrak{C}) = \infty$ for $h(t) = t/L^s(1/t)$.*

The same reason as above argument, we derive the following problem.

Problem 3.3. Whether we can find a gauge function such that the Hausdorff measure of parameter rays respect to which is ∞.

But the method in this paper can not work for above problem. Because the parabolic domain $\chi_{p,\tau}$ used to separate the endpoint from the escaping parameter is not enough. We need a accurately separation for studying the Hausdorff measure is ∞.

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