TAIWANESE JOURNAL OF MATHEMATICS
Vol. 22, No. 6, pp. 1479–1508, December 2018
DOI: 10.11650/tjm/180302

Global Existence, Finite Time Blow-up and Vacuum Isolating Phenomena for Semilinear Parabolic Equation with Conical Degeneration

Guangyu Xu

Abstract. This paper is devoted to studying a semilinear parabolic equation with conical degeneration. First, we extend previous results on the vacuum isolating of solution with initial energy $J(u_0) < d$, where d is the mountain pass level. Concretely, we obtain the explicit vacuum region, the global existence region and the blow-up region. Moreover, as far as the blow-up solution is concerned, we estimate the upper bound of the blow-up time and blow-up rate. Second, for all $p > 1$, we get a new sufficient condition, which demonstrates the finite time blow-up for arbitrary initial energy, and the upper bound estimate of blow-up time is obtained.

1. Introduction

1.1. The model and literature overview

Let X be an $(n-1)$-dimensional closed compact C^∞-smooth manifold, which is regarded as the local model near the conical points. $\mathbb{B} = [0,1) \times X$, $\partial\mathbb{B} = \{0\} \times X$, near $\partial\mathbb{B}$ we use the coordinates $(x_1, x') = (x_1, x_2, \ldots, x_n)$ for $0 \leq x_1 < 1$, $x' \in X$. We denote by $\mathbb{B}_0$ the interior of $\mathbb{B}$.

In this paper, we consider the following degenerate parabolic equation

$$\begin{cases} u_t - \Delta_{\mathbb{B}} u = |u|^{p-1}u, & t > 0,\ x \in \mathbb{B}_0, \\ u(x,t) = 0, & t \geq 0,\ x \in \partial\mathbb{B}, \\ u(x,0) = u_0(x), & x \in \mathbb{B}_0, \end{cases} \tag{1.1}$$

where u_0 is a nonnegative, nontrivial function and belongs to the weighted Sobolev space $\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})$. We will introduce the weighted Sobolev space later. p satisfies some appropriate assumptions, the Fuchsian type Laplace operator is defined as

$$\Delta_{\mathbb{B}} = (x_1\partial_{x_1})^2 + \partial_{x_2}^2 + \cdots + \partial_{x_n}^2,$$

Received October 30, 2017; Accepted March 11, 2018.
Communicated by Eiji Yanagida.
2010 *Mathematics Subject Classification.* 35K61, 35B44, 35K10, 35K55, 35D30.
Key words and phrases. cone Sobolev space, vacuum isolating phenomena, global existence, finite time blow-up, blow-up time, blow-up rate.

which is an elliptic operator with conical degeneration on the boundary $x_1 = 0$. The corresponding gradient operator is $\nabla_{\mathbb{B}} = (x_1\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n})$.

In recent years, degenerate type parabolic equation has been attracting considerable attention in the research field of analysis of nonlinear PDEs. The papers [7, 8, 10, 26, 27] introduced a class of weighted Sobolev spaces and proved the corresponding cone Sobolev inequality and Poincaré inequality. Later on, by applying those precursory results, many academicians investigated semilinear parabolic and hyperbolic equations with conical degeneration and their analogue, for example [1–3, 6, 9, 12, 13, 29].

As for problem (1.1), by the classical Galerkin method and potential well method, the authors in [4] considered the solution with low initial energy case ($J(u_0) < d$) and critical initial energy case ($J(u_0) = d$). In order to introduce their and our main results, we give some necessary notations and definitions.

Throughout the paper, the norm in space $\mathcal{L}_p^{n/p}(\mathbb{B}) = \mathcal{H}_{p,0}^{0,n/p}(\mathbb{B})$ is defined by

$$\|u\|_{\mathcal{L}_p^{n/p}(\mathbb{B})} = \left(\int_{\mathbb{B}} |u|^p \frac{dx_1}{x_1}\, dx' \right)^{1/p}, \quad \forall\, p \in (1, +\infty).$$

As in [4], we define

$$\begin{aligned} J(u) &= \frac{1}{2} \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{dx_1}{x_1}\, dx' - \frac{1}{p+1} \int_{\mathbb{B}} |u|^{p+1} \frac{dx_1}{x_1}\, dx', \\ K(u) &= \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{dx_1}{x_1}\, dx' - \int_{\mathbb{B}} |u|^{p+1} \frac{dx_1}{x_1}\, dx', \\ \mathcal{N} &= \left\{ u \in \mathcal{H}_{2,0}^{1,n/2}(\mathbb{B}) : K(u) = 0, \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{dx_1}{x_1}\, dx' \neq 0 \right\}. \end{aligned}$$

If $1 < p < (n+2)/(n-2)$, by [4, Propositions 2.2], we know that the embedding $\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B}) \hookrightarrow \mathcal{L}_{p+1}^{n/(p+1)}(\mathbb{B})$ is continuous, then for any $u \in \mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})$, $u \neq 0$, there exists a optimal positive constant C_* such that

$$\|u\|_{\mathcal{L}_{p+1}^{n/(p+1)}(\mathbb{B})} \le C_* \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{n/2}(\mathbb{B})}. \tag{1.2}$$

Let

$$\alpha_1 = C_*^{-(p+1)/(p-1)}. \tag{1.3}$$

Further, we define the mountain-pass level by

$$d = \inf \left\{ \sup_{\lambda \ge 0} J(\lambda u), u \in \mathcal{H}_{2,0}^{1,n/2}(\mathbb{B}), \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{dx_1}{x_1}\, dx' \neq 0 \right\} = \inf_{u \in \mathcal{N}} J(u),$$

it follows from [4, Lemma 3.3(iii)] that

$$d = \frac{p-1}{2(p+1)} C_*^{-2(p+1)/(p-1)} = \frac{p-1}{2(p+1)} \alpha_1^2. \tag{1.4}$$

Let λ_1 be the first nonzero eigenvalue of the following Dirichlet problem

$$\begin{cases} -\Delta_{\mathbb{B}}\phi(x) = \lambda\phi(x), & x \in \mathbb{B}_0, \\ \phi(x) = 0, & x \in \partial\mathbb{B}, \end{cases}$$

then by [4, Proposition 2.3] we know that $\lambda_1 > 0$ and satisfies the following inequality:

$$\lambda_1 \|u\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})} \le \|\nabla_{\mathbb{B}} u\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}. \tag{1.5}$$

For $\delta > 0$, we define

$$\begin{aligned} K_\delta(u) &= \delta \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{dx_1}{x_1}\, dx' - \int_{\mathbb{B}} |u|^{p+1} \frac{dx_1}{x_1}\, dx', \\ \mathcal{N}_\delta &= \left\{ u \in \mathcal{H}^{1,n/2}_{2,0}(\mathbb{B}) : K_\delta(u) = 0, \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{dx_1}{x_1}\, dx' \neq 0 \right\}, \\ d(\delta) &= \inf_{u \in \mathcal{N}_\delta} J(u). \end{aligned} \tag{1.6}$$

For $\delta > 0$, $u \in \mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})$, we let

$$\begin{aligned} \mathcal{B}_\delta &= \left\{ \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{n/2}(\mathbb{B})} < C_*^{-(p+1)/(p-1)} \delta^{1/(p-1)} \right\}, \\ \overline{\mathcal{B}}_\delta &= \mathcal{B}_\delta \cup \partial\mathcal{B}_\delta = \left\{ \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{n/2}(\mathbb{B})} \le C_*^{-(p+1)/(p-1)} \delta^{1/(p-1)} \right\}, \\ \mathcal{B}^c_\delta &= \left\{ \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{n/2}(\mathbb{B})} > C_*^{-(p+1)/(p-1)} \delta^{1/(p-1)} \right\}, \\ \overline{\mathcal{B}}^c_\delta &= \mathcal{B}^c_\delta \cup \partial\mathcal{B}^c_\delta = \left\{ \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{n/2}(\mathbb{B})} \ge C_*^{-(p+1)/(p-1)} \delta^{1/(p-1)} \right\}. \end{aligned} \tag{1.7}$$

Next, we give the definition of weak solution of problem (1.1).

Definition 1.1. [4, Definition 1.1] A function $u = u(x,t)$ is called a weak solution of problem (1.1) on $[0,T) \times \mathbb{B}_0$ in which T is either infinity or the limit of the existence interval of solution, if it satisfies

(i) $u \in L^\infty(0,T; \mathcal{H}^{1,n/2}_{2,0}(\mathbb{B}))$ and $u_t \in L^2(0,T; \mathcal{L}_2^{n/2}(\mathbb{B}))$;

(ii) For any $t \in [0,T)$,

$$\int_0^t \|u_\tau\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\, d\tau + J(u(t)) \le J(u_0); \tag{1.8}$$

(iii) For any $v \in \mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})$ and $t \in (0,T)$, u satisfies:

$$\int_{\mathbb{B}} u_\tau \cdot v \frac{dx_1}{x_1}\, dx' + \int_{\mathbb{B}} \nabla_{\mathbb{B}} u \cdot \nabla_{\mathbb{B}} v \frac{dx_1}{x_1}\, dx' = \int_{\mathbb{B}} |u|^{p-1} u \cdot v \frac{dx_1}{x_1}\, dx'. \tag{1.9}$$

Definition 1.2. Let $u(x,t)$ be a weak solution of problem (1.1). We define the maximal existence time T of $u(x,t)$ as follows:

(i) If $u(x,t)$ exists for all $t \in [0,+\infty)$, then $T = +\infty$;

(ii) If there exists a $t_0 \in (0,+\infty)$ such that $u(x,t)$ exists for $t \in [0,t_0)$, but doesn't exist at $t = t_0$, then $T = t_0$.

For the readers' convenience, we summarize the main results obtained in [4] as follows, which are relevant to the work in this paper.

Theorem 1.3. [4] *Let $u_0 \in \mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})$, $1 < p < (n+2)/(n-2)$.*

(i) *If $J(u_0) < d$, $K(u_0) > 0$, then the solution $u = u(x,t)$ of problem* (1.1) *exists globally. Moreover, there exist constants $C_0 > 0$, $\lambda_0 > 0$ such that*

$$\|\nabla_{\mathbb{B}} u\|_{\mathcal{L}^{n/2}_2(\mathbb{B})} \le C_0 e^{-\lambda_0 t}, \quad \forall\, t \in [0,+\infty).$$

(ii) *If $J(u_0) = d$, $K(u_0) \ge 0$, then the solution $u = u(x,t)$ of problem* (1.1) *exists globally. Moreover, there exist constants $C_1 > 0$, $\widetilde{\lambda}_0 > 0$ and $t_1 > 0$ such that*

$$\|\nabla_{\mathbb{B}} u\|_{\mathcal{L}^{n/2}_2(\mathbb{B})} \le C_1 e^{-\widetilde{\lambda}_0 t}, \quad \forall\, t \in [t_1,+\infty).$$

(iii) *If $J(u_0) \le d$, $K(u_0) < 0$, then the solution of problem* (1.1) *blows up in finite time.*

By Theorem 1.3, we know that the relation between global existence and finite time blow-up is derived as a sharp condition. Namely, the sign of $K(u_0)$ determines the solution exists globally or blows up in finite time. When $K(u_0) > 0$, by Theorem 1.3(i) and (ii) we see the solution exists globally and decays exponentially with $\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})$-norm. When $K(u_0) < 0$, Theorem 1.3(iii) shows that the solution blows up in finite time. However, for the blow-up solution, they do not consider the blow-up time and blow-up rate. Moreover, it is important and curious to ask a further question like: How does the blow-up solution grow for $t \in [0,T)$, algebraically or exponentially?

From above, we also note that Theorem 1.3 holds under the assumptions $J(u_0) \le d$ and $p \in (1,(n+2)/(n-2))$. While, the interval $(1,(n+2)/(n-2))$ is very small for sufficiently large space dimension n. In other words, there are no results on blow-up in finite time for problem (1.1) under the condition $p \in (1,+\infty)$. On the other hand, for the high initial energy case ($J(u_0) > d$), it is still open that whether the global solution exists or the solution blows up in finite time. It is worthwhile to point out that the high initial energy case are very few seen.

More important, by the [4, Remark 3.2(b)], we know that there is a vacuum region U_e for the solution of problem (1.1) with initial energy $0 < J(u_0) \le e < d$, i.e., any solution is isolated by U_e, and it is defined by

$$U_e = \bigcup_{\delta_1<\delta<\delta_2} \mathcal{N}_\delta, \tag{1.10}$$

where δ_1, δ_2 are the two positive roots of $d(\delta) = e$, $\mathcal{N}_\delta$ is given by (1.6). But, it is easy to see that the set is so abstract and we can not get the explicit information about U_e. Furthermore, it is natural to ask what the asymptotic behavior of U_e when $e \to 0^+$? Is there any vacuum region of solution when $J(u_0) \le 0$?

1.2. Main results

The main purpose of this paper is to study the problems stated above. We first consider the vacuum isolating phenomena of solution with $0 < J(u_0) \le e < d$ and $J(u_0) \le 0$ respectively. Moreover, as far as the blow-up solution is concerned, we get the estimates of blow-up time and blow-up rate. In particular, we study the asymptotic behavior of the blow-up solution for $t \in [0, T)$, and we prove that the solution grows exponentially with $\mathcal{L}_{p+1}^{n/(p+1)}(\mathbb{B})$-norm. As for more works about the vacuum isolating behavior of solution, we mention the papers [5, 16–21, 23] for the study of other evolution equations.

The first result focus on the vacuum region U_e with $0 < J(u_0) \le e < d$. We will show that the vacuum region is an annulus and splits the space $\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})$ into inner ball $\overline{\mathcal{B}}_{\delta_1}$ and corresponding outer region $\overline{\mathcal{B}}_{\delta_2}^c$.

Theorem 1.4. *Let $e \in (0, d)$, $\delta_1 < \delta_2$ are the two positive roots of equation $d(\delta) = e$, $1 < p < (n+2)/(n-2)$, then for all solution $u(t) = u(x,t)$ of problem (1.1) with $0 < J(u_0) \le e$, there is a bounded region*

$$U_e = \left\{ u \in \mathcal{H}_{2,0}^{1,n/2}(\mathbb{B}) : C_*^{-(p+1)/(p-1)} \delta_1^{1/(p-1)} < \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{n/2}(\mathbb{B})} < C_*^{-(p+1)/(p-1)} \delta_2^{1/(p-1)} \right\}$$

such that $\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B}) = U_e \cup \overline{\mathcal{B}}_{\delta_1} \cup \overline{\mathcal{B}}_{\delta_2}^c$, where C_* is defined in (1.2). Moreover, we have the following conclusions:

(i) *U_e is a vacuum region, i.e., $u(t) \notin U_e$ for all $t \in [0, T)$. Moreover, $\overline{\mathcal{B}}_{\delta_1}$ and $\overline{\mathcal{B}}_{\delta_2}^c$ are two invariant sets;*

(ii) *If $u_0 \in \overline{\mathcal{B}}_{\delta_1}$, then $u(t)$ exists globally and decays exponentially with $\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B})$-norm;*

(iii) *If $u_0 \in \overline{\mathcal{B}}_{\delta_2}^c$, then $u(t)$ blows up at finite time T with $\mathcal{L}_2^{n/2}(\mathbb{B})$-norm and we can estimate T as follows*

$$T \le \frac{(p+1)|\mathbb{B}|^{(p-1)/2} \|u_0\|_{\mathcal{L}_2^{n/2}(\mathbb{B})}^{1-p}}{(p-1)^2 \left[1 - \left((p+1)\left(\frac{1}{2} - \frac{J(u_0)}{\alpha_1^2} \right) \right)^{-(p+1)/(p-1)} \right]},$$

the blow-up rate can be estimated by

$$\|u(\,\cdot\,,t)\|_{\mathcal{L}_2^{n/2}(\mathbb{B})} < \frac{(p-1)^{2/(1-p)}\sqrt{|\mathbb{B}|}}{(p+1)^{1/(1-p)}} \times \left[1-\left((p+1)\left(\frac{1}{2}-\frac{J(u_0)}{\alpha_1^2}\right)\right)^{-(p+1)/(p-1)}\right]^{1/(1-p)} (T-t)^{-1/(p-1)},$$

where $|\mathbb{B}|$ *is the measure of* $\mathbb{B}$, $\overline{\mathcal{B}}_\delta$ *and* $\overline{\mathcal{B}}_\delta^c$ *are defined in* (1.7) *for all* $\delta \in (0,(p+1)/2)$, α_1 *is defined in* (1.3). *Moreover,* $u(t)$ *grows exponentially with* $\mathcal{L}_{p+1}^{n/(p+1)}(\mathbb{B})$*-norm for* $t \in [0,T)$.

Remark 1.5. We give two remarks about Theorem 1.4.

(i) It follows from $J(u_0) < d$, the definition of d and α_1 in (1.4) and (1.3) respectively, that

$$(p+1)\left(\frac{1}{2}-\frac{2J(u_0)}{\alpha_1^2}\right) > (p+1)\left(\frac{1}{2}-\frac{2d}{\alpha_1^2}\right) = 1.$$

Then the right-hand sides of the two inequalities in Theorem 1.4(iii) make sense.

(ii) By Lemma 2.5, we also know that δ_1 decreases to 0 and δ_2 increases to $(p+1)/2$ as e decreasing to 0, which implies the vacuum region U_e expands as e decreasing. As the limit case, we guess the vacuum region for the nontrivial solutions with $J(u_0) = 0$ is

$$\begin{aligned} U_0 &= \lim_{e\to 0} U_e \\ &= \left\{u \in \mathcal{H}_{2,0}^{1,n/2}(\mathbb{B}) : 0 < \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{n/2}(\mathbb{B})} < C_*^{-(p+1)/(p-1)}\left(\frac{p+1}{2}\right)^{1/(p-1)}\right\} \\ &= \left\{u \in \mathcal{H}_{2,0}^{1,n/2}(\mathbb{B}) : \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{n/2}(\mathbb{B})} < C_*^{-(p+1)/(p-1)}\left(\frac{p+1}{2}\right)^{1/(p-1)}\right\} \end{aligned} \tag{1.11}$$

(by Lemma 2.4), and we will prove the above conjecture in the next theorem.

In order to state our next theorem, we consider a function $G(r)$ which is defined by

$$G(r) = \frac{1}{p+1}C_*^{p+1}r^{p+1} - \frac{1}{2}r^2 + J(u_0), \tag{1.12}$$

where C_* is defined in (1.2), $J(u_0) \le 0$. Obviously, by $p > 1$, we know that the equation $G(r) = 0$ admits a unique positive root $r^* = r^*(J(u_0))$, and $r^*(J(u_0))$ proposes the following properties:

(i) $G(r) \ge 0$ if and only if $r \ge r^*(J(u_0))$;

(ii) $r^*(0) = C_*^{-(p+1)/(p-1)}\left(\frac{p+1}{2}\right)^{1/(p-1)}$;

(iii) $r^*(J(u_0))$ is increasing as $J(u_0)$ decrease and $\lim_{J(u_0)\to-\infty} r^*(J(u_0)) = +\infty$.

The second result focus on the vacuum isolating phenomena of solution with $J(u_0) \le 0$. For the blow-up solution, we also get the estimates of blow-up time and blow-up rate, and it should be pointed out that our blow-up result holds for all $p > 1$ when $J(u_0) < 0$. Moreover, we will show the solution grows exponentially with $\mathcal{L}_{p+1}^{n/(p+1)}(\mathbb{B})$-norm.

Theorem 1.6. *Suppose $J(u_0) \le 0$, $r^* = r^*(J(u_0))$ is the unique positive root of $G(r)$, where $G(r)$ is defined by (1.12). Let $u(t) = u(x,t)$ be the solution of problem (1.1), then there exists a bounded region*

$$U_{r^*} = \left\{ u \in \mathcal{H}_{2,0}^{1,n/2}(\mathbb{B}) : \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{n/2}(\mathbb{B})} < r^* \right\} \tag{1.13}$$

such that

(i) *U_{r^*} is a vacuum region, i.e., $u(t) \notin U_{r^*}$ for all $t \in [0,T)$;*

(ii) *$\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B}) \setminus U_{r^*}$ is an invariant region. If*

$$u_0 \in \begin{cases} \mathcal{H}_{2,0}^{1,n/2}(\mathbb{B}) \setminus U_{r^*},\ 1 < p < \frac{n+2}{n-2} & \text{when } J(u_0) = 0, \\ \mathcal{H}_{2,0}^{1,n/2}(\mathbb{B}) \setminus U_{r^*},\ p > 1, & \text{when } J(u_0) < 0, \end{cases}$$

then $u(t)$ blows up at finite time T with $\mathcal{L}_2^{n/2}(\mathbb{B})$-norm and we can estimate T as follows

$$T \le \begin{cases} \dfrac{(p+1)|\mathbb{B}|^{(p-1)/2}\|u_0\|^{1-p}_{\mathcal{L}_2^{n/2}(\mathbb{B})}}{(p-1)^2\left[1-\left(\frac{p+1}{2}\right)^{-(p+1)/(p-1)}\right]} & \text{when } J(u_0) = 0, \\ \dfrac{1}{1-p^2}\dfrac{\|u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}}{J(u_0)} & \text{when } J(u_0) < 0. \end{cases}$$

The blow-up rate can be estimated by

$$\|u(\,\cdot\,,t)\|_{\mathcal{L}_2^{n/2}(\mathbb{B})} \le \begin{cases} \dfrac{(p-1)^{2/(1-p)}\sqrt{|\mathbb{B}|}}{(p+1)^{1/(1-p)}}\left[1-\left(\dfrac{p+1}{2}\right)^{-\frac{p+1}{p-1}}\right]^{1/(1-p)}(T-t)^{-1/(p-1)} & \text{when } J(u_0) = 0, \\ \left[\dfrac{(1-p^2)J(u_0)}{\|u_0\|^{p+1}_{\mathcal{L}_2^{n/2}(\mathbb{B})}}\right]^{1/(1-p)}(T-t)^{-1/(p-1)} & \text{when } J(u_0) < 0. \end{cases}$$

Moreover, $u(t)$ grow exponentially with $\mathcal{L}_{p+1}^{n/(p+1)}(\mathbb{B})$-norm for $t \in [0,T)$.

Remark 1.7. We make two remarks about the vacuum isolating phenomena for $J(u_0) \le e < d$.

(i) By the properties of $r^*(J(u_0))$, we know that $U_{r^*} = U_0$ when $J(u_0) = 0$, where U_0 is defined in (1.11). U_{r^*} expands as $J(u_0)$ decreasing, and

$$\lim_{J(u_0)\to -\infty} U_{r^*} = \mathcal{H}_{2,0}^{1,n/2}(\mathbb{B}).$$

(ii) We can define three sets $\widetilde{U}_e$, G_e and B_e as follows:

$$\widetilde{U}_e \triangleq \begin{cases} U_e & \text{if } 0 < J(u_0) \le e, \\ U_{r^*} & \text{if } J(u_0) \le 0; \end{cases} \qquad G_e \triangleq \begin{cases} \overline{\mathcal{B}}_{\delta_1} & \text{if } 0 < J(u_0) \le e, \\ \emptyset & \text{if } J(u_0) \le 0; \end{cases}$$

$$B_e \triangleq \begin{cases} \overline{\mathcal{B}}^c_{\delta_2} & \text{if } 0 < J(u_0) \le e, \\ \mathcal{H}^{1,n/2}_{2,0}(\mathbb{B}) \setminus U_{r^*} & \text{if } J(u_0) \le 0. \end{cases}$$

Then combining the results of Theorems 1.4 and 1.6, we can see the statements in the abstract hold.

Finally, for all $p > 1$, we get a blow-up condition with arbitrary initial energy. Furthermore, we get a upper bound estimate of the blow-up time.

Theorem 1.8. *Assume $u(t) = u(x,t)$ is a solution of problem* (1.1)*, $p > 1$ and λ_1 is given as in* (1.5)*. If the initial value satisfies*

$$J(u_0) < \frac{\lambda_1(p-1)}{2(p+1)} \|u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})},$$

then $u(t)$ blow up at some finite time T. Moreover, T can be estimated by

$$T \le \frac{8(p+1)\|u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}}{(p-1)^2 \left[\lambda_1(p-1)\|u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})} - 2(p+1)J(u_0)\right]}.$$

The rest of this paper is organized as follows. For reader's convenience, in Section 2, we first introduce some definitions and properties of cone Sobolev spaces. Secondly, we give some important preliminary lemmas, which will be used in the proofs of main results. In Section 3, we consider the vacuum isolating phenomena of solution for problem (1.1), and we give the proofs of Theorems 1.4 and 1.6. In Section 4, we study the solution of problem (1.1) with arbitrary initial energy and prove Theorem 1.8.

2. Preliminaries

2.1. Some definitions and properties of cone Sobolev spaces

The detail research of manifold with conical singularities and the corresponding cone Sobolev spaces can be found in [7,8]. In this subsection, we shall introduce some definitions and properties of cone Sobolev spaces briefly, which is enough to make our paper readable.

Let X be a closed, compact, C^∞ manifold, we set $X^\triangle = (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)$ as a local model interpreted as a cone with the base X. We denote $X^\nabla = \mathbb{R}_+ \times X$ as the corresponding open stretched cone with the base X. An n-dimensional manifold B with conical singularities is a topological space with a finite subset $B_0 = \{b_1, \dots, b_M\} \subset B$ of conical singularities. For simplicity, we assume that the manifold B has only one conical point on the boundary. Thus, near the conical point, we have a stretched manifold $\mathbb{B}$, associated with B.

Definition 2.1. Let $\mathbb{B} = [0,1) \times X$ be the stretched manifold of the manifold B with conical singularity, then for any cut-off function ω, supported by a collar neighborhood of $(0,1) \times \partial\mathbb{B}$, the cone Sobolev space $\mathcal{H}_p^{m,\gamma}(\mathbb{B})$, for $m \in \mathbb{N}$, $\gamma \in \mathbb{R}$ and $1 < p < +\infty$, is defined as follows

$$\mathcal{H}_p^{m,\gamma}(\mathbb{B}) = \left\{u \in W_{\text{loc}}^{m,p}(\mathbb{B}_0) \mid \omega u \in \mathcal{H}_p^{m,\gamma}(X^\triangledown)\right\}.$$

Moreover, the subspace $\mathcal{H}_{p,0}^{m,\gamma}(\mathbb{B})$ of $\mathcal{H}_p^{m,\gamma}(\mathbb{B})$ is defined by

$$\mathcal{H}_{p,0}^{m,\gamma}(\mathbb{B}) = \omega \mathcal{H}_{p,0}^{m,\gamma}(X^\triangledown) + (1-\omega) W_0^{m,p}(\mathbb{B}_0),$$

where $W_0^{m,p}(\mathbb{B}_0)$ denotes the closure of $C_0^\infty(\mathbb{B}_0)$ in Sobolev spaces $W^{m,p}(\widetilde{X})$, here $\widetilde{X}$ is a closed compact C^∞ manifold of dimension n that containing $\mathbb{B}$ as a sub-manifold with boundary.

Definition 2.2. We say $u(x) \in \mathcal{L}_p^\gamma(\mathbb{B})$ with $1 < p < +\infty$ and $\gamma \in \mathbb{R}$ if

$$\|u\|^p_{\mathcal{L}_p^\gamma(\mathbb{B})} = \int_{\mathbb{B}} x_1^n |x_1^{-\gamma} u(x)|^p \, \frac{dx_1}{x_1} \, dx' < +\infty.$$

Observe that if $u(x) \in \mathcal{L}_p^{n/p}(\mathbb{B})$, $v(x) \in \mathcal{L}_q^{n/q}(\mathbb{B})$ with $p, q \in (1, +\infty)$ and $1/p + 1/q = 1$, then we have the following Hölder's inequality

$$\int_{\mathbb{B}} |u(x)v(x)| \, \frac{dx_1}{x_1} \, dx' \le \|u\|_{\mathcal{L}_p^{n/p}(\mathbb{B})} \|v\|_{\mathcal{L}_q^{n/q}(\mathbb{B})}.$$

Lemma 2.3 (Poincaré inequality)**.** *Let $\mathbb{B} = [0,1) \times X$ be a bounded subspace in $\mathbb{R}_+^n$ with $X \in \mathbb{R}^{n-1}$, and $1 < p < +\infty$, $\theta \in \mathbb{R}$. If $u(x) \in \mathcal{H}_{p,0}^{1,\theta}(\mathbb{B})$, then*

$$\|u(x)\|_{\mathcal{L}_p^\theta(\mathbb{B})} \le \mu \|\nabla_{\mathbb{B}} u(x)\|_{\mathcal{L}_p^\theta(\mathbb{B})},$$

where μ is a positive constant depending only on $\mathbb{B}$.

2.2. Some auxiliary lemmas

Lemma 2.4. *Let $u(t) = u(x,t)$ be a nontrivial solution of problem* (1.1) *with $J(u_0) \le 0$, then we have $\|\nabla_{\mathbb{B}} u(t)\|_{\mathcal{L}_2^{n/2}(\mathbb{B})} > 0$ for all $t \in [0, T)$.*

Proof. Since $J(u_0) \leq 0$, it follows from (1.8) that $J(u(t)) \leq J(u_0) \leq 0$ for $t \in [0,T)$. Obviously, $\|\nabla_{\mathbb{B}} u_0\|_{\mathcal{L}_2^{n/2}(\mathbb{B})} > 0$ because of u_0 is nontrivial. Then we can prove the lemma by contradiction. If the conclusion is not true, then there exists a $t_0 \in (0,T)$ such that $\|\nabla_{\mathbb{B}} u(t)\|_{\mathcal{L}_2^{n/2}(\mathbb{B})} > 0$ for all $t \in [0,t_0)$ and $\|\nabla_{\mathbb{B}} u(t_0)\|_{\mathcal{L}_2^{n/2}(\mathbb{B})} = 0$.

On the other hand, it follows from the definition of $J(u)$ and C_* that

$$\frac{1}{2}\|\nabla_{\mathbb{B}} u\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})} \leq \frac{1}{p+1}\|u\|^{p+1}_{\mathcal{L}_{p+1}^{n/(p+1)}(\mathbb{B})} \leq \frac{1}{p+1} C_*^{p+1}\|\nabla_{\mathbb{B}} u\|^{p+1}_{\mathcal{L}_2^{n/2}(\mathbb{B})},$$

so it is easy to see that

$$\|\nabla_{\mathbb{B}} u(t)\|_{\mathcal{L}_2^{n/2}(\mathbb{B})} \geq \gamma = \left(\frac{p+1}{2} C_*^{-(p+1)}\right)^{1/(p-1)}, \quad \forall\, t \in [0,t_0).$$

Then by the continuity of $\|\nabla_{\mathbb{B}} u(t)\|_{\mathcal{L}_2^{n/2}(\mathbb{B})}$ with respect to t, we get $\|\nabla_{\mathbb{B}} u(t_0)\|_{\mathcal{L}_2^{n/2}(\mathbb{B})} \geq \gamma > 0$, which contradicts $\|\nabla_{\mathbb{B}} u(t_0)\|_{\mathcal{L}_2^{n/2}(\mathbb{B})} = 0$. □

Lemma 2.5. [4, Lemma 3.4 and Remark 3.1] *Let $0 < \delta < (p+1)/2$, $d(\delta)$ defined in* (1.6)*, then*

$$d(\delta) = \delta^{2/(p-1)} \frac{p+1-2\delta}{p-1} d.$$

Moreover, $d(\delta)$ satisfies the following properties:

(i) $\lim_{\delta \to 0} d(\delta) = 0$*, $d((p+1)/2) = 0$ and $d(\delta) < 0$ for all $\delta > (p+1)/2$;*

(ii) *$d(\delta)$ is increasing on $0 < \delta \leq 1$, decreasing on $1 \leq \delta \leq (p+1)/2$ and it takes the maximum $d = d(1)$ at $\delta = 1$.*

By Lemma 2.5 and the value of d in (1.4), we get

$$d(\delta) = \frac{p+1-2\delta}{2(p+1)} \delta^{2/(p-1)} C_*^{-2(p+1)/(p-1)}. \tag{2.1}$$

Lemma 2.6. *Assume $J(u) < d(\delta)$, $0 < \delta < (p+1)/2$, then we have*

(i) *$K_\delta(u) > 0$ if and only if*

$$0 < \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{n/2}(\mathbb{B})} < C_*^{-(p+1)/(p-1)} \delta^{1/(p-1)}; \tag{2.2}$$

(ii) *$K_\delta(u) < 0$ if and only if*

$$\|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{n/2}(\mathbb{B})} > C_*^{-(p+1)/(p-1)} \delta^{1/(p-1)}; \tag{2.3}$$

(iii) *$K_\delta(u) = 0$ if and only if*

$$\|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{n/2}(\mathbb{B})} = C_*^{-(p+1)/(p-1)} \delta^{1/(p-1)}.$$

Proof. (i) If (2.2) holds, then by the definition of C_* we have

$$\|u\|^{p+1}_{\mathcal{L}^{n/(p+1)}_{p+1}(\mathbb{B})} \le C_*^{p+1}\|\nabla_{\mathbb{B}} u\|^{p-1}_{\mathcal{L}^{n/2}_2(\mathbb{B})}\|\nabla_{\mathbb{B}} u\|^2_{\mathcal{L}^{n/2}_2(\mathbb{B})} < \delta\|\nabla_{\mathbb{B}} u\|^2_{\mathcal{L}^{n/2}_2(\mathbb{B})},$$

then

$$K_\delta(u) = \delta\|\nabla_{\mathbb{B}} u\|^2_{\mathcal{L}^{n/2}_2(\mathbb{B})} - \|u\|^{p+1}_{\mathcal{L}^{n/(p+1)}_{p+1}(\mathbb{B})} > 0.$$

On the other hand, we assume $K_\delta(u) > 0$, then by the definition of $K_\delta(u)$, we know that $\|\nabla_{\mathbb{B}} u\|_{\mathcal{L}^{n/2}_2(\mathbb{B})} > 0$. Since $0 < \delta < (p+1)/2$, then it follows from the definitions of $J(u)$ and $K_\delta(u)$ that

$$J(u) = \left(\frac{1}{2} - \frac{\delta}{p+1}\right)\|\nabla_{\mathbb{B}} u\|^2_{\mathcal{L}^{n/2}_2(\mathbb{B})} + \frac{1}{p+1}K_\delta(u), \tag{2.4}$$

this joins the assumption $J(u) < d(\delta)$, and the value of $d(\delta)$ given as in (2.1) entails

$$\frac{p+1-2\delta}{2(p+1)}\delta^{2/(p-1)}C_*^{-2(p+1)/(p-1)} > J(u) > \left(\frac{1}{2} - \frac{\delta}{p+1}\right)\|\nabla_{\mathbb{E}} u\|^2_{\mathcal{L}^{(n+1)/2}_2(\mathbb{E})},$$

which yields (2.2).

(ii) If (2.3) holds, then we have

$$\left(\frac{1}{2} - \frac{\delta}{p+1}\right)\|\nabla_{\mathbb{B}} u\|^2_{\mathcal{L}^{n/2}_2(\mathbb{B})} > \frac{p+1-2\delta}{2(p+1)}\delta^{2/(p-1)}C_*^{-2(p+1)/(p-1)} = d(\delta).$$

Combining (2.4) and $J(u) < d(\delta)$, we get $K_\delta(u) < 0$.

On the other hand, we assume $K_\delta(u) < 0$. By the definition of $K_\delta(u)$ and C_*, we can deduce that

$$\delta\|\nabla_{\mathbb{B}} u\|^2_{\mathcal{L}^{n/2}_2(\mathbb{B})} < \|u\|^{p+1}_{\mathcal{L}^{n/(p+1)}_{p+1}(\mathbb{B})} \le C_*^{p+1}\|\nabla_{\mathbb{B}} u\|^{p+1}_{\mathcal{L}^{n/2}_2(\mathbb{B})},$$

which implies (2.3).

(iii) By (i) and (ii), we get (iii) immediately. □

Next lemma focus on the blow-up in finite time of solution with negative initial energy, moreover, we can get the upper bound estimate of blow-up time and blow-up rate. The idea of the following proof comes from [15, 22, 24, 25, 28].

Lemma 2.7. *Let $u = u(x,t)$ be a weak solution of problem* (1.1) *with $J(u_0) < 0$, $p > 1$, then $u(t)$ blows up at finite time T with $\mathcal{L}^{n/2}_2(\mathbb{B})$-norm and we can estimate T as follows*

$$T < \frac{1}{1-p^2}\frac{\|u_0\|^2_{\mathcal{L}^{n/2}_2(\mathbb{B})}}{J(u_0)}.$$

Moreover, the blow-up rate can be estimated by

$$\|u(\,\cdot\,,t)\|_{\mathcal{L}^{n/2}_2(\mathbb{B})} \le \left[\frac{(1-p^2)J(u_0)}{\|u_0\|^{p+1}_{\mathcal{L}^{n/2}_2(\mathbb{B})}}\right]^{1/(1-p)}(T-t)^{-1/(p-1)}.$$

Proof. We define

$$f(t) = \frac{1}{2}\int_{\mathbb{B}} |u|^2 \frac{dx_1}{x_1}\, dx' \tag{2.5}$$

and

$$g(t) = -(p+1)J(u(t)) = -\frac{p+1}{2}\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{dx_1}{x_1}\, dx' + \int_{\mathbb{B}} |u|^{p+1} \frac{dx_1}{x_1}\, dx'. \tag{2.6}$$

By the definition of $f(t)$ and take $v = u$ in (1.9) we get

$$f'(t) = \int_{\mathbb{B}} u u_t \frac{dx_1}{x_1}\, dx' = -\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{dx_1}{x_1}\, dx' + \int_{\mathbb{B}} |u|^{p+1} \frac{dx_1}{x_1}\, dx'. \tag{2.7}$$

Combining the definition of $g(t)$ and (1.8) we have

$$g'(t) = -(p+1)\frac{d}{dt}J(u(t)) \geq (p+1)\|u_t\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})} \geq 0. \tag{2.8}$$

By $J(u_0) < 0$ and the definition of $g(t)$ we have $g(0) > 0$, and it follows from above inequality that $g(t) > 0$ for all $t \in [0, T)$. By $p > 1$, (2.6), (2.7) and Lemma 2.4 we know

$$f'(t) > g(t) > 0, \quad \forall\, t \in [0, T), \tag{2.9}$$

then $f(t) > 0$ for all $t \in [0, T)$. Combining (2.5), (2.8), Schwartz's inequality, (2.7) and (2.9) we obtain

$$\begin{aligned} f(t)g'(t) &\geq \frac{p+1}{2}\|u\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\|u_t\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})} \geq \frac{p+1}{2}\left(\int_{\mathbb{B}} u u_t \frac{dx_1}{x_1}\, dx'\right)^2 \\ &= \frac{p+1}{2}[f'(t)]^2 > \frac{p+1}{2} f'(t) g(t), \end{aligned}$$

which can be rewritten as

$$\frac{g'(t)}{g(t)} > \frac{p+1}{2}\frac{f'(t)}{f(t)}.$$

Integrating above inequality from 0 to t we get

$$\frac{g(t)}{[f(t)]^{(p+1)/2}} > \frac{g(0)}{[f(0)]^{(p+1)/2}},$$

then by (2.9), we have

$$\frac{f'(t)}{[f(t)]^{(p+1)/2}} > \frac{g(0)}{[f(0)]^{(p+1)/2}}. \tag{2.10}$$

Integrating inequality (2.10) from 0 to t, we see

$$\frac{1}{[f(t)]^{(p-1)/2}} < \frac{1}{[f(0)]^{(p-1)/2}} - \frac{p-1}{2}\frac{g(0)}{[f(0)]^{(p+1)/2}} t. \tag{2.11}$$

Clearly, (2.11) can not hold for all time, this means $f(t)$ blows up at some finite time T, i.e.,

$$f(T) = +\infty. \tag{2.12}$$

By the definition of $f(t)$ in (2.5), we know $u(t)$ blows up at some finite time T with $\mathcal{L}_2^{n/2}(\mathbb{B})$-norm.

Next, we estimate T and blow-up rate. Let $t \to T$ in (2.11), by (2.12) and the definitions of $f(t), g(t)$ we get

$$T < \frac{2}{p-1}\frac{f(0)}{g(0)} = \frac{1}{1-p^2}\frac{\|u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}}{J(u_0)}.$$

Moreover, integrating inequality (2.10) from t to T, by (2.12) we have

$$f(t) < (T-t)^{-2/(p-1)}\left[\frac{(p-1)g(0)}{2[f(0)]^{(p+1)/2}}\right]^{2/(1-p)},$$

then it follows from the definitions of $f(t)$ and $g(t)$ that

$$\|u(\,\cdot\,,t)\|_{\mathcal{L}_2^{n/2}(\mathbb{B})} < \left[\frac{(1-p^2)J(u_0)}{\|u_0\|^{p+1}_{\mathcal{L}_2^{n/2}(\mathbb{B})}}\right]^{1/(1-p)} (T-t)^{-1/(p-1)}. \qquad \square$$

Lemma 2.8. *Let $u(t) = u(x,t)$ be a weak solution of problem (1.1) with $J(u_0) < d$, $K(u_0) < 0$ and $1 < p < (n+2)/(n-2)$, then*

$$\|\nabla_{\mathbb{B}} u_0\|_{\mathcal{L}_2^{n/2}(\mathbb{B})} > \alpha_1, \tag{2.13}$$

where α_1 is defined in (1.3). There exists a positive constant $\alpha_2 > \alpha_1$ such that

$$\|\nabla_{\mathbb{B}} u(t)\|_{\mathcal{L}_2^{n/2}(\mathbb{B})} \geq \alpha_2, \quad \forall\, t \geq 0, \tag{2.14}$$

and

$$\|u(t)\|_{\mathcal{L}_{p+1}^{n/(p+1)}(\mathbb{B})} \geq C_* \alpha_2, \quad \forall\, t \geq 0. \tag{2.15}$$

Moreover, if $J(u_0) \geq 0$, then the following inequality holds

$$\frac{\alpha_2}{\alpha_1} \geq \left[(p+1)\left(\frac{1}{2} - \frac{J(u_0)}{\alpha_1^2}\right)\right]^{1/(p-1)} > 1. \tag{2.16}$$

Proof. It follows from $K(u_0) < 0$ and (1.2) that

$$\|\nabla_{\mathbb{B}} u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})} < \|u_0\|^{p+1}_{\mathcal{L}_{p+1}^{n/(p+1)}(\mathbb{B})} \leq C_*^{p+1}\|\nabla_{\mathbb{B}} u_0\|^{p+1}_{\mathcal{L}_2^{n/2}(\mathbb{B})},$$

which implies (2.13). For any $t \in [0,T)$, combining (1.2) and the definition of $J(u)$ we know that

$$
\begin{aligned}
J(u(t)) &\geq \frac{1}{2}\|\nabla_{\mathbb{B}} u(t)\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})} - \frac{1}{p+1} C_*^{p+1} \|\nabla_{\mathbb{B}} u(t)\|^{p+1}_{\mathcal{L}_2^{n/2}(\mathbb{B})} \\
&= \frac{1}{2}\alpha^2 - \frac{1}{p+1} C_*^{p+1} \alpha^{p+1} \\
&= g(\alpha),
\end{aligned}
\tag{2.17}
$$

where $\alpha = \|\nabla_{\mathbb{B}} u(t)\|_{\mathcal{L}_2^{n/2}(\mathbb{B})}$. Since $p > 1$, it is easy to see that $g(\alpha)$ is increasing for $\alpha \in (0, \alpha_1)$, decreasing for $\alpha \in (\alpha_1, +\infty)$ and takes the maximum at $\alpha = \alpha_1$, $g(\alpha_1) = d$. Since $J(u_0) < d$, then there exists a positive constant $\alpha_2 > \alpha_1$ such that $g(\alpha_2) = J(u_0)$.

Let $\alpha_0 = \|\nabla_{\mathbb{B}} u_0\|_{\mathcal{L}_2^{n/2}(\mathbb{B})}$, it follows from (2.17) that $g(\alpha_0) \leq J(u_0) = g(\alpha_2)$. By (2.13) we know that $\alpha_0 > \alpha_1$, then coupled with $g(\alpha)$ is decreasing for $\alpha \in (\alpha_1, +\infty)$, leads to $\alpha_0 \geq \alpha_2$, i.e., (2.14) holds for $t = 0$. We prove (2.14) holds for all $t > 0$ by contradiction. Suppose on the contrary that $\|\nabla_{\mathbb{B}} u(\,\cdot\,, t_0)\|_{\mathcal{L}_2^{n/2}(\mathbb{B})} < \alpha_2$ for some $t_0 > 0$. By the continuity of $\|\nabla_{\mathbb{B}} u(\,\cdot\,, t)\|_{\mathcal{L}_2^{n/2}(\mathbb{B})}$ we can choose t_0 such that $\alpha_1 < \|\nabla_{\mathbb{B}} u(\,\cdot\,, t_0)\|_{\mathcal{L}_2^{n/2}(\mathbb{B})} < \alpha_2$, then it follows from the monotonicity properties of $g(\alpha)$ and (2.17) that

$$
J(u_0) = g(\alpha_2) < g(\|\nabla_{\mathbb{B}} u(\,\cdot\,, t_0)\|_{\mathcal{L}_2^{n/2}(\mathbb{B})}) \leq J(u(t_0)),
$$

which contradicts (1.8), hence we get (2.14).

On the other hand, by (1.8), we get

$$
J(u_0) \geq J(u(t)) = \frac{1}{2}\|\nabla_{\mathbb{B}} u\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})} - \frac{1}{p+1}\|u\|^{p+1}_{\mathcal{L}_{p+1}^{n/(p+1)}(\mathbb{B})}.
\tag{2.18}
$$

Since $J(u_0) = g(\alpha_2) = \frac{1}{2}\alpha_2^2 - \frac{1}{p+1} C_*^{p+1} \alpha_2^{p+1}$, then coupled with (2.14) and (2.18), leads to

$$
\begin{aligned}
\frac{1}{p+1}\|u\|^{p+1}_{\mathcal{L}_{p+1}^{n/(p+1)}(\mathbb{B})} &\geq \frac{1}{2}\|\nabla_{\mathbb{B}} u\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})} - J(u_0) \\
&\geq \frac{1}{2}\alpha_2^2 - \frac{1}{2}\alpha_2^2 + \frac{1}{p+1} C_*^{p+1} \alpha_2^{p+1} \\
&= \frac{1}{p+1} C_*^{p+1} \alpha_2^{p+1},
\end{aligned}
$$

so we get (2.15).

Finally, we prove inequality (2.16). To this end, we denote $\beta = \alpha_2/\alpha_1$, then $\beta > 1$ by the fact that $\alpha_2 > \alpha_1$. So it follows from $J(u_0) = g(\alpha_2)$, (2.17) and $\alpha_1 = C_*^{-(p+1)/(p-1)}$ that

$$
\begin{aligned}
J(u_0) = g(\beta\alpha_1) &= \frac{1}{2}(\beta\alpha_1)^2 - \frac{1}{p+1} C_*^{p+1} (\beta\alpha_1)^{p+1} \\
&= (\beta\alpha_1)^2 \left(\frac{1}{2} - \frac{1}{p+1} C_*^{p+1} (\beta\alpha_1)^{p-1}\right) = (\beta\alpha_1)^2 \left(\frac{1}{2} - \frac{1}{p+1}\beta^{p-1}\right).
\end{aligned}
$$

Since $J(u_0) \geq 0$ and $\beta > 1$, then we can infer from the above inequality that

$$\frac{1}{2} - \frac{1}{p+1}\beta^{p-1} = \frac{J(u_0)}{(\beta\alpha_1)^2} \leq \frac{J(u_0)}{\alpha_1^2},$$

then we have

$$\beta = \frac{\alpha_2}{\alpha_1} \geq \left[(p+1)\left(\frac{1}{2} - \frac{J(u_0)}{\alpha_1^2}\right)\right]^{1/(p-1)},$$

which combining Remark 1.5(i) deduce to (2.16). □

Our next lemma is about the upper bound estimate of blow-up time and blow-up rate for nonnegative initial energy, the method in the proof of the following lemma comes from [11, 14, 30–32].

Lemma 2.9. *Let $1 < p < (n+2)/(n-2)$, $u = u(x,t)$ be a weak solution of problem* (1.1) *with $0 \leq J(u_0) < d$, $K(u_0) < 0$, then $u(t)$ blows up at finite time T with $\mathcal{L}_2^{n/2}(\mathbb{B})$-norm and we can estimate T as follows*

$$T \leq \frac{(p+1)|\mathbb{B}|^{(p-1)/2}\|u_0\|_{\mathcal{L}_2^{n/2}(\mathbb{B})}^{1-p}}{(p-1)^2\left[1 - \left((p+1)\left(\frac{1}{2} - \frac{J(u_0)}{\alpha_1^2}\right)\right)^{-(p+1)/(p-1)}\right]}. \tag{2.19}$$

Moreover, the blow-up rate can be estimated by

$$\begin{aligned} &\|u(\,\cdot\,,t)\|_{\mathcal{L}_2^{n/2}(\mathbb{B})} \\ &< \frac{(p-1)^{2/(1-p)}\sqrt{|\mathbb{B}|}}{(p+1)^{1/(1-p)}}\left[1 - \left((p+1)\left(\frac{1}{2} - \frac{J(u_0)}{\alpha_1^2}\right)\right)^{-(p+1)/(p-1)}\right]^{1/(1-p)} (T-t)^{-1/(p-1)}, \end{aligned} \tag{2.20}$$

where $|\mathbb{B}|$ is the measure of $\mathbb{B}$, α_1 is given by (1.3).

Proof. We define a functional $f(t)$ the same as in (2.5), and let

$$H(t) = d - J(u(t)). \tag{2.21}$$

By (1.8), we know that energy functional $J(u(t))$ is nonincreasing with respect to t, then coupled with $J(u_0) < d$, (2.21) leads to

$$H(t) > 0, \quad \forall\, t \in [0,T). \tag{2.22}$$

Combining (2.7), the definitions of $J(u)$ and $H(t)$, we get

$$\begin{aligned} f'(t) &= -\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \,\frac{dx_1}{x_1}\,dx' + \int_{\mathbb{B}} |u|^{p+1}\,\frac{dx_1}{x_1}\,dx' \\ &= \frac{p-1}{p+1}\int_{\mathbb{B}} |u|^{p+1}\,\frac{dx_1}{x_1}\,dx' - 2J(u(t)) \\ &= \frac{p-1}{p+1}\int_{\mathbb{B}} |u|^{p+1}\,\frac{dx_1}{x_1}\,dx' - 2d + 2H(t). \end{aligned} \tag{2.23}$$

By the value of d in (1.4), $\alpha_1 = C_*^{-(p+1)/(p-1)}$ and (2.15) we get

$$2d = \frac{p-1}{p+1} C_*^{-2(p+1)/(p-1)} = \frac{\alpha_1^{p+1}}{\alpha_2^{p+1}} \frac{p-1}{p+1} C_*^{p+1} \alpha_2^{p+1} \leq \frac{\alpha_1^{p+1}}{\alpha_2^{p+1}} \frac{p-1}{p+1} \|u\|^{p+1}_{\mathcal{L}^{n/(p+1)}_{p+1}(\mathbb{B})}.$$

Since $H(t) > 0$, then substituting the above inequality into (2.23) we obtain

$$f'(t) \geq \left(1 - \frac{\alpha_1^{p+1}}{\alpha_2^{p+1}}\right) \frac{p-1}{p+1} \int_{\mathbb{B}} |u|^{p+1} \frac{dx_1}{x_1}\, dx' + 2H(t) > C_0 \int_{\mathbb{B}} |u|^{p+1} \frac{dx_1}{x_1}\, dx', \tag{2.24}$$

where $C_0 = \left(1 - \frac{\alpha_1^{p+1}}{\alpha_2^{p+1}}\right) \frac{p-1}{p+1} > 0$ due to the fact $\alpha_2 > \alpha_1$.

By Hölder's inequality, we can get

$$f^{(p+1)/2}(t) = \left(\frac{1}{2} \int_{\mathbb{B}} |u|^2 \frac{dx_1}{x_1}\, dx'\right)^{(p+1)/2} \leq \widetilde{C} \int_{\mathbb{B}} |u|^{p+1} \frac{dx_1}{x_1}\, dx',$$

where $\widetilde{C} = 2^{-(p+1)/2} |\mathbb{B}|^{(p-1)/2}$. Combining (2.24) and above inequality we have

$$f'(t) > \gamma f^{(p+1)/2}(t), \tag{2.25}$$

where $\gamma = C_0/\widetilde{C} > 0$. Integrating the inequality (2.25) from 0 to t, we get

$$f(t) > \left(f^{(1-p)/2}(0) - \frac{p-1}{2}\gamma t\right)^{-2/(p-1)} = \left(2^{(p-1)/2} \|u_0\|^{1-p}_{\mathcal{L}^{n/2}_2(\mathbb{B})} - \frac{p-1}{2}\gamma t\right)^{-2/(p-1)}. \tag{2.26}$$

Let

$$T_* = \frac{2^{(p+1)/2}}{(p-1)\gamma} \|u_0\|^{1-p}_{\mathcal{L}^{n/2}_2(\mathbb{B})} \in (0, +\infty), \tag{2.27}$$

then by (2.26) we know that $f(t)$ blows up at some finite time $T \leq T_*$, and so does $u(t)$.

Next, we estimate T. By (2.16), (2.27) and the values of C_0, $\widetilde{C}$, γ, we get

$$\begin{aligned}
T &\leq \frac{2^{(p+1)/2}}{(p-1)\gamma} \|u_0\|^{1-p}_{\mathcal{L}^{n/2}_2(\mathbb{B})} = \frac{2^{(p+1)/2}\widetilde{C}}{(p-1)C_0} \|u_0\|^{1-p}_{\mathcal{L}^{n/2}_2(\mathbb{B})} \\
&= \frac{(p+1)|\mathbb{B}|^{(p-1)/2}}{(p-1)^2 \left[1 - \left(\frac{\alpha_1}{\alpha_2}\right)^{p+1}\right]} \|u_0\|^{1-p}_{\mathcal{L}^{n/2}_2(\mathbb{B})} \\
&\leq \frac{(p+1)|\mathbb{B}|^{(p-1)/2} \|u_0\|^{1-p}_{\mathcal{L}^{n/2}_2(\mathbb{B})}}{(p-1)^2 \left[1 - \left((p+1)\left(\frac{1}{2} - \frac{J(u_0)}{\alpha_1^2}\right)\right)^{-(p+1)/(p-1)}\right]}.
\end{aligned}$$

Finally, we estimate the blow-up rate. Integrating the inequality (2.25) from t to T, then by $f(T) = +\infty$ we can conclude that

$$f(t) < \left(\frac{p-1}{2}\gamma\right)^{2/(1-p)} (T-t)^{-2/(p-1)},$$

so by the definition of $f(t)$, (2.16) and the values of C_0, $\widetilde{C}$, γ, we have

$$\begin{aligned}
&\frac{1}{2}\|u(\cdot,t)\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\\
&< \left(\frac{p-1}{2}\frac{C_0}{\widetilde{C}}\right)^{2/(1-p)}(T-t)^{-2/(p-1)}\\
&= \left[\frac{2^{(p-1)/2}(p-1)^2}{(p+1)|\mathbb{B}|^{(p-1)/2}}\left(1-\left(\frac{\alpha_1}{\alpha_2}\right)^{p+1}\right)\right]^{2/(1-p)}(T-t)^{-2/(p-1)}\\
&\le \frac{(p-1)^{4/(1-p)}|\mathbb{B}|}{2(p+1)^{2/(1-p)}}\left[1-\left((p+1)\left(\frac{1}{2}-\frac{J(u_0)}{\alpha_1^2}\right)\right)^{-(p+1)/(p-1)}\right]^{2/(1-p)}(T-t)^{-2/(p-1)},
\end{aligned}$$

then

$$\begin{aligned}
&\|u(\cdot,t)\|_{\mathcal{L}_2^{n/2}(\mathbb{B})}\\
&< \frac{(p-1)^{2/(1-p)}\sqrt{|\mathbb{B}|}}{(p+1)^{1/(1-p)}}\left[1-\left((p+1)\left(\frac{1}{2}-\frac{J(u_0)}{\alpha_1^2}\right)\right)^{-\frac{p+1}{p-1}}\right]^{1/(1-p)}(T-t)^{-1/(p-1)}. \quad \square
\end{aligned}$$

Next, we give a lemma about the asymptotic behavior of blow-up solution. We prove that the blow-up solution grows exponentially for $t\in[0,T)$, and we point out that our following result holds for $J(u_0)\le d$.

Lemma 2.10. *Let $1<p<(n+2)/(n-2)$, $u(t)=(x,t)$ be a weak solution of problem* (1.1) *with $J(u_0)\le d$, $K(u_0)<0$, then $u(t)$ increases exponentially in $\mathcal{L}_{p+1}^{n/(p+1)}(\mathbb{B})$-norm for $t\in[0,T)$.*

Proof. The proof is divided into two steps.

Step 1*: the case ($J(u_0)<d$).* We define a functional

$$G(t)=H(t)+f(t),\quad \forall\, t\in[0,T),$$

where $H(t)$, $f(t)$ are defined in (2.21) and (2.5) respectively, then by (1.8), (2.7) and the definition of $K(u)$ we get

$$\begin{aligned}
G'(t)&=-\frac{d}{dt}J(u(t))+f'(t)\\
&\ge \|u_t\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}-\|\nabla_{\mathbb{B}}u\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}+\|u\|^{p+1}_{\mathcal{L}_{p+1}^{n/(p+1)}(\mathbb{B})}\\
&=\|u_t\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}-K(u).
\end{aligned}\tag{2.28}$$

Using the definitions of $J(u)$, $K(u)$, $H(t)$, we obtain

$$\begin{aligned}
K(u)&=(p+1)J(u)-\frac{p-1}{2}\|\nabla_{\mathbb{B}}u\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\\
&=(p+1)d-(p+1)H(t)-\frac{p-1}{2}\|\nabla_{\mathbb{B}}u\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}.
\end{aligned}\tag{2.29}$$

By (2.14) we can deduce that

$$\begin{aligned}\frac{p-1}{2}\|\nabla_{\mathbb{B}}u\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})} &= \frac{p-1}{2}\left(\frac{\alpha_2^2-\alpha_1^2}{\alpha_2^2}\|\nabla_{\mathbb{B}}u\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}+\frac{\alpha_1^2}{\alpha_2^2}\|\nabla_{\mathbb{B}}u\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\right)\\ &\geq \frac{p-1}{2}\left(\frac{\alpha_2^2-\alpha_1^2}{\alpha_2^2}\right)\|\nabla_{\mathbb{B}}u\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}+\frac{p-1}{2}\alpha_1^2.\end{aligned} \tag{2.30}$$

Then combining (2.28)–(2.30) and $d=\frac{p-1}{2(p+1)}\alpha_1^2$ we have

$$\begin{aligned}G'(t) &\geq \|u_t\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}-(p+1)d+(p+1)H(t)+\frac{p-1}{2}\|\nabla_{\mathbb{B}}u\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\\ &\geq (p+1)H(t)+\frac{p-1}{2}\left(\frac{\alpha_2^2-\alpha_1^2}{\alpha_2^2}\right)\|\nabla_{\mathbb{B}}u\|^2_{\mathcal{L}_2^{n/2}(\mathbb{E})}.\end{aligned}$$

By (2.22), we can take $C_1=\min\left\{p+1,\frac{p-1}{2}\left(\frac{\alpha_2^2-\alpha_1^2}{\alpha_2^2}\right)\right\}>0$ such that

$$G'(t)\geq C_1\left(H(t)+\|\nabla_{\mathbb{B}}u\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\right). \tag{2.31}$$

On the other hand, by Lemma 2.3 we have

$$G(t)=H(t)+\frac{1}{2}\|u\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\leq H(t)+\frac{1}{2}\mu^2\|\nabla_{\mathbb{B}}u\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})},$$

where μ is a positive constant depending only on $\mathbb{B}$. Taking $C_2=\max\left\{1,\frac{1}{2}\mu^2\right\}>0$, then

$$G(t)\leq C_2\left(H(t)+\|\nabla_{\mathbb{B}}u\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\right).$$

Combining (2.31) and above inequality we get

$$G'(t)\geq C_3G(t),$$

where $C_3=C_1/C_2>0$. It follows from Gronwall's inequality that

$$G(t)\geq G(0)e^{C_3t},\quad \forall\, t\in[0,T). \tag{2.32}$$

By the definitions of $G(t)$, $J(u)$ and Hölder's inequality, we have

$$\begin{aligned}G(t) &= d-\frac{1}{2}\|\nabla_{\mathbb{B}}\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}+\frac{1}{p+1}\|u\|^{p+1}_{\mathcal{L}_{p+1}^{n/(p+1)}(\mathbb{B})}+\frac{1}{2}\|u\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\\ &\leq d+\frac{1}{p+1}\|u\|^{p+1}_{\mathcal{L}_{p+1}^{n/(p+1)}(\mathbb{B})}+\frac{1}{2}|\mathbb{B}|^{(p-1)/(p+1)}\|u\|^2_{\mathcal{L}_{p+1}^{n/(p+1)}(\mathbb{B})}.\end{aligned}$$

Taking $C_4=\max\left\{\frac{1}{p+1},\frac{1}{2}|\mathbb{B}|^{(p-1)/(p+1)}\right\}>0$, then

$$\|u\|^{p+1}_{\mathcal{L}_{p+1}^{n/(p+1)}(\mathbb{B})}+\|u\|^2_{\mathcal{L}_{p+1}^{n/(p+1)}(\mathbb{B})}\geq\frac{1}{C_4}(G(t)-d).$$

This, coupled with (2.32) and the definition of $G(t)$, leads to

$$\|u(t)\|^{p+1}_{\mathcal{L}^{n/(p+1)}_{p+1}(\mathbb{B})} + \|u(t)\|^2_{\mathcal{L}^{n/(p+1)}_{p+1}(\mathbb{B})} \geq C_5 e^{C_3 t} - \frac{d}{C_4}, \quad \forall\, t \in [0,T),$$

where

$$C_5 = \frac{1}{C_4}\left(d - J(u_0) + \frac{1}{2}\|u_0\|^2_{\mathcal{L}^{n/2}_2(\mathbb{B})}\right) > 0.$$

Then we know that the solution of problem (1.1) grows as an exponential function in $\mathcal{L}^{n/(p+1)}_{p+1}(\mathbb{B})$-norm.

Step 2: the case ($J(u_0) = d$). By $J(u_0) = d > 0$, $K(u_0) < 0$ and the continuity of $J(u(t))$ and $K(u(t))$ with respect to t, we know that there exists a sufficiently small $t_1 > 0$ such that $J(u(t_1)) > 0$ and $K(u(t)) < 0$ for $0 \leq t \leq t_1$. Then for $0 \leq t \leq t_1$ it follows from (2.7) that

$$\int_{\mathbb{B}} u u_t \, \frac{dx_1}{x_1} \, dx' = -K(u) > 0. \tag{2.33}$$

On the other hand, by Hölder's inequality it is easy to see that

$$\int_{\mathbb{B}} |u u_t| \, \frac{dx_1}{x_1} \, dx' \leq \|u_t\|_{\mathcal{L}^{n/2}_2(\mathbb{B})} \|u\|_{\mathcal{L}^{n/2}_2(\mathbb{B})},$$

then by (2.33), for $0 \leq t \leq t_1$, we have $\|u_t\|^2_{\mathcal{L}^{n/2}_2(\mathbb{B})} > 0$. Then it follows from the continuity of $\int_0^t \|u_\tau\|^2_{\mathcal{L}^{n/2}_2(\mathbb{B})} \, d\tau$ that

$$0 < d - \int_0^{t_1} \|u_\tau\|^2_{\mathcal{L}^{n/2}_2(\mathbb{B})} \, d\tau < d.$$

So from (1.8) we can get

$$J(u(t_1)) \leq d - \int_0^{t_1} \|u_\tau\|^2_{\mathcal{L}^{n/2}_2(\mathbb{B})} \, d\tau < d.$$

Hence, taking $t = t_1$ as the initial time, we have $J(u(t_1)) < d$, $K(u(t_1)) < 0$, the remainder of the proof is similar as that in Step 1 and we have

$$\|u(t)\|^{p+1}_{\mathcal{L}^{n/(p+1)}_{p+1}(\mathbb{B})} + \|u(t)\|^2_{\mathcal{L}^{n/(p+1)}_{p+1}(\mathbb{B})} \geq C_6 e^{C_3 t} - \frac{d}{C_4}, \quad \forall\, t \in [t_1,T),$$

where

$$C_6 = \frac{1}{C_4}\left(d - J(u(t_1)) + \frac{1}{2}\|u(t_1)\|^2_{\mathcal{L}^{n/2}_2(\mathbb{B})}\right) > 0.$$

Namely, the solution of problem (1.1) grows exponentially in $\mathcal{L}^{n/(p+1)}_{p+1}(\mathbb{B})$-norm, then Lemma 2.10 is proved. □

3. Vacuum isolating phenomena of solution

Proof of Theorem 1.4. (i) Let U_e be the set defined in (1.10), then by the definition of $\mathcal{N}_\delta$ and Lemma 2.6(iii), we have

$$\begin{aligned} U_e &= \bigcup_{\delta_1<\delta<\delta_2} \mathcal{N}_\delta \\ &= \bigcup_{\delta_1<\delta<\delta_2} \left\{ u \in \mathcal{H}^{1,n/2}_{2,0}(\mathbb{B}) : \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}^{n/2}_2(\mathbb{B})} = C_*^{-(p+1)/(p-1)} \delta^{1/(p-1)} \right\} \\ &= \left\{ u \in \mathcal{H}^{1,n/2}_{2,0}(\mathbb{B}) : C_*^{-(p+1)/(p-1)} \delta_1^{1/(p-1)} < \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}^{n/2}_2(\mathbb{B})} < C_*^{-(p+1)/(p-1)} \delta_2^{1/(p-1)} \right\}. \end{aligned}$$

So by [4, Remark 3.2(b)], we know U_e is a vacuum region. Namely, for the initial energy $0 < J(u_0) \le e$, there is no solution of problem (1.1) in U_e and all solutions are isolated by U_e, then $\overline{\mathcal{B}}_{\delta_1}$ and $\overline{\mathcal{B}}^c_{\delta_2}$ are both invariant regions.

(ii) We only consider the nontrivial solutions of problem (1.1) (otherwise, the conclusion obviously hold). If $u_0 \in \overline{\mathcal{B}}_{\delta_1}$, then the corresponding solution $u(t) \in \overline{\mathcal{B}}_{\delta_1}$ for all $t \in [0,T)$ since $\overline{\mathcal{B}}_{\delta_1}$ is an invariant region. By the definition of $\overline{\mathcal{B}}_{\delta_1}$ in (1.7) we have

$$\|\nabla_{\mathbb{B}} u(t)\|_{\mathcal{L}^{n/2}_2(\mathbb{B})} \le C_*^{-(p+1)/(p-1)} \delta_1^{1/(p-1)}, \tag{3.1}$$

which means that $u(t)$ exist globally.

Next, we prove $u(t)$ decays exponentially with $\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})$-norm. To this end, we first claim that

$$\|\nabla_{\mathbb{B}} u(t)\|_{\mathcal{L}^{n/2}_2(\mathbb{B})} > 0, \quad \forall\, t \in [0,+\infty).$$

Arguing by contradiction, if there exists a t_0 such that $\|\nabla_{\mathbb{B}} u(t_0)\|_{\mathcal{L}^{n/2}_2(\mathbb{B})} = 0$, then by the definition of $J(u)$, we have $J(u(t_0)) < 0$. Then take t_0 as initial time, it follows from Lemma 2.7 that the solution blows up in finite time, so we get a contradiction.

Hence, by (3.1) and Lemma 2.6(i)(iii) we have $K_{\delta_1}(u(t)) \ge 0$ for all $t \ge 0$, especially, we have $K_{\delta_1}(u_0) \ge 0$. Since $\delta_1 < 1$ and $\|\nabla_{\mathbb{B}} u_0\|_{\mathcal{L}^{n/2}_2(\mathbb{B})} > 0$, by the definition of $K_{\delta_1}(u)$ in (1.6) we have

$$K(u_0) = K_1(u_0) > K_{\delta_1}(u_0) \ge 0.$$

Then by Theorem 1.3(i), we know that there exist constants $C_0 > 0$, $\lambda_0 > 0$ such that

$$\|\nabla_{\mathbb{B}} u(t)\|_{\mathcal{L}^{n/2}_2(\mathbb{B})} \le C_0 e^{-\lambda_0 t}, \quad \forall\, t \in [0,+\infty).$$

Namely, $u(t)$ decays exponentially with $\mathcal{H}^{1,n/2}_{2,0}(\mathbb{B})$-norm.

(iii) If $u_0 \in \overline{\mathcal{B}}^c_{\delta_2}$, since $\overline{\mathcal{B}}^c_{\delta_2}$ is an invariance set, then we get $u(t) \in \overline{\mathcal{B}}^c_{\delta_2}$ for all $t \in [0,T)$, i.e.,

$$\|\nabla_{\mathbb{B}} u(t)\|_{\mathcal{L}^{n/2}_2(\mathbb{B})} \ge C_*^{-(p+1)/(p-1)} \delta_2^{1/(p-1)}, \quad \forall\, t \in [0,T).$$

Combining Lemma 2.6(ii)(iii) we know that

$$K_{\delta_2}(u(t)) \le 0, \quad \forall\, t \in [0,T).$$

Since $\delta_2 > 1$, then by the definition of $K_{\delta_2}(u)$ and the above inequality, we have $K(u(t)) < 0$ for all $t \in [0,T)$, especially, $K(u_0) < 0$. Then the assumptions of Lemmas 2.9 and 2.10 hold, then Theorem 1.4 is proved. □

Proof of Theorem 1.6. (i) Since $J(u(t))$ is nonincreasing with respect to t and $J(u_0) \le 0$, then we have $J(u(t)) \le 0$ for all $t \in [0,T)$. By

$$J(u_0) \ge J(u(t)) = \frac{1}{2}\|\nabla_{\mathbb{B}} u(t)\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})} - \frac{1}{p+1}\|u(t)\|^{p+1}_{\mathcal{L}_{p+1}^{n/(p+1)}(\mathbb{B})},$$

we get

$$\frac{1}{p+1}\|u(t)\|^{p+1}_{\mathcal{L}_{p+1}^{n/(p+1)}(\mathbb{B})} \ge \frac{1}{2}\|\nabla_{\mathbb{B}} u(t)\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})} - J(u_0).$$

Then combining the definition of C_*, we can deduce that

$$\frac{1}{p+1} C_*^{p+1}\|\nabla_{\mathbb{B}} u(t)\|^{p+1}_{\mathcal{L}_2^{n/2}(\mathbb{B})} \ge \frac{1}{2}\|\nabla_{\mathbb{B}} u(t)\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})} - J(u_0),$$

which implies $G(\|\nabla_{\mathbb{B}} u(t)\|_{\mathcal{L}_2^{n/2}(\mathbb{B})}) \ge 0$, where G is defined in (1.12). Then by the properties of G we get $\|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{n/2}(\mathbb{B})} \ge r^*$, where $r^* = r^*(J(u_0))$ is the unique positive root of $G(r) = 0$. So the set U_{r^*} defined in (1.13) is a vacuum region such that all solutions are isolated by U_{r^*}, and then $\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B}) \setminus U_{r^*}$ is a invariant set.

(ii) If $u_0 \in \mathcal{H}_{2,0}^{1,n/2}(\mathbb{B}) \setminus U_{r^*}$, by $\mathcal{H}_{2,0}^{1,n/2}(\mathbb{B}) \setminus U_{r^*}$ is an invariant regions, we know that the corresponding solution $u(t) \in \mathcal{H}_{2,0}^{1,n/2}(\mathbb{B}) \setminus U_{r^*}$ for all $t \in [0,T)$. Since $J(u_0) \le 0$, then by the definition of $J(u_0)$ and $p > 1$ we get

$$\frac{1}{2}\|\nabla_{\mathbb{B}} u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})} \le \frac{1}{p+1}\|u_0\|^{p+1}_{\mathcal{L}_{p+1}^{n/(p+1)}(\mathbb{B})} < \frac{1}{2}\|u_0\|^{p+1}_{\mathcal{L}_{p+1}^{n/(p+1)}(\mathbb{B})},$$

then by the definition of $K(u_0)$, we get $K(u_0) < 0$. So we obtain

$$J(u_0) \le 0 < d, \quad K(u_0) < 0,$$

then the assumptions of Lemmas 2.9 and 2.10 hold, so we get $u(t)$ blows up at finite time T with $\mathcal{L}_2^{n/2}(\mathbb{B})$-norm, and $u(t)$ grow exponentially with $\mathcal{L}_{p+1}^{n/(p+1)}(\mathbb{B})$-norm for $t \in [0,T)$. Moreover, when $J(u_0) = 0$, $1 < p < (n+2)/(n-2)$, by (2.19) and (2.20), we can have the following estimate:

$$T \le \frac{(p+1)|\mathbb{B}|^{(p-1)/2}\|u_0\|^{1-p}_{\mathcal{L}_2^{n/2}(\mathbb{B})}}{(p-1)^2\left[1 - \left(\frac{p+1}{2}\right)^{-(p+1)/(p-1)}\right]},$$

$$\|u(\cdot,t)\|_{\mathcal{L}_2^{n/2}(\mathbb{B})} < \frac{(p-1)^{2/(1-p)}\sqrt{|\mathbb{B}|}}{(p+1)^{1/(1-p)}}\left[1 - \left(\frac{p+1}{2}\right)^{-(p+1)/(p-1)}\right]^{1/(1-p)} (T-t)^{-1/(p-1)}.$$

When $J(u_0) < 0$, $p > 1$, the assumption of Lemma 2.7 holds, so Theorem 1.6 is proved. □

4. Blow-up for arbitrary initial energy

Proof of Theorem 1.8. Let $u(t)=u(x,t)$ be a weak solution of problem (1.1), $p>1$ and

$$J(u_0)<\frac{\lambda_1(p-1)}{2(p+1)}\|u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}. \tag{4.1}$$

We first prove that $u(t)$ blows up in finite time. In fact, by Lemma 2.7, we can see that if there is a t_0 such that $J(u(t_0))<0$, then take t_0 as the initial time, the solution blows up in finite time. Hence, we only need to consider $J(u(t))>0$ for all $t\geq 0$ in the following proof.

The proof proceeds by contradiction, we assume $u(t)$ exists globally. It is easy to see that

$$\int_0^t \|u_\tau\|_{\mathcal{L}_2^{n/2}(\mathbb{B})}\,d\tau \geq \left\|\int_0^t u_\tau\,d\tau\right\|_{\mathcal{L}_2^{n/2}(\mathbb{B})} = \|u(t)-u_0\|_{\mathcal{L}_2^{n/2}(\mathbb{B})} \geq \|u(t)\|_{\mathcal{L}_2^{n/2}(\mathbb{B})}-\|u_0\|_{\mathcal{L}_2^{n/2}(\mathbb{B})},$$

then combining the above inequality, Hölder's inequality with (1.8) we obtain

$$\begin{aligned}\|u(t)\|_{\mathcal{L}_2^{n/2}(\mathbb{B})} &\leq \|u_0\|_{\mathcal{L}_2^{n/2}(\mathbb{B})}+\int_0^t \|u_\tau\|_{\mathcal{L}_2^{n/2}(\mathbb{B})}\,d\tau\\ &\leq \|u_0\|_{\mathcal{L}_2^{n/2}(\mathbb{B})}+t^{1/2}\left(\int_0^t \|u_\tau\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\,d\tau\right)^{1/2}\\ &\leq \|u_0\|_{\mathcal{L}_2^{n/2}(\mathbb{B})}+t^{1/2}[J(u_0)-J(u(t))]^{1/2}.\end{aligned}$$

It follows from (1.8) that $J(u(t))$ is nonincreasing with respect to t, then $0<J(u(t))\leq J(u_0)$, which implies

$$\|u(t)\|_{\mathcal{L}_2^{n/2}(\mathbb{B})}\leq \|u_0\|_{\mathcal{L}_2^{n/2}(\mathbb{B})}+t^{1/2}J(u_0)^{1/2}. \tag{4.2}$$

On the other hand, by (2.7), we know

$$\frac{d}{dt}\left(\frac{1}{2}\|u(t)\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\right)=\|u(t)\|^{p+1}_{\mathcal{L}_{p+1}^{n/(p+1)}(\mathbb{B})}-\|\nabla_{\mathbb{B}}u(t)\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}, \tag{4.3}$$

then combining the definition of $J(u)$ and (1.5) we have

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\|u(t)\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\right) &= \frac{p-1}{2}\|\nabla_{\mathbb{B}}u(t)\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}-(p+1)J(u(t))\\ &\geq \frac{p-1}{2}\lambda_1\|u(t)\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}-(p+1)J(u(t))\\ &= (p-1)\lambda_1\left[\frac{1}{2}\|u(t)\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}-\frac{p+1}{\lambda_1(p-1)}J(u(t))\right].\end{aligned} \tag{4.4}$$

Since $\frac{d}{dt}J(u(t))\le 0$, then we can deduce that

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\|u(t)\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}-\frac{p+1}{\lambda_1(p-1)}J(u(t))\right)\\ \ge{}&\frac{d}{dt}\left(\frac{1}{2}\|u(t)\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\right)\\ \ge{}&\lambda_1(p-1)\left[\frac{1}{2}\|u(t)\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}-\frac{p+1}{\lambda_1(p-1)}J(u(t))\right].\end{aligned}$$

Let $\overline{H}(t)=\frac{1}{2}\|u(t)\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}-\frac{p+1}{\lambda_1(p-1)}J(u(t))$, then it follows from the above inequality and Gronwall's inequality that

$$\|u(t)\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\ge\frac{2(p+1)}{\lambda_1(p-1)}J(u(t))+2e^{\lambda_1(p-1)t}\overline{H}(0),$$

where $\overline{H}(0)=\frac{1}{2}\|u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}-\frac{p+1}{\lambda_1(p-1)}J(u_0)>0$ due to (4.1). By $J(u(t))>0$ for all $t\ge 0$ we know

$$\|u(t)\|_{\mathcal{L}_2^{n/2}(\mathbb{B})}\ge(2\overline{H}(0))^{1/2}e^{\frac{\lambda_1(p-1)}{2}t},$$

which contradicts (4.2) for t sufficiently large, then $u(t)$ blows up at some finite time T.

Next, we give a upper bound estimate of T. To this end, we first claim that $K(u(t))<0$ for all $t\in[0,T)$. In fact, by the definition of $J(u)$, (4.1) and (1.5) we know

$$\begin{aligned}\frac{1}{2}\|\nabla_{\mathbb{B}}u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}-\frac{1}{p+1}\|u_0\|^{p+1}_{\mathcal{L}_{p+1}^{n/(p+1)}(\mathbb{B})}&<\frac{\lambda_1(p-1)}{2(p+1)}\|u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\\&\le\left(\frac{1}{2}-\frac{1}{p+1}\right)\|\nabla_{\mathbb{B}}u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})},\end{aligned}$$

hence, $K(u_0)=\|\nabla_{\mathbb{B}}u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}-\|u_0\|^{p+1}_{\mathcal{L}_{p+1}^{n/(p+1)}(\mathbb{B})}<0$. We assume there exists a $t_0\in(0,T)$ such that

$$K(u(t_0))=0,\quad K(u(t))<0,\quad\forall\,t\in[0,t_0).$$

Then by (4.3) we know

$$\frac{d}{dt}\left(\frac{1}{2}\|u(t)\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\right)=-K(u(t))>0,\quad\forall\,t\in[0,t_0).$$

Namely, $\|u(t)\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}$ is strictly increasing on $[0,t_0)$, so combine (4.1) we have

$$J(u_0)<\frac{\lambda_1(p-1)}{2(p+1)}\|u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}<\frac{\lambda_1(p-1)}{2(p+1)}\|u(t_0)\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}.\tag{4.5}$$

On the other hand, since $J(u(t))$ is nonincreasing with respect to t, and combine the definition of $J(u)$, $K(u)$ and (1.5), we get

$$\begin{aligned}J(u_0)\ge J(u(t_0))&=\frac{p-1}{2(p+1)}\|\nabla_{\mathbb{B}}u(t_0)\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}+\frac{1}{p+1}K(u(t_0))\\&\ge\frac{\lambda_1(p-1)}{2(p+1)}\|u(t_0)\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})},\end{aligned}$$

which contradicts (4.5). So $K(u(t)) < 0$, $\|u(t)\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}$ is strictly increasing for all $t \in [0, T)$.

For any $\widetilde{T} \in (0, T)$, we define a function

$$F(t) = \int_0^t \|u(\tau)\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\, d\tau + (T - t)\|u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})} + \beta(t+\gamma)^2, \quad \forall\, t \in [0, \widetilde{T}],$$

where β, γ are two positive constants which will be determined later, so

$$\begin{aligned} F'(t) &= \|u(t)\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})} - \|u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})} + 2\beta(t+\gamma) \\ &= \int_0^t \frac{d}{d\tau}\|u(\tau)\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\, d\tau + 2\beta(t+\gamma). \end{aligned} \tag{4.6}$$

Then combining (4.4) and (1.8) we can deduce that

$$\begin{aligned} F''(t) &= \frac{d}{dt}\|u(t)\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})} + 2\beta \\ &= (p-1)\|\nabla_{\mathbb{B}} u(t)\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})} - 2(p+1)J(u(t)) + 2\beta \\ &\geq (p-1)\|\nabla_{\mathbb{B}} u(t)\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})} - 2(p+1)J(u_0) \\ &\quad + 2(p+1)\int_0^t \|u_\tau\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\, d\tau + 2\beta. \end{aligned} \tag{4.7}$$

By the definition of $F(t)$, we note that $F(0) > 0$. Since $\|u(t)\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}$ is strictly increasing for all $t \in [0, T)$, it follows from (4.6) that $F'(t) > 0$ for all $t \in [0, \widetilde{T}]$, which implies $F(t) > 0$ and $F(t)$ is strictly increasing for any $t \in [0, \widetilde{T}]$.

Next, for any $t \in [0, \widetilde{T}]$, we define

$$\begin{aligned} \xi(t) = &\left[\int_0^t \|u(\tau)\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\, d\tau + \beta(t+\gamma)^2\right]\left[\int_0^t \|u_\tau\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\, d\tau + \beta\right] \\ &- \left[\frac{1}{2}\int_0^t \frac{d}{d\tau}\|u(\tau)\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\, d\tau + \beta(t+\gamma)\right]^2. \end{aligned}$$

Using Hölder's inequality we have

$$\frac{1}{2}\frac{d}{d\tau}\|u(\tau)\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})} = \int_{\mathbb{B}} u u_\tau \frac{dx_1}{x_1}\, dx' \leq \|u\|_{\mathcal{L}_2^{n/2}(\mathbb{B})}\|u_\tau\|_{\mathcal{L}_2^{n/2}(\mathbb{B})}.$$

Then, by Hölder's inequality again we obtain

$$\frac{1}{2}\int_0^t \frac{d}{d\tau}\|u(\tau)\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\, d\tau \leq \left(\int_0^t \|u\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\, d\tau\right)^{1/2}\left(\int_0^t \|u_\tau\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\, d\tau\right)^{1/2}.$$

Let

$$A = \int_0^t \|u(\tau)\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\, d\tau \quad \text{and} \quad B = \int_0^t \|u_\tau\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\, d\tau,$$

then it follows from the above inequalities and the definition of $\xi(t)$ that

$$\begin{aligned}\xi(t) &\geq (A+\beta(t+\gamma)^2)(B+\beta) - (A^{1/2}B^{1/2}+\beta(t+\gamma))^2 \\ &= \beta A + \beta(t+\gamma)^2 B - 2\beta(t+\gamma)A^{1/2}B^{1/2}.\end{aligned}$$

By Young's inequality, we get

$$2\beta(t+\gamma)A^{1/2}B^{1/2} \leq \beta A + \beta(t+\gamma)^2 B.$$

So the above two inequalities imply $\xi(t) \geq 0$.

For any positive constant $\mu > 0$, we have

$$\begin{aligned}FF'' - \mu(F')^2 &= FF'' - 4\mu\left(\frac{1}{2}\int_0^t \frac{d}{d\tau}\|u(\tau)\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\,d\tau + \beta(t+\gamma)\right)^2 \\ &= FF'' + 4\mu\Bigg[\left(\int_0^t \|u(\tau)\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\,d\tau + \beta(t+\gamma)^2\right)\left(\int_0^t \|u_\tau\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\,d\tau + \beta\right) \\ &\qquad - \left(\frac{1}{2}\int_0^t \frac{d}{d\tau}\|u(\tau)\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\,d\tau + \beta(t+\gamma)\right)^2 \\ &\qquad - \left(F - (T-t)\|u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\right)\left(\int_0^t \|u_\tau\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\,d\tau + \beta\right)\Bigg] \\ &= FF'' + 4\mu\xi(t) - 4\mu\left(F - (T-t)\|u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\right)\left(\int_0^t \|u_\tau\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\,d\tau + \beta\right).\end{aligned}$$

Since $\xi(t) \geq 0$, $F(t) > 0$ for all $t \in [0, \widetilde{T}]$, by above inequality, (4.7), (1.5) and $\|u(t)\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}$ is strictly increasing for all $t \in [0, T)$ we obtain

$$\begin{aligned}&FF'' - \mu(F')^2 \\ &\geq FF'' - 4\mu F\left(\int_0^t \|u_\tau\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\,d\tau + \beta\right) \\ &= F\left(F'' - 4\mu\int_0^t \|u_\tau\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\,d\tau - 4\mu\beta\right) \\ &\geq F\left((p-1)\|\nabla_{\mathbb{B}} u\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})} - 2(p+1)J(u_0) + 2\beta + [2(p+1) - 4\mu]\int_0^t \|u_\tau\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\,d\tau - 4\mu\beta\right) \\ &\geq F\left[2(p+1)\left(\frac{\lambda_1(p-1)}{2(p+1)}\|u(t)\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})} - J(u_0)\right) + [2(p+1) - 4\mu]\int_0^t \|u_\tau\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\,d\tau - 4\mu\beta\right] \\ &> F\left[2(p+1)\left(\frac{\lambda_1(p-1)}{2(p+1)}\|u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})} - J(u_0)\right) + [2(p+1) - 4\mu]\int_0^t \|u_\tau\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\,d\tau - 4\mu\beta\right].\end{aligned}$$

Taking $\mu = (p+1)/2$ in the above inequations and restricting β satisfy:

$$0 < \beta \leq \frac{\lambda_1(p-1)}{2(p+1)}\|u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})} - J(u_0), \tag{4.8}$$

we have

$$FF'' - \frac{p+1}{2}(F')^2 > 2(p+1)F\left(\frac{\lambda_1(p-1)}{2(p+1)}\|u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})} - J(u_0) - \beta\right) \geq 0$$

holds for all $t \in [0, \widetilde{T}]$. Define $G(t) = F^{(1-p)/2}(t)$ for $t \in [0, \widetilde{T}]$, then by $F(t) > 0$, $F'(t) > 0$ and the above inequality we get

$$\begin{aligned} G'(t) &= \frac{1-p}{2} F^{-(p+1)/2}(t) F'(t) < 0, \\ G''(t) &= \frac{1-p}{2} F^{-(p+1)/2-1}(t) \left(F F'' - \frac{p+1}{2} (F')^2 \right) < 0 \end{aligned}$$

for all $t \in [0, \widetilde{T}]$. It follows from $G''(t) < 0$ that

$$G(\widetilde{T}) - G(0) = \int_0^1 G'(\theta \widetilde{T}) \, d\theta \widetilde{T} < G'(0) \widetilde{T}, \tag{4.9}$$

where

$$\begin{aligned} G(0) = F^{(1-p)/2}(0) > 0, \quad G(\widetilde{T}) = F^{(1-p)/2}(\widetilde{T}) > 0, \\ G'(0) = \frac{1-p}{2} F^{-(p+1)/2}(0) F'(0) = (1-p)\beta\gamma F^{-(p+1)/2}(0) < 0. \end{aligned}$$

By (4.9) and the above inequalities we get

$$\widetilde{T} < \frac{G(\widetilde{T})}{G'(0)} - \frac{G(0)}{G'(0)} < -\frac{G(0)}{G'(0)} = \frac{F(0)}{(p-1)\beta\gamma}.$$

Then by the definition of $F(t)$ and the above inequality we have

$$\widetilde{T} < \frac{T \|u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})} + \beta\gamma^2}{(p-1)\beta\gamma} = \frac{\|u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}}{(p-1)\beta\gamma} T + \frac{\gamma}{p-1}, \quad \forall \widetilde{T} \in [0, T).$$

Hence, letting $\widetilde{T} \to T$, we get

$$T \leq \frac{\|u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}}{(p-1)\beta\gamma} T + \frac{\gamma}{p-1}. \tag{4.10}$$

For any β satisfying (4.8), let γ be large enough such that

$$\frac{\|u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}}{(p-1)\beta} < \gamma < +\infty, \tag{4.11}$$

then

$$0 < \frac{\|u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}}{(p-1)\beta\gamma} < 1.$$

Hence, by (4.10) we lead to

$$T \leq \frac{\gamma}{p-1} \left(1 - \frac{\|u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}}{(p-1)\beta\gamma} \right)^{-1} = \frac{\beta\gamma^2}{(p-1)\beta\gamma - \|u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}}. \tag{4.12}$$

We define a set Φ by

$$\Phi = \{(\beta, \gamma) : \beta, \gamma \text{ satisfy (4.8) and (4.11) respectively}\}.$$

Then by (4.12) we obtain

$$T \leq \min_{(\beta,\gamma)\in\Phi} H(\beta, \gamma),$$

where

$$H(\beta, \gamma) = \frac{\beta\gamma^2}{(p-1)\beta\gamma - \|u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}}.$$

Since

$$H'_\beta(\beta, \gamma) = -\frac{\gamma^2 \|u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}}{\left[(p-1)\beta\gamma - \|u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}\right]^2} < 0,$$

namely, $H(\beta, \gamma)$ is decreasing with respect to β. Then we get

$$\min_{(\beta,\gamma)\in\Phi} H(\beta, \gamma) = \min_{(\beta,\gamma)\in\Phi} H(\beta, \gamma)\Big|_{\beta = \frac{\lambda_1(p-1)}{2(p+1)}\|u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})} - J(u_0)} = \min_{\gamma\in\Psi} \widetilde{H}(\gamma), \tag{4.13}$$

where

$$\widetilde{H}(\gamma) = \frac{\gamma^2 \left[\lambda_1(p-1)\|u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})} - 2(p+1)J(u_0)\right]}{\gamma \left[\lambda_1(p-1)^2\|u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})} - 2(p^2-1)J(u_0)\right] - 2(p+1)\|u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}}$$

and

$$\Psi = \left\{\gamma : \frac{2(p+1)\|u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}}{\lambda_1(p-1)^2\|u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})} - 2(p^2-1)J(u_0)} < \gamma < +\infty\right\}.$$

Through simple calculation, it is easy to see that $\widetilde{H}(\gamma)$ achieves its minimum at $\widetilde{\gamma} \in \Psi$, and

$$\widetilde{H}(\widetilde{\gamma}) = \frac{8(p+1)\|u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}}{(p-1)^2\left[\lambda_1(p-1)\|u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})} - 2(p+1)J(u_0)\right]}$$

and

$$\widetilde{\gamma} = \frac{4(p+1)\|u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}}{\lambda_1(p-1)^2\|u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})} - 2(p^2-1)J(u_0)}.$$

Hence, by (4.13), we get

$$T \leq \frac{8(p+1)\|u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})}}{(p-1)^2\left[\lambda_1(p-1)\|u_0\|^2_{\mathcal{L}_2^{n/2}(\mathbb{B})} - 2(p+1)J(u_0)\right]}.$$ □

References

[1] M. Alimohammady, C. Cattani and M. K. Kalleji, *Invariance and existence analysis for semilinear hyperbolic equations with damping and conical singularity*, J. Math. Anal. Appl. **455** (2017), no. 1, 569–591.

[2] M. Alimohammady and M. K. Kalleji, *Existence result for a class of semilinear totally characteristic hypoelliptic equations with conical degeneration*, J Funct. Anal. **265** (2013), no. 10, 2331–2356.

[3] ______, *Blow up property for viscoelastic evolution equations on manifolds with conical degeneration*, (2016). arXiv:1602.02593v1

[4] H. Chen and G. Liu, *Global existence and nonexistence for semilinear parabolic equations with conical degeneration*, J. Pseudo-Differ. Oper. Appl. **3** (2012), no. 3, 329–349.

[5] ______, *Global existence, uniform decay and exponential growth for a class of semilinear wave equation with strong damping*, Acta Math. Sci. Ser. B Engl. Ed. **33** (2013), no. 1, 41–58.

[6] H. Chen and N. Liu, *Asymptotic stability and blow-up of solutions for semi-linear edge-degenerate parabolic equations with singular potentials*, Discrete Contin. Dyn. Syst. **36** (2016), no. 2, 661–682.

[7] H. Chen, X. Liu and Y. Wei, *Existence theorem for a class of semilinear totally characteristic elliptic equations with critical cone Sobolev exponents*, Ann. Global Anal. Geom. **39** (2011), no. 1, 27–43.

[8] ______, *Cone Sobolev inequality and Dirichlet problem for nonlinear elliptic equations on a manifold with conical singularities*, Calc. Var. Partial Differential Equations **43** (2012), no. 3-4, 463–484.

[9] ______, *Dirichlet problem for semilinear edge-degenerate elliptic equations with singular potential term*, J. Differential Equations **252** (2012), no. 7, 4289–4314.

[10] S. Coriasco, E. Schrohe and J. Seiler, *Realizations of differential operators on conic manifolds with boundary*, Ann. Global Anal. Geom. **31** (2007), no. 3, 223–285.

[11] W. Gao and Y. Han, *Blow-up of a nonlocal semilinear parabolic equation with positive initial energy*, Appl. Math. Lett. **24** (2011), no. 5, 784–788.

[12] A. Hao and J. Zhou, *A new blow-up condition for semi-linear edge degenerate parabolic equation with singular potentials*, Appl. Anal. **96** (2017), no. 3, 363–374.

[13] A. A. Jafari and M. Alimohammady, *Existence theorem and global solution for semilinear edge-degenerate hypoelliptic equations*, Accepted in J. Pseudo-Differ. Oper. Appl., (2017), 1–27.

[14] Y. Jian and Z. Yang, *Bounds for the blow-up time and blow-up rate estimates for nonlinear parabolic equations with Dirichlet or Neumann boundary conditions*, British J. Math. Comput. Sci. **12** (2016), no. 2, 1–12.

[15] A. Khelghati and K. Baghaei, *Blow-up phenomena for a nonlocal semilinear parabolic equation with positive initial energy*, Comput. Math. Appl. **70** (2015), no. 5, 896–902.

[16] G. Li, J. Yu and W. Liu, *Global existence, exponential decay and finite time blow-up of solutions for a class of semilinear pseudo-parabolic equations with conical degeneration*, J. Pseudo-Differ. Oper. Appl. **8** (2017), no. 4, 629–660.

[17] Y. Liu, *On potential wells and vacuum isolating of solutions for semilinear wave equations*, J. Differential Equations **192** (2003), no. 1, 155–169.

[18] Y. Liu and R. Xu, *A class of fourth order wave equations with dissipative and nonlinear strain terms*, J. Differential Equations **244** (2008), no. 1, 200–228.

[19] ______, *Global existence and blow up of solutions for Cauchy problem of generalized Boussinesq equation*, Phys. D **237** (2008), no. 6, 721–731.

[20] ______, *Potential well method for Cauchy problem of generalized double dispersion equations*, J. Math. Anal. Appl. **338** (2008), no. 2, 1169–1187.

[21] Y. Liu and J. Zhao, *On potential wells and applications to semilinear hyperbolic equations and parabolic equations*, Nonlinear Anal. **64** (2006), no. 12, 2665–2687.

[22] P. Luo, *Blow-up phenomena for a pseudo-parabolic equation*, Math. Methods Appl. Sci. **38** (2015), no. 12, 2636–2641.

[23] N. Pan, P. Pucci and B. Zhang, *Degenerate Kirchhoff-type hyperbolic problems involving the fractional Laplacian*, Accepted in J. Evol. Equ., (2017), 1–25.

[24] L. E. Payne and P. W. Schaefer, *Bounds for the blow-up time for the heat equation under nonlinear boundary conditions*, Proc. Roy. Soc. Edinburgh Sect. A **139** (2009), no. 6, 1289–1296.

[25] X. Peng, Y. Shang and X. Zheng, *Blow-up phenomena for some nonlinear pseudo-parabolic equations*, Appl. Math. Lett. **56** (2016), 17–22.

[26] E. Schrohe and J. Seiler, *Ellipticity and invertibility in the cone algebra on L_p-Sobolev spaces*, Integral Equations Operator Theory **41** (2001), no. 1, 93–114.

[27] B.-W. Schulze, *Boundary Value Problems and Singular Pseudo-differential Operators*, Pure and Applied Mathematics (New York), John Wiley & SOns, Chichester, 1999.

[28] X. Song and X. Lv, *Bounds for the blowup time and blowup rate estimates for a type of parabolic equations with weighted source*, Appl. Math. Comput. **236** (2014), 78–92.

[29] G. Xu and J. Zhou, *Upper bounds of blow-up time and blow-up rate for a semi-linear edge-degenerate parabolic equation*, Appl. Math. Lett. **73** (2017), 1–7.

[30] ———, *Global existence and blow-up for a fourth order parabolic equation involving the Hessian*, NoDEA Nonlinear Differential Equations Appl. **24** (2017), no. 4, Art. 41, 12 pp.

[31] J. Zhou, *Blow-up for a thin-film equation with positive initial energy*, J. Math. Anal. Appl. **446** (2017), no. 1, 1133–1138.

[32] J. Zhou and D. Yang, *Upper bound estimate for the blow-up time of an evolution m-Laplace equation involving variable source and positive initial energy*, Comput. Math. Appl. **69** (2015), no. 12, 1463–1469.

Guangyu Xu
College of Mathematics and Statistics, Chongqing University, Chongqing, 401331,
P. R. China
E-mail address: guangyuswu@126.com