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Maximal Averages over Certain Non-smooth and Non-convex Hypersurfaces

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Abstract. We consider the maximal operators whose averages are taken over some non-smooth and non-convex hypersurfaces. For each $1 \leq i \leq d-1$, let $\phi_i \colon [-1,1] \to \mathbb{R}$ be a continuous function satisfying some derivative conditions, and let $\phi(y) = \sum_{i=1}^{d-1} \phi_i(y_i)$. We prove the L^p boundedness of the maximal operators associated with the graph of ϕ which is a non-smooth and non-convex hypersurface in $\mathbb{R}^d$, $d \geq 3$.

1. Introduction and statement of results

Let Σ be a hypersurface in $\mathbb{R}^d$, $d \geq 3$. Let η be a compactly supported $C^\infty(\mathbb{R}^d)$ function. For $t > 0$ we define averages T_t by

$$\mathrm{T}_t f(x) = \int_\Sigma f(x - ty)\, d\mu(y)$$

where $d\mu$ is a Borel measure supported in a compact subset $\Sigma_0 = \operatorname{supp}(\eta) \cap \Sigma$. The maximal operator of T_t is defined by

$$\mathrm{T}^* f(x) = \sup_{t>0} |\mathrm{T}_t f(x)|.$$

When Σ is the sphere and $d\mu$ is the spherical measure, E. M. Stein [11, 13] proved that the maximal operator T^* is bounded on $L^p(\mathbb{R}^d)$, $d \geq 3$ if and only if $p > d/(d-1)$. J. Bourgain [1] proved later the analogous result in two dimensions. In addition, various classes of maximal operators associated with sub-varieties have been studied for forty years (see Stein's monograph [12]). It is well known that the estimates of the maximal operators T^* are intimately connected with the decay of the Fourier transform $\widehat{d\mu}$ to estimate an oscillatory integral. These in turn are closely related to geometric properties of the surface Σ. A. Greenleaf [4] proved that T^* is bounded on $L^p(\mathbb{R}^d)$, $d \geq 3$ and $p > d/(d-1)$, provided that Σ is a smooth hypersurface having everywhere non-vanishing

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Gaussian curvature and $d\mu$ is the surface measure. In contrast, the results for the case where the Gaussian curvature vanishes at some points are still open with the exception of the two dimensional case (see [7–10]). For a smooth convex hypersurface of finite type, A. Nagel, A. Seeger, and S. Wainger [9] expressed the decay of the Fourier transform $\widehat{d\mu}$ by using the caps

$$B(x,\delta) := \{y \in \Sigma : \mathrm{dist}(y, H_x(\Sigma)) < \delta\},$$

where $H_x(\Sigma)$ denotes the tangent plane at $x \in \Sigma$. The estimate is

$$|\widehat{d\mu}(\xi)| \leq C\left(|B(x_+, |\xi|^{-1})| + |B(x_-, |\xi|^{-1})|\right),$$

where $x_\pm$ are the points on Σ for which ξ is a normal vector (see also [2]). They obtained sharp results for the maximal operator T^* in higher dimensions $d \geq 3$ by using an L^q-norm of a family of nonisotropic balls on Σ. For example, the maximal operators T^* associated with the surface $\Sigma \subset \mathbb{R}^d$ given as a graph

$$x_d = c + \sum_{i=1}^{d-1} |x_i|^{a_i} \quad (2 \leq a_1 \leq \cdots \leq a_{d-1} \leq d) \tag{1.1}$$

where the a_i are even integers, is bounded on $L^p(\mathbb{R}^d)$, $d \geq 3$ for $p > d/(d-1)$. Based on calculations in [9] on examples of the form (1.1), A. Isoveich, E. Sawyer and A. Seeger in [8] conjectured the below and obtained partial results and drew a fairly complete pictures where $d = 3$.

Conjecture 1.1. *Let Σ be a convex surface in $\mathbb{R}^{d-1}$, and let $P \in \Sigma$ and $\mathfrak{a} = (a_1, \dots, a_{d-1})$ be the multitype at P. Define ν_k by*

$$\nu_k = \sum_{j=k}^{d-1} \frac{1}{a_j}, \quad k = 1, \dots, d-1; \quad \nu_d = 0.$$

The maximal operator T^* is bounded on $L^p(\mathbb{R}^d)$ if the support η is contained in a sufficiently small neighborhood of P and if

$$p > \max_{k=1,\dots,d} \left(\frac{k}{k-1+\nu_k}\right). \tag{1.2}$$

As for non-convex hypersurfaces Σ, A. Ikromov, M. Kempe and D. Müller [5,6] proved the sharp L^p boundedness of the maximal operator T^*, provided that Σ is a smooth hypersurface of finite type on $\mathbb{R}^3$. This study was done under a transversality assumption on the underlying hypersurface Σ, saying that for every point $x_0 \in \Sigma$, the affine tangent plane $x_0 + T_{x_0}\Sigma$ does not pass through the origin (see also [15] for the related results of maximal averages over hypersurfaces not satisfying the transversality condition). In this case the decay estimates of the Fourier transform of surface-carried measure on a smooth convex hypersurface Σ of finite type in [9] fail to be true. Thus they estimated it in terms of Newton polyhedron associated to the given hypersurface (see [6, 14]). In light of the above, one is naturally led to the following question:

Is T^* bounded on $L^p(\mathbb{R}^d)$ for non-smooth and non-convex hypersurfaces?

In this paper we provide estimates which not only give an affirmative answer for certain non-smooth and non-convex hypersurface Σ in $\mathbb{R}^d$, $d \geq 3$, but also reveal how the main theorem is related to the range of p in (1.2) of Conjecture 1.1.

We would also like to emphasize that in the process of proving L^p boundedness of T^*, we do not use any information of the Fourier decay estimates for smooth hypersurfaces in [6, 8, 9]. We rely on microlocal decomposition of the measure in both the space and the frequency variables, and we divide the range of p into finite pieces bases on the singularities of Σ to obtain the desired estimates.

We begin with a definition and some examples to state the main results.

Definition 1.2. Let $I = [\alpha, \beta]$ be a closed interval in $\mathbb{R}$. For each positive integer m, let $t_1, \ldots, t_m$ be arbitrary m distinct points in (α, β). For each positive integer $N \geq 2$, let $\phi \in C^1(I) \cap C^N(I \setminus \{t_1, \ldots, t_m\})$ be a real valued function, then we say that ϕ has type $(b_1, \ldots, b_m; N)$ at $(t_1, \ldots, t_m)$ if there are real numbers $b_i > 1$ ($1 \leq i \leq m$) and positive constants $C_2', C_2, C_3, \ldots, C_N$ such that

(1) $C_2' \left(\prod_{i=1}^m |t - t_i|^{b_i - 2}\right) \leq |\phi''(t)|$,

(2) $|\phi^{(\ell)}(t)| < C_\ell \left(\prod_{i=1}^m |t - t_i|^{-\ell+1}\right)$, $\ell = 2, \ldots, N$,

for all $t \in I \setminus \{t_1, \ldots, t_m\}$. Define $\mathfrak{C}^N(I; t_1, \ldots, t_m; b_1, \ldots, b_m)$ to be the collection of all real valued functions that have type $(b_1, \ldots, b_m; N)$ at $(t_1, \ldots, t_m)$.

1.1. Examples for Definition 1.2

(1) Let $\phi(t) = |t|^a$ for some real number $a > 1$, then for each $\ell \geq 2$ we have

$$|\phi^{(\ell)}(t)| = |a(a-1)\cdots(a-\ell+1)||t|^{-\ell+a} \quad \text{if } t \neq 0. \tag{1.3}$$

Thus $\phi \in \mathfrak{C}^N([-1, 1]; 0; a)$ for any $N \geq 2$. This example shows that Definition 1.2(2) is not optimized. That is to say, the range of p in Theorem 1.3 is not improved even though (1.3) is used in proving Theorem 1.3.

(2) Let ϕ be a polynomial with $\phi''(t) = (t+1)t^2(t-1)^3(t-3)^4$, then for any $N \geq 2$

$$\phi \in \mathfrak{C}^N([-2, 2]; -1, 0, 1; 3, 4, 5) \quad \text{and} \quad \phi \in \mathfrak{C}^N([-2, 4]; -1, 0, 1, 3; 3, 4, 5, 6).$$

(3) Let $a > 1$ and $b > 1$ be real numbers. Let $\phi \in C^1([-1,1])$ be a real valued function such that

$$C_2' \left|t+\frac{1}{2}\right|^{a-2} \left|t-\frac{1}{2}\right|^{b-2} \le |\phi''(t)| \le C_2 \left|t+\frac{1}{2}\right|^{-1} \left|t-\frac{1}{2}\right|^{-1} \quad \text{if } t \neq \pm\frac{1}{2},$$

for some positive constants C_2' and C_2, then $\phi \in \mathfrak{C}^2([-1,1]; -\frac{1}{2}, \frac{1}{2}; a, b)$.

We now proceed to introduce our maximal operators. For each $1 \le i \le d-1$, let $\phi_i\colon [-1,1] \to \mathbb{R}$ be a continuous function, and define $\phi(y) := \sum_{i=1}^{d-1} \phi_i(y_i)$ if $y = (y_1, \ldots, y_{d-1}) \in [-1,1]^{d-1}$. Then the hypersurface Σ is given by the graph

$$\Sigma = \left\{(x, \phi(x)) : x \in [-1,1]^{d-1}\right\}.$$

Suppose that μ is the Borel measure on $\mathbb{R}^d$ given by

$$\mu(F) = \int_{\mathbb{R}^{d-1}} \chi_F(x, \phi(x))\eta(x)\,dx,$$

where η is a positive smooth function supported on $[-1,1]^{d-1}$. Then for each $t > 0$, we define the average T_t associated with the measure μ by

$$\mathrm{T}_t f(x) = \int_\Sigma f(x - tz)\,d\mu(z) = \int_{[-1,1]^{d-1}} f(x - t(y, \phi(y)))\eta(y)\,dy$$

and its maximal operator T^* by

$$\mathrm{T}^* f(x) = \sup_{t>0} |\mathrm{T}_t f(x)|.$$

We are interested in the L^p-boundedness properties of T^*, i.e.,

$$\|\mathrm{T}^* f\|_p \le C\|f\|_p. \tag{1.4}$$

We shall prove the following.

Theorem 1.3. *Let $d \ge 3$ and $N \ge 2$. For each $1 \le i \le d-1$, suppose that $\phi_i \in \mathfrak{C}^N([-1,1]; t_i; a_i)$ for some real number a_i with $1 < a_1 \le \cdots \le a_{d-1}$. Define ν_k by*

$$\nu_k = \sum_{j=k}^{d-1} \frac{1}{a_j}, \quad k = 1, \ldots, d-1; \quad \nu_d = 0. \tag{1.5}$$

Let $0 \le n \le d-1$ be the largest integer so that $1 = a_0 < a_1 \le \cdots \le a_n < 2$. Define

$$p_0 := \max_{k=n+1}^{d} \left(\frac{k}{k-1+\nu_k}\right).$$

If $N \ge 3$ and $\nu_1 > 1/2$, then T^* is bounded on $L^p(\mathbb{R}^d)$ for $p > \max\left(\frac{2N-2}{2N-3}, p_0\right)$. Moreover, if $N \ge 2$ and $\nu_1 \le 1/2$, then T^* is bounded on $L^p(\mathbb{R}^d)$ for $p > 1/\nu_1$.

Remark 1.4. (1) It is well known (see [5,6]) that the behavior of the maximal operator essentially depends on so-called the transversality condition. It means that the affine tangent plane $x+T_x\Sigma$ to Σ through x does not pass through the origin $\mathbb{R}^d$ for every $x \in \Sigma$. In particular, without this condition our result of Theorem 1.3 can not be sharp. See E. Zimmermann [15] for details.

(2) The results of Theorem 1.3 are sharp if we take $\phi(y) = \sum_{j=1}^{d-1} y_j^{a_j} + c$ $(c \neq 0)$ where $2 \le a_1 \le \cdots \le a_{d-1}$ (a_i positive even integer) as in [9].

(3) Let $N \ge 3$. For each $1 \le i \le d-1$ if $\phi_i \in \mathfrak{C}^N([-1,1]; t_i; a_i)$ and $1 < a_1 \le \cdots \le a_{d-1} \le d$, then T^* is bounded on $L^p(\mathbb{R}^d)$, $d \ge 3$ for $p > \max\left(\frac{2N-2}{2N-3}, \frac{d}{d-1}\right)$.

(4) When N is sufficiently large in the condition of above (3), we see that the maximal operator T^* is bounded on $L^p(\mathbb{R}^d)$ for $p > \max\left(\frac{2N-2}{2N-3}, \frac{d}{d-1}\right) = \frac{d}{d-1}$. In particular if $\phi(x_0) = \nabla\phi(x_0) = 0$ and $\det\left[\frac{\partial^2\phi}{\partial_{x_i}\partial_{x_j}}\right](x_0) \neq 0$, then by Morse's lemma there exists a diffeomorphism from a small neighborhood of x_0 in the x-space, to a neighborhood of the origin in the y-space, under which ϕ is transformed into

$$\sum_{j=1}^{m} y_j^2 - \sum_{j=m+1}^{d-1} y_j^2$$

for some $0 \le m \le d-1$. Thus this recovers a result of A. Greenleaf [4] if Σ is a smooth hypersurface in $\mathbb{R}^d$, $d \ge 3$ and Σ has everywhere non-vanishing Gaussian curvature.

Corollary 1.5. *Let $d \ge 3$ and $N \ge 2$. For each $1 \le i \le d-1$, suppose that*

$$\phi_i \in \mathfrak{C}^N([-1,1]; t_{i,1}, \ldots, t_{i,m_i}; b_{i,1}, \ldots, b_{i,m_i}) \quad \textit{and} \quad a_i := \max(b_{i,1}, \ldots, b_{i,m_i}).$$

Without loss of generality we assume $1 < a_1 \le \cdots \le a_{d-1}$. Let ν_k be as in (1.5) and let $0 \le n \le d-1$ be the largest integer so that $1 = a_0 < a_1 \le \cdots \le a_n < 2$. If $N \ge 3$ and $\nu_1 > 1/2$ then T^* is bounded on $L^p(\mathbb{R}^d)$ for

$$p > \max\left(\frac{2N-2}{2N-3}, \max_{k=n+1}^{d}\left(\frac{k}{k-1+\nu_k}\right)\right).$$

In addition, if $N \ge 2$ and $\nu_1 \le 1/2$, then T^* is bounded on $L^p(\mathbb{R}^d)$ for $p > 1/\nu_1$.

Proof. By using partitions of unity we dominate $\mathrm{T}^*f(x)$ as a finite sum of maximal operators $\mathrm{T}^*_{j_1,\ldots,j_{d-1}}(|f|)(x)$ associated with measures

$$\mu_{j_1,\ldots,j_{d-1}}(F) = \int_{\mathbb{R}^{d-1}} \chi_F(x, \phi(x))\eta_{j_1,\ldots,j_{d-1}}(x)\,dx,$$

where $\eta_{j_1,\dots,j_{d-1}}$ is a positive smooth function supported in $I_{j_1}\times\cdots\times I_{j_{d-1}}\subset[-1,1]^{d-1}$ for some closed intervals $I_{j_i}\subset[-1,1]$, and $\varphi_i\in\mathfrak{C}^N(I_{j_i};t_{i,j_i};b_{i,j_i})$. Next let $b_{1,j_1},\dots,b_{d-1,j_{d-1}}$ be one of the $(d-1)$-tuples appearing in the definition of the maximal operators $\mathrm{T}^*_{j_1,\dots,j_{d-1}}$, then we have $b_{i,j_i}\le a_i$. Let $a'_1,\dots,a'_{d-1}$ be the increasing rearrangement of $b_{1,j_1},\dots,b_{d-1,j_{d-1}}$ such that $1<a'_1\le\cdots\le a'_l<2\le a'_{l+1}\le\cdots\le a'_{d-1}$, then $a'_i\le a_i$ and $n\le l$. Define

$$\nu'_k=\sum_{j=k}^{d-1}\frac{1}{a_j},\quad k=1,\dots,d-1;\quad \nu'_d=0,$$

then $\nu_k\le\nu'_k$ and so we have

$$\frac{1}{\nu_1}\ge\frac{1}{\nu'_1}\quad\text{and}\quad\max_{k=n+1}^{d}\left(\frac{k}{k-1+\nu_k}\right)\ge\max_{k=n+1}^{d}\left(\frac{k}{k-1+\nu'_k}\right).$$

Hence by applying Theorem 1.3 for the maximal operators $\mathrm{T}^*_{j_1,\dots,j_{d-1}}$, we obtain the desired result. □

Notation. The Fourier transform of f is denoted by $\widehat{f}$. For each positive real numbers A and B, we will use $A\lesssim B$ to denote the estimate $A\le CB$, where C is a constant depending only on the dimension d. We will denote $A\sim B$ if $A\lesssim B$ and $B\lesssim A$.

2. Proof of Theorem 1.3: maximal averages over the interval $E=[1,2]$

In order to prove (1.4), we dominate $|\eta(x)|$ by $\eta_1(x_1)\cdots\eta_{d-1}(x_{d-1})$ for some positive smooth functions η_i $(1\le i\le d-1)$ supported in $[-1,1]$. We also note that

$$|\mathrm{T}_tf(x)|\le\int_{[-1,1]^{d-1}}|f(x-t(y,\phi(y)))|\,\eta_1(y_1)\cdots\eta_{d-1}(y_{d-1})\,dy.$$

From now on we may assume that $\eta(x)=\eta_1(x_1)\cdots\eta_{d-1}(x_{d-1})$ for further discussions. For each subset $E\subset\mathbb{R}^+$, we define the maximal operator by

$$\mathrm{T}_Ef(x):=\sup_{t\in E}|\mathrm{T}_tf(x)|$$

and then this implies $\mathrm{T}^*f=\mathrm{T}_{(0,\infty)}f$. First we will prove the L^p boundedness of the maximal operator T_E with $E=[1,2]$, and the general case T_E with $E=(0,\infty)$ will follow by the argument of M. Christ as in [3] (see Section 3).

2.1. The case $E=[1,2]$ and $\nu_1>1/2$

Choose $\psi_0\in C^\infty(\mathbb{R})$ such that

$$\widehat{\psi_0}(\xi)=\begin{cases}1 & \text{if } |\xi|\le1,\\ 0 & \text{if } |\xi|\ge2.\end{cases}$$

Let $\widehat{\psi}(\xi) = \widehat{\psi}_0(\xi) - \widehat{\psi}_0(2\xi)$ then $\widehat{\psi}$ is supported in $\{1/2 < |\xi| < 2\}$ and

$$1 = \widehat{\psi}_0(\xi) + \sum_{j=1}^{\infty} \widehat{\psi}(2^{-j}\xi) \quad \text{for all } \xi.$$

Now we have

$$(\mathrm{T}_t f)^\wedge(\xi) = \widehat{f}(\xi)\widehat{\mu}(t\xi)\left(\widehat{\psi}_0(\xi) + \sum_{j=1}^{\infty} \widehat{\psi}(2^{-j}\xi)\right) =: (\mathrm{T}_t^0 f)^\wedge(\xi) + \sum_{j=1}^{\infty} (\mathrm{T}_t^j f)^\wedge(\xi).$$

It is easy to see that

$$\sup_{t\in E} |\mathrm{T}_t^0 f(x)| \le C\mathrm{M}f(x)$$

where M denotes the Hardy-Littlewood maximal operator. Hence for each $1 < p \le \infty$ we have

$$\|\mathrm{T}_E f\|_p \le C\|f\|_p + \sum_{j=1}^{\infty} \left\| \sup_{t\in E} |\mathrm{T}_t^j f| \right\|_p.$$

Let φ be a $C^\infty(\mathbb{R})$ function that is supported in $\{1/2 < |s| < 2\}$ and

$$\sum_{k=1}^{\infty} \varphi(2^k s) = 1 \quad \text{for all } s \in [-1,1] \setminus \{0\}.$$

Then we have

$$\begin{aligned} \mathrm{T}_t^j f(x, x_d) &= \sum_{\vec{k}} \int_{\mathbb{R}^{d-1}} f * \psi_j(x - ty, x_d - t\phi(y)) \prod_{i=1}^{d-1} \Big(\varphi(2^{k_i}(y_i - t_i))\eta_i(y_i)\Big)\, dy \\ &:= \sum_{\vec{k}} \int_{\mathbb{R}^d} f * \psi_j(x - ty, x_d - ty_d)\, d\mu_{\vec{k}}(y, y_d) \\ &:= \sum_{\vec{k}} \mathrm{T}_t^{j,\vec{k}} f(x, x_d) \end{aligned}$$

where $\psi_j(\cdot) = 2^{jd}\psi(2^j\cdot)$ and $\vec{k} = (k_1, \ldots, k_{d-1}) \in (\mathbb{Z}^+)^{d-1}$.

Lemma 2.1 (van der Corput's Lemma)**.** *Suppose ω is real-valued and $\omega \in C^k(a,b)$, and that $|\omega^{(k)}(t)| \ge 1$ for all $t \in (a,b)$. Then we have*

$$\left| \int_a^b \mathrm{e}^{\mathrm{i}\lambda\omega(t)}\eta(t)\, dt \right| \le c_k \lambda^{-1/k} \left(|\eta(b)| + \int_a^b |\eta'(t)|\, dt \right)$$

when $k \ge 2$ or $k = 1$ and $\omega'(t)$ is monotonic. The bound c_k is independent of ω and λ.

Lemma 2.2. *Let $2 \leq N \in \mathbb{Z}^+$ and $\phi_i \in \mathfrak{C}^N([-1,1];t_i;a_i)$, $i = 1,\dots,d-1$. Then for each $1 \leq k_1,\dots,k_{d-1} < \infty$ and multi-index α we have*

(2.1)

$$|\partial_\xi^\alpha \widehat{\mu_{\vec{k}}}(\xi)| \leq \begin{cases} C_\alpha 2^{-k_1-\cdots-k_{d-1}} \prod_{i=1}^{d-1} \min\left(|\xi|^{-\frac{1}{2}} 2^{\frac{a_i}{2}k_i}, 1\right) & \text{if } |\xi_d| \geq |(\xi_1,\dots,\xi_{d-1})|, \\ C_\alpha 2^{-k_1-\cdots-k_{d-1}} \min\left(|\xi|^{-(N-1)} 2^{k_i(N-1)}, 1\right) & \text{if } |\xi_d| \leq |(\xi_1,\dots,\xi_{d-1})| \sim |\xi_i| \text{ for some } i. \end{cases}$$

Hence we have

$$|\partial_\xi^\alpha \widehat{\mu_{\vec{k}}}(\xi)| \leq C_\alpha 2^{-k_1-\cdots-k_{d-1}} \left(\prod_{i=1}^{d-1} \min\left(|\xi|^{-\frac{1}{2}} 2^{\frac{a_i}{2}k_i}, 1\right) + \sum_{i=1}^{d-1} \min\left(|\xi|^{-(N-1)} 2^{k_i(N-1)}, 1\right) \right).$$

Proof. For the proof we use van der Corput's lemma (see [12, pp. 332–334]). It suffices to show the case $\alpha = 0$, since the other cases are similar. Note that

$$|\widehat{\mu_{\vec{k}}}(\xi)| = \prod_{i=1}^{d-1} \left| \int_{\mathbb{R}} \mathrm{e}^{-2\pi \mathrm{i}(\xi_i y + \xi_d \phi_i(y))} \varphi(2^{k_i}(y-t_i)) \eta_i(y)\, dy \right|, \quad \xi = (\xi_1,\dots,\xi_{d-1},\xi_d).$$

By Definition 1.2(1), for each $1 \leq i \leq d-1$, we have

$$\left|\partial_y^2 (\xi_i y + \xi_d \phi_i(y))\right| \gtrsim |\xi_d| |y - t_i|^{a_i - 2}.$$

Hence by Lemma 2.1 with $k = 2$

$$\left| \int_{\mathbb{R}} \mathrm{e}^{-2\pi \mathrm{i}(\xi_i y + \xi_d \phi_i(y))} \varphi(2^{k_i}(y-t_i)) \eta_i(y)\, dy \right| \leq C |\xi_d|^{-\frac{1}{2}} 2^{\frac{a_i - 2}{2} k_i}. \tag{2.2}$$

If $|\xi_i| \gtrsim |\xi_d|$ for some $1 \leq i \leq d-1$, then since $\phi_i \in C^1([-1,1])$

$$|\xi_i + \xi_d \phi_i'(y)| \gtrsim |\xi_i|.$$

Hence if we integrate by parts $(N-1)$ times, by using the identity

$$\mathrm{e}^{-2\pi \mathrm{i}(\xi_i y + \xi_d \phi_i(y))} = \frac{1}{-2\pi \mathrm{i}(\xi_i + \xi_d \phi_i'(y))} \frac{d}{dy} \left(\mathrm{e}^{-2\pi \mathrm{i}(\xi_i y + \xi_d \phi_i(y))} \right)$$

together with Definition 1.2(2), if $|\xi_i| \gtrsim |\xi_d|$ then we have

$$\left| \int_{\mathbb{R}} \mathrm{e}^{-2\pi \mathrm{i}(\xi_i y + \xi_d \phi_i(y))} \varphi(2^{k_i}(y-t_i)) \eta_i(y)\, dy \right| \leq C |\xi_i|^{-(N-1)} 2^{k_i(N-2)}. \tag{2.3}$$

The proof of (2.1) follows from (2.2) and (2.3) together with the trivial estimates

$$\left| \int_{\mathbb{R}} \mathrm{e}^{-2\pi \mathrm{i}(\xi_i y + \xi_d \phi_i(y))} \varphi(2^{k_i}(y-t_i)) \eta_i(y)\, dy \right| \leq C 2^{-k_i}.$$

□

Lemma 2.3. *For each $j \in \mathbb{Z}^+$ and $\vec{k} = (k_1, \dots, k_{d-1}) \in (\mathbb{Z}^+)^{d-1}$ define*

$$\mathrm{B}(j, \vec{k}) := \sup_{|\alpha| \le 1} \sup_{\xi \in \mathbb{R}^d} \left| \widehat{\psi}(2^{-j}\xi) \partial_\xi^\alpha \widehat{\mu_{\vec{k}}}(\xi) \right|.$$

Then for each $t, t' \in [1,2]$, we have the following:

(1) $\mathrm{B}(j, \vec{k}) \le C\, 2^{-k_1 - \cdots - k_{d-1}} \left(\prod_{i=1}^{d-1} \left(2^{\min(-\frac{j}{2} + \frac{a_i}{2} k_i, 0)} \right) + \sum_{i=1}^{d-1} 2^{(N-1)\min(-j + k_i, 0)} \right).$

(2) *For $1 \le p \le 2$,*

$$\left\| \mathrm{T}_t^{j,\vec{k}} f \right\|_p \le C (2^{-k_1 - \cdots - k_{d-1}})^{\frac{2}{p} - 1} \mathrm{B}(j, \vec{k})^{2 - \frac{2}{p}} \|f\|_p,$$
$$\left\| \frac{d}{dt} \mathrm{T}_t^{j,\vec{k}} f \right\|_p \le C (2^{-k_1 - \cdots - k_{d-1}})^{\frac{2}{p} - 1}\, 2^j\, \mathrm{B}(j, \vec{k})^{2 - \frac{2}{p}} \|f\|_p.$$

(3) *For $2 \le p < \infty$,*

$$\left\| \mathrm{T}_t^{j,\vec{k}} f \right\|_p \le C (2^{-k_1 - \cdots - k_{d-1}})^{1 - \frac{2}{p}}\, \mathrm{B}(j, \vec{k})^{2/p} \|f\|_p,$$
$$\left\| \frac{d}{dt} \mathrm{T}_t^{j,\vec{k}} f \right\|_p \le C (2^{-k_1 - \cdots - k_{d-1}})^{1 - \frac{2}{p}}\, 2^j\, \mathrm{B}(j, \vec{k})^{2/p} \|f\|_p.$$

Proof. (1) follows from Lemma 2.2. Recall that

$$\mathrm{T}_t^{j,\vec{k}} f(x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(z)\, 2^{jd} \psi(2^j(x - ty - z))\, d\mu_{\vec{k}}(y)\, dz,$$

and it is easy to see that

$$\begin{aligned} &\left\| \mathrm{T}_t^{j,\vec{k}} f \right\|_1 \le C\, 2^{-k_1 - \cdots - k_{d-1}} \|f\|_1, && \left\| \frac{d}{dt} \mathrm{T}_t^{j,\vec{k}} f \right\|_1 \le C\, 2^j\, 2^{-k_1 - \cdots - k_{d-1}} \|f\|_1, \\ &\left\| \mathrm{T}_t^{j,\vec{k}} f \right\|_\infty \le C\, 2^{-k_1 - \cdots - k_{d-1}} \|f\|_\infty, && \left\| \frac{d}{dt} \mathrm{T}_t^{j,\vec{k}} f \right\|_\infty \le C\, 2^j\, 2^{-k_1 - \cdots - k_{d-1}} \|f\|_\infty. \end{aligned} \tag{2.4}$$

Also note that

$$\mathrm{T}_t^{j,\vec{k}} f(x) = \int_{\mathbb{R}^d} \mathrm{e}^{2\pi \mathrm{i} x \cdot \xi}\, \widehat{f}(\xi)\, \widehat{\psi}(2^{-j}\xi)\, \widehat{\mu_{\vec{k}}}(t\xi)\, d\xi,$$

and by Plancherel's identity we have

$$\left\| \mathrm{T}_t^{j,\vec{k}} f \right\|_2 \le C\, \mathrm{B}(j, \vec{k}) \|f\|_2, \quad \left\| \frac{d}{dt} \mathrm{T}_t^{j,\vec{k}} f \right\|_2 \le C\, 2^j\, \mathrm{B}(j, \vec{k}) \|f\|_2. \tag{2.5}$$

By interpolating (2.4) and (2.5) we have (2) and (3). □

Lemma 2.4. *Suppose that $S \in C^1([1,2])$. Let $t_0 = 1 < t_1 < \cdots < t_m = 2$ be a partition of $[1,2]$ with $|t_{i+1} - t_i| \sim 2^{-j}$ for $i = 0, 1, \dots, m-1$. Then for $1 \le p < \infty$,*

$$\sup_{t \in [1,2]} |S(t)| \lesssim \left(\sum_{i=0}^{m-1} |S(t_i)|^p \right)^{1/p} + 2^{-j/p'} \left(\int_1^2 |S'(u)|^p\, du \right)^{1/p}.$$

Proof. For each $t \in [t_i, t_{i+1}]$, we have $S(t) = S(t_i) + \int_{t_i}^t S'(u)\,du$. And by Hölder's inequality for $1 \le p < \infty$

$$|S(t)| \le |S(t_i)| + |t_{i+1} - t_i|^{1/p'} \left(\int_{t_i}^{t_{i+1}} |S'(u)|^p \, du \right)^{1/p},$$

and so

$$\begin{aligned} \sup_{t \in [1,2]} |S(t)| &\lesssim \sup_i |S(t_i)| + 2^{-j/p'} \left(\int_1^2 |S'(u)|^p \, du \right)^{1/p}, \\ &\lesssim \left(\sum_i |S(t_i)|^p \right)^{1/p} + 2^{-j/p'} \left(\int_1^2 |S'(u)|^p \, du \right)^{1/p}. \end{aligned}$$

□

Now we proceed to prove Theorem 1.3 for the case of $E = [1, 2]$. By Lemma 2.4, for $1 \le p < \infty$ we have

$$\left\| \sup_{t \in [1,2]} |\mathrm{T}_t^{j,\vec{k}} f| \right\|_p \le \left(\sum_i \|\mathrm{T}_{t_i}^{j,\vec{k}} f\|_p^p \right)^{1/p} + 2^{-j/p'} \left(\int_1^2 \left\| \frac{d}{dt} \mathrm{T}_t^{j,\vec{k}} f \right\|_p^p \, dt \right)^{1/p}. \tag{2.6}$$

By Lemma 2.3(2), for $1 \le p \le 2$, the right-hand side of (2.6) is dominated by

$$C \left[\sum_{\vec{k}} 2^{j/p} (2^{-k_1 - \cdots - k_{d-1}})^{\frac{2}{p} - 1} B(j, \vec{k})^{2 - \frac{2}{p}} \right] \|f\|_p \le C[\mathrm{I}(j) + \mathrm{II}(j)] \, \|f\|_p,$$

where

$$\begin{aligned} \mathrm{I}(j) &:= 2^{j/p} \sum_{\vec{k}} (2^{-k_1 - \cdots - k_{d-1}}) \prod_{i=1}^{d-1} \left(2^{\min(-\frac{j}{2} + \frac{a_i}{2} k_i, 0)} \right)^{2 - \frac{2}{p}}, \\ \mathrm{II}(j) &:= 2^{j/p} \sum_{i=1}^{d-1} \sum_{\vec{k}} (2^{-k_1 - \cdots - k_{d-1}}) \left(2^{(N-1) \min(-j + k_i, 0)} \right)^{2 - \frac{2}{p}}. \end{aligned}$$

Note that

$$\mathrm{II}(j) = 2^{j/p} \sum_{i=1}^{d-1} \sum_{k_i} \left(2^{-k_i} \prod_{\ell \ne i} \left(\sum_{k_\ell} 2^{-k_\ell} \right) \right) \left(2^{(N-1) \min(-j + k_i, 0)} \right)^{2 - \frac{2}{p}}.$$

Hence if

$$p > \frac{2N - 2}{2N - 3} \iff (N - 1) \left(2 - \frac{2}{p} \right) > 1, \tag{2.7}$$

then

$$\mathrm{II}(j) \le C \, 2^{j/p} \sum_{i=1}^{d-1} \sum_{k_i} 2^{-k_i} \left(2^{(N-1) \min(-j + k_i, 0)} \right)^{2 - \frac{2}{p}}$$

$$\le C\,2^{j/p}\sum_{i=1}^{d-1}\left(\sum_{k_i\ge j}2^{-k_i}+\sum_{k_i<j}2^{-j(N-1)(2-\frac{2}{p})}2^{k_i(N-1)(2-\frac{2}{p})-k_i}\right)$$
$$\le C\,2^{j(\frac{1}{p}-1)},$$

and $\sum_{j=1}^{\infty}\mathrm{II}(j)<\infty$. Next we estimate $\mathrm{I}(j)$. Note that

$$\mathrm{I}(j)=2^{j/p}\prod_{i=1}^{d-1}\left(\sum_{k_i=1}^{\infty}(2^{-k_i})2^{\min(-\frac{j}{2}+\frac{a_i}{2}k_i,0)(2-\frac{2}{p})}\right). \tag{2.8}$$

For every $1\le i\le d-1$ by considering k_i in two cases $k_i\le j/a_i$ and $k_i>j/a_i$ we obtain that

$$\sum_{k_i=1}^{\infty}(2^{-k_i})2^{\min(-\frac{j}{2}+\frac{a_i}{2}k_i,0)(2-\frac{2}{p})}\lesssim 2^{-\frac{j}{a_i}}+2^{-j(1-\frac{1}{p})}\lesssim\begin{cases}2^{-\frac{j}{a_i}} & \text{if } p\ge\frac{a_i}{a_i-1},\\ 2^{-j(1-\frac{1}{p})} & \text{if } p<\frac{a_i}{a_i-1}.\end{cases} \tag{2.9}$$

In particular when $1\le i\le n$ since $a_i<2$, we have $p\le 2<\frac{a_i}{a_i-1}$. Hence by (2.9)

$$\sum_{k_i=1}^{\infty}(2^{-k_i})2^{\min(-\frac{j}{2}+\frac{a_i}{2}k_i,0)(2-\frac{2}{p})}\le C\,2^{-j(1-\frac{1}{p})}\quad\text{for } 1\le i\le n. \tag{2.10}$$

Note that

$$1<\frac{a_{d-1}}{a_{d-1}-1}\le\frac{a_{d-2}}{a_{d-2}-1}\le\cdots\le\frac{a_{n+1}}{a_{n+1}-1}<2. \tag{2.11}$$

We consider $1<p\le 2$ as a union of $(d-n)$ subintervals

$$1<p\le\frac{a_{d-1}}{a_{d-1}-1},\quad \frac{a_l}{a_\ell-1}<p\le\frac{a_{\ell-1}}{a_{\ell-1}-1},\ \ell=n+2,\dots,d-1,\quad \frac{a_{n+1}}{a_{n+1}-1}<p\le 2.$$

And for each interval, we estimate $\mathrm{I}(j)$ by using (2.8), (2.9), (2.10) and (2.11).

Case 1: $1<p\le\frac{a_{d-1}}{a_{d-1}-1}$.

$$\mathrm{I}(j)\le C\,2^{j(\frac{d}{p}-(d-1))}$$

and the series converges when

$$p\in\mathcal{F}_d:=\left\{p:1<p\le\frac{a_{d-1}}{a_{d-1}-1}\text{ and } p>\frac{d}{d-1}\right\}. \tag{2.12}$$

Case 2: $\frac{a_\ell}{a_\ell-1}<p\le\frac{a_{\ell-1}}{a_{\ell-1}-1}$, $n+2\le\ell\le d-1$.

$$\mathrm{I}(j)\le C\,2^{j/p}2^{-j(1-\frac{1}{p})n}2^{-j(1-\frac{1}{p})(\ell-n-1)}2^{-\nu_\ell j}\le C\,2^{j(\frac{\ell}{p}-(\ell-1+\nu_\ell))}$$

and the series converges when

$$p\in\mathcal{F}_\ell:=\left\{p:\frac{a_\ell}{a_\ell-1}<p\le\frac{a_{\ell-1}}{a_{\ell-1}-1}\text{ and } p>\frac{\ell}{\ell-1+\nu_\ell}\right\}. \tag{2.13}$$

Case 3: $\frac{a_{n+1}}{a_{n+1}-1} < p \le 2$.

$$\mathrm{I}(j) \le C\, 2^{j/p} 2^{-j(1-\frac{1}{p})n} 2^{-\nu_{n+1} j} \le C\, 2^{j(\frac{n+1}{p} - (n+\nu_{n+1}))}$$

and the series converges when

$$p \in \mathcal{F}_{n+1} := \left\{ p : \frac{a_{n+1}}{a_{n+1}-1} < p \le 2 \text{ and } p > \frac{n+1}{n+\nu_{n+1}} \right\}. \tag{2.14}$$

Hence from (2.12), (2.13) and (2.14) the series $\mathrm{I}(j)$ converges if

$$p \in \bigcup_{k=n+1}^{d} \mathcal{F}_k.$$

Lemma 2.5. *Let $\nu_1 > 1/2$. For each $n+1 \le k \le d$, let $\mathcal{F}_k$ be as in* (2.12), (2.13) *and* (2.14)*. Then*

$$\bigcup_{k=n+1}^{d} \mathcal{F}_k = \left\{ p : \max_{k=n+1}^{d} \left(\frac{k}{k-1+\nu_k} \right) < p \le 2 \right\}. \tag{2.15}$$

For the moments we assume Lemma 2.5, then the series $\mathrm{I}(j)$ converges if

$$p > p_0 := \max_{k=n+1}^{d} \left(\frac{k}{k-1+\nu_k} \right),$$

and so by (2.7) the series $\mathrm{I}(j) + \mathrm{II}(j)$ converges if

$$\max \left(p_0, \frac{2N-2}{2N-3} \right) < p \le 2, \tag{2.16}$$

which is the desired estimate for the case $E = [1,2]$ and $\nu_1 > 1/2$.

Proof of Lemma 2.5. Let

$$\max_{k=n+1}^{d} \left(\frac{k}{k-1+\nu_k} \right) = \frac{\ell}{\ell-1+\nu_\ell}$$

for some $n+1 \le \ell \le d$, then we have

$$\frac{\ell}{\ell-1+\nu_\ell} \ge \frac{j}{j-1+\nu_j} \quad \text{for all } j = n+1, \dots, d.$$

This condition is equivalent to

$$\ell(\nu_j - 1) \ge j(\nu_\ell - 1) \quad \text{for all } j = n+1, \dots, d. \tag{2.17}$$

We claim that

$$\mathcal{F}_j = \emptyset \quad \text{for all } \ell < j \le d. \tag{2.18}$$

To see this, by applying (2.17) with $j > \ell$, we have

$$\ell(\nu_j - 1) \geq j(\nu_\ell - 1) = j\left(\frac{1}{a_\ell} + \cdots + \frac{1}{a_{j-1}} + \nu_j - 1\right) \geq j\left(\frac{j-\ell}{a_{j-1}} + \nu_j - 1\right)$$

and so

$$a_{j-1}(1 - \nu_j) \geq j \quad \text{if } j > \ell.$$

This is equivalent to

$$\frac{a_{j-1}}{a_{j-1} - 1} \leq \frac{j}{j - 1 + \nu_j},$$

and so we have $\mathcal{F}_j = \emptyset$ for $\ell < j \leq d$.

Case 1*:* $\ell = n+1$. By applying (2.17) with $j = n+2$, we have

$$a_{n+1}(1 - \nu_{n+1}) \geq n + 1$$

and this is equivalent to

$$\frac{a_{n+1}}{a_{n+1} - 1} \leq \frac{n+1}{n + \nu_{n+1}}.$$

Hence we have

$$\mathcal{F}_{n+1} = \left\{p : \frac{n+1}{n + \nu_{n+1}} < p \leq 2\right\} \neq \emptyset$$

and by (2.18) we have $\bigcup_{k=n+1}^{d} \mathcal{F}_k = \mathcal{F}_{n+1}$ and so we have (2.15).

Case 2*:* $n+2 \leq \ell \leq d-1$. We will show that

$$\mathcal{F}_{n+1} = \left\{p : \frac{a_{n+1}}{a_{n+1} - 1} < p \leq 2\right\}, \tag{2.19}$$

$$\mathcal{F}_j = \left\{p : \frac{a_j}{a_j - 1} < p \leq \frac{a_{j-1}}{a_{j-1} - 1}\right\} \quad \text{if } n+2 \leq j < \ell, \tag{2.20}$$

and

$$\mathcal{F}_\ell = \left\{p : \frac{\ell}{\ell - 1 + \nu_\ell} < p \leq \frac{a_{\ell-1}}{a_{\ell-1} - 1}\right\}. \tag{2.21}$$

Then from the condition

$$\frac{a_{d-1}}{a_{d-1} - 1} \leq \frac{a_{d-2}}{a_{d-2} - 1} \leq \cdots \leq \frac{a_n}{a_n - 1},$$

we have

$$\bigcup_{j=n+1}^{d} \mathcal{F}_j = \left\{p : \frac{\ell}{\ell - 1 + \nu_\ell} < p \leq 2\right\} \neq \emptyset.$$

By applying (2.17) with $n+1 \leq j < \ell$, we have

$$\ell(\nu_j - 1) \geq j(\nu_\ell - 1) = j\left(-\frac{1}{a_j} - \cdots - \frac{1}{a_{\ell-1}} + \nu_j - 1\right).$$

From this we have

$$(\ell - j)(1-\nu_j) \le j\left(\frac{1}{a_j} + \cdots + \frac{1}{a_{\ell-1}}\right) \le j\frac{(\ell-j)}{a_j}$$

and so $a_j(1-\nu_j) \le j$. This is equivalent to

$$\frac{j}{j-1+\nu_j} \le \frac{a_j}{a_j - 1}$$

and we have (2.19) and (2.20). To show (2.21), by applying (2.17) with $j = \ell - 1$ and $j = \ell + 1$ we have

$$a_\ell(1-\nu_\ell) \ge \ell \quad \text{and} \quad a_{\ell-1}(1-\nu_\ell) \le \ell.$$

These are equivalent to

$$\frac{a_\ell}{a_\ell - 1} \le \frac{\ell}{\ell - 1 + \nu_\ell} \le \frac{a_{\ell-1}}{a_{\ell-1}-1}$$

and we have (2.21).

Case 3*:* $\ell = d$. By applying (2.17) with $j = d-1$ we have $a_{d-1} \le d$, hence $\frac{d}{d-1} \le \frac{a_{d-1}}{a_{d-1}-1}$ and

$$\mathcal{F}_d = \left\{p : \frac{d}{d-1} < p \le \frac{a_{d-1}}{a_{d-1}-1}\right\}.$$

And by (2.20) we have

$$\bigcup_{k=n+1}^{d} \mathcal{F}_k = \left\{p : \frac{d}{d-1} < p \le 2\right\} \ne \emptyset. \qquad \square$$

2.2. The case $E = [1,2]$ and $\nu_1 \le 1/2$

As in Section 2.1, for each $2 \le p < \infty$, the right-hand side of (2.6) is dominated by

$$C\left[2^{j/p}\sum_{\vec{k}}(2^{-k_1-\cdots-k_{d-1}})^{1-\frac{2}{p}}\,\mathrm{B}(j,\vec{k})^{2/p}\right]\|f\|_p. \tag{2.22}$$

By applying Lemma 2.3(1) with $N = 2$ we have

$$(2.22) \le C\big[\mathrm{I}(j) + \mathrm{II}(j)\big]\,\|f\|_p$$

where

$$\mathrm{I}(j) = 2^{j/p}\sum_{i=1}^{d-1}\sum_{\vec{k}}(2^{-k_1-\cdots-k_{d-1}})\left(2^{\min(-j+k_i,0)}\right)^{2/p},$$

$$\mathrm{II}(j) = 2^{j/p}\sum_{\vec{k}}(2^{-k_1-\cdots-k_{d-1}})\prod_{i=1}^{d-1}\left(2^{\min(-\frac{j}{2}+\frac{a_i}{2}k_i,0)}\right)^{2/p}.$$

For $p > 2$, it is easy to see that

$$\mathrm{I}(j) \le C\, 2^{-j/p}. \tag{2.23}$$

Note that

$$\mathrm{II}(j) = 2^{j/p} \prod_{i=1}^{d-1} \left(\sum_{k_i=1}^{\infty} (2^{-k_i})\, 2^{\min(-\frac{j}{2}+\frac{a_i}{2}k_i,0)\frac{2}{p}} \right).$$

By considering k_i in two cases $k_i \le j/a_i$ and $k_i > j/a_i$, we have

$$\sum_{k_i=1}^{\infty} (2^{-k_i}) 2^{\min(-\frac{j}{2}+\frac{a_i}{2}k_i,0)\frac{2}{p}} \le \begin{cases} C\, 2^{-\frac{j}{a_i}} & \text{if } p \le a_i, \\ C\, 2^{-\frac{j}{p}} & \text{if } p > a_i. \end{cases} \tag{2.24}$$

Since $d \ge 3$ we have $1/\nu_1 < a_1$. Hence by using the estimates (2.24) with $2 < 1/\nu_1 < p < a_1$ we have

$$\mathrm{II}(j) \le C\, 2^{j(\frac{1}{p}-\nu_1)}. \tag{2.25}$$

By (2.23) and (2.25), for each $1/\nu_1 < p < a_1$ we have

$$\mathrm{I}(j) + \mathrm{II}(j) \le C \left(2^{j(\frac{1}{p}-\nu_1)} + 2^{-\frac{j}{p}} \right)$$

and the series converges when

$$\frac{1}{\nu_1} < p < a_1.$$

3. Proof of Theorem 1.3: maximal averages over the interval $E = (0, \infty)$

For the general case $E = (0, \infty)$, the L^p boundedness of the maximal operator T_E follows by the argument of M. Christ as in [3].

3.1. The general case $E = (0, \infty)$ and $\nu_1 > 1/2$

Let $E = (0, \infty)$ and define $E_l = [2^{-l}, 2^{-l+1}]$. Then we have

$$\mathrm{T}_E f(x) = \sup_{l \in \mathbb{Z}} |\mathrm{T}_{E_l} f(x)|.$$

And for each $t \in E_l$, we write

$$(\mathrm{T}_t f)^\wedge(\xi) = \widehat{f}(\xi) \widehat{\mu}(t\xi) \left(\widehat{\psi}_0(2^{-l}\xi) + \sum_{j \ge 1} \widehat{\psi}(2^{-j-l}\xi) \right),$$

then by using the condition $t \in E_l$ we have

$$\mathrm{T}_{E_l} f(x) \le C\mathrm{M}f(x) + \sum_{j \ge 1} \mathrm{T}_{E_l}(f * \psi_{j+l})(x), \tag{3.1}$$

where M denotes the Hardy-Littlewood maximal operator. It is easy to see that

$$\mathrm{T}_{E_l}(f * \psi_{j+l})(x) = \sup_{t \in 2^l E_l} |\mathrm{T}_t^j(f(2^{-l}\cdot))(2^l x)|$$

where T_t^j is the same as in Section 2.1. Hence we have

$$\|\mathrm{T}_{E_l}(f * \psi_{j+l})\|_p \le \left\| \sup_{t \in 2^l E_l} |\mathrm{T}_t^j| \right\|_{L^p \to L^p} \|f\|_p := C_p(j,l)\|f\|_p. \tag{3.2}$$

Define $C_p(j) := \sup_l C_p(j,l)$. Note that $C_p(j) < \infty$ and the series $C_p(j)$ converges if p satisfies the condition (2.16). For a fixed positive integer $\mathcal{N}$, define the operator

$$\mathrm{T}^*_{\mathcal{N}} := \sup_{|l| \le \mathcal{N}} |\mathrm{T}_{E_l}|.$$

And let $A_p(\mathcal{N})$ be such that

$$\|\mathrm{T}^*_{\mathcal{N}}(f)\|_p \le A_p(\mathcal{N})\|f\|_p. \tag{3.3}$$

We need to prove that $A_p(\mathcal{N})$ is actually bounded by a constant independent of $\mathcal{N}$. Define the vector-valued operator

$$\mathbf{T}\colon \{f_l\}_{l=-\mathcal{N}}^{\mathcal{N}} \to \{\mathrm{T}_{E_l}(f_l * \psi_{j+l})\}_{l=-\mathcal{N}}^{\mathcal{N}}.$$

Then by assumption (3.3) and $|f * \psi_{j+l}(x)| \le C\mathrm{M}f(x)$ we have

$$\begin{aligned} \|\mathbf{T}(\{f_l\})\|_{L^p(\ell^\infty)} &= \left\| \sup_{|l| \le \mathcal{N}} |\mathrm{T}_{E_l}(f_l * \psi_{j+l})| \right\|_p \le \left\| \sup_{|l| \le \mathcal{N}} |\mathrm{T}^*_{\mathcal{N}}(\mathrm{M}(f_l))| \right\|_p \\ &\le \left\| \mathrm{T}^*_{\mathcal{N}}\Big(\mathrm{M}(\sup_{|l| \le \mathcal{N}} |f_l|)\Big) \right\|_p \le A_p(\mathcal{N})\|\{f_l\}\|_{L^p(\ell^\infty)}. \end{aligned} \tag{3.4}$$

Also by (3.2) we have

$$\|\mathbf{T}(\{f_l\})\|_{L^p(\ell^p)} \le C_p(j)\|\{f_l\}\|_{L^p(\ell^p)}. \tag{3.5}$$

Hence by interpolating (3.4) and (3.5) under the condition $1 < p \le 2$, we have

$$\|\mathbf{T}(\{f_l\})\|_{L^p(\ell^2)} \le A_p(\mathcal{N})^{1-\frac{p}{2}} C_p(j)^{p/2} \|\{f_l\}\|_{L^p(\ell^2)}. \tag{3.6}$$

Choose $\widehat{\Psi} \in C^\infty(\mathbb{R}^d)$ which is supported in $\{1/8 < |\xi| < 4\}$ and $\widehat{\Psi}(\xi) \equiv 1$ on $\{1/4 < |\xi| < 2\}$. Then we have

$$\widehat{\Psi}(2^{-j-l}\xi) = \widehat{\Psi}(2^{-j-l}\xi)\widehat{\psi}(2^{-j-l}\xi).$$

Hence from (3.6), we have

$$\begin{aligned}\left\|\sup_{|l|\le\mathcal{N}}|\mathrm{T}_{E_l}(f*\psi_{j+l})|\right\|_p &= \left\|\sup_{|l|\le\mathcal{N}}|\mathrm{T}_{E_l}(f*\Psi_{j+l}*\psi_{j+l})|\right\|_p \\ &\le \left\|\left(\sum_{|l|\le\mathcal{N}}|\mathrm{T}_{E_l}(f*\Psi_{j+l}*\psi_{j+l})|^2\right)^{1/2}\right\|_p \\ &\le A_p(\mathcal{N})^{1-\frac{p}{2}}C_p(j)^{p/2}\left\|\left(\sum_{|l|\le\mathcal{N}}|f*\Psi_{j+l}|^2\right)^{1/2}\right\|_p \\ &\le A_p(\mathcal{N})^{1-\frac{p}{2}}C_p(j)^{p/2}\|f\|_p.\end{aligned}$$

Adding in j and comparing this with (3.1) and (3.3), we have

$$A_p(\mathcal{N}) \le C + A_p(\mathcal{N})^{1-\frac{p}{2}}\sum_{j\ge1}C_p(j)^{p/2}.$$

If p satisfies the condition (2.16), then the series $C_p(j)^{p/2}$ converges and thus $A_p(\mathcal{N}) \le C$.

3.2. The general case $E=(0,\infty)$ and $\nu_1 \le 1/2$

For $2 \le p < \infty$, we have

$$\begin{aligned}\left\|\sup_l|\mathrm{T}_{E_l}(f*\psi_{j+l})|\right\|_p &= \left\|\sup_l|\mathrm{T}_{E_l}(f*\Psi_{j+l}*\psi_{j+l})|\right\|_p \\ &\le \sum_l\|\mathrm{T}_{E_l}(f*\Psi_{j+l}*\psi_{j+l})\|_p \le C_p(j)^p\sum_l\|f*\Psi_{j+l}\|_p \\ &\le C_p(j)^p\left\|\left(\sum_l|f*\Psi_{j+l}|^2\right)^{1/2}\right\|_p \le C_p(j)^p\|f\|_p\end{aligned}$$

and the series $C_p(j)^p$ converges when $1/\nu_1 < p < a_1$.

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