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Extensions to Chen's Minimizing Equal Mass Parallelogram Solutions

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Abstract. In this paper, we study the extension of the minimizing equal mass parallelogram solutions which was derived by Chen in 2001 [2]. Chen's solution was minimizing for one quarter of the period $[0,T]$, where numerical integration had been used in his proof. In this paper we extend Chen's solution in the reduced space to $[0,4T]$ and we show that this extension is also minimizing over the intervals $[0,2T]$ and $[0,4T]$. The minimizing property of the extension is proved without using numerical integration.

1. Introduction

The planar four body problem describes the motion of four masses $m_1,\dots,m_4>0$ moving in $\mathbb{R}^2$ in accordance with Newton's law:

$$m_i\ddot{q}_i = \frac{\partial}{\partial q_i}U(q), \quad i=1,\dots,4,$$

where $q_i\in\mathbb{R}^2$ denotes the position, and m_i the mass of the ith particle. With configuration $q=(q_1,q_2,q_3,q_4)$, the force function $U(q)$ (negative of the potential energy) is defined as

$$U(q)=\sum_{i<j}\frac{m_im_j}{r_{ij}},$$

where $r_{ij}=|q_i-q_j|$ measures the distance between the ith and jth bodies and $|\cdot|$ denotes the standard Euclidean norm. The moment of inertia I and the kinetic energy K are given by

$$I(q)=\sum_{i=1}^{4}m_i|q_i|^2 \quad \text{and} \quad K(\dot{q})=\frac{1}{2}\sum_{i=1}^{4}m_i|\dot{q}_i|^2,$$

while the Hamiltonian governing the equations of motion is

$$H(q,\dot{q})=K(\dot{q})-U(q).$$

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For the parallelogram four body problem (having equal masses normalized to one), we consider the configuration space $\mathcal{M}$

$$\mathcal{M} = \{q = (q_1, \ldots, q_4) \in (\mathbb{R}^2)^4 \mid q_3 = -q_1, q_4 = -q_2, q_i \neq q_j, \forall\, i \neq j\}.$$

Notice that the vectors q_i in $\mathcal{M}$ automatically satisfy the constraint $\sum_{i=1}^4 q_i = 0$ (i.e., the center of mass is set at the origin). Moreover, it is clear that $\mathcal{M}$ is a four dimensional manifold, since it is globally parametrized by the vectors q_1 and q_2 (for example). The Lagrangian $L\colon T\mathcal{M} \to \mathbb{R}$ is

$$L(q, \dot{q}) = \frac{1}{2} \sum_{i=1}^{4} |\dot{q}_i|^2 + U(q),$$

and the action defined on loops in the configuration manifold, $q(t) \in H^1([0,T], \mathcal{M})$ is

$$\mathcal{A}(q) = \int_0^T L(q, \dot{q})\, dt,$$

where $H^1([0,T], \mathcal{M})$ denotes the Sobolev space of absolutely continuous T-periodic loops in the configuration manifold $\mathcal{M}$ with square-integrable weak derivatives.

Motivated by the methods and results of Chenciner and Montogomery [3], Chen [2] gave the following theorem:

Theorem 1.1. *For a positive real number T, there exists a periodic solution $q(t) = (q_1(t), q_2(t), q_3(t), q_4(t)) \in \mathbb{R}^8$ of the Newtonian four body problem with equal masses of minimum period $8T$ with the following properties:*

(a) $q_1(t) = -q_3(t)$, $q_2(t) = -q_4(t)$ *for any* $t \in \mathbb{R}$*;*

(b) $q_i(t) \neq q_j(t)$ *for any* $t \in \mathbb{R}$ *and* $i \neq j$*;*

(c) $q_i(t + 4T) = -q_i(t)$ *for any* $t \in \mathbb{R}$*;*

(d) $q_i(0)$ *are vertices of a square and* $q_i(T)$ *are collinear;*

(e) $q_1(t) : t \in [0, 8T]$ *is a star-shaped simple closed curve.*

In [4] the author proved that the elliptic Keplerian orbit minimize the Lagrangian action of the two body problem with periodic boundary conditions. In [11,12] the authors have shown that the Eulerian and Lagrangian elliptical solutions for the planar three body problem are the variational minimizers of the Lagrangian action functional.

The authors in [6–8] have shown that the homographic solutions to the rhombus four body problem are the variational minimizers of the action functional restricted to a certain loop spaces. Chen has studied the existence of a new family of periodic solutions for the planar four body problem with equal masses. He showed the existence of the periodic orbit where pairs of masses move on two isometric star-shaped simple closed curves (see Figure 1.1). These two closed curves intersect at vertices of a square with their major axes perpendicular to each other. An important feature of Chen's parallelogram solutions is that the masses sit on two separate closed curves and the motion is non-homographic. The configuration of the equal masses changes from square to collinear periodically and remains a parallelogram for all time. Chen's solution was only considered for one quarter of the period $[0, T]$ using the constraints $u(0) \in S$, $u(T) \in C$, where S and C are respectively square and colinear configurations.

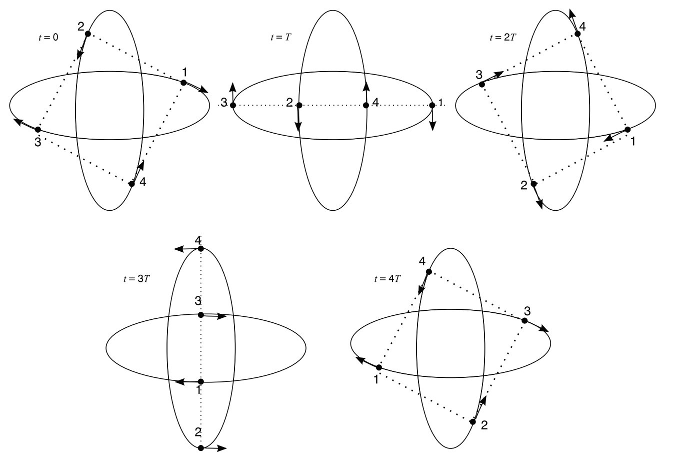

Figure 1.1

In this paper we extend Chen's solution in the reduced space from $[0, T]$ to $[0, 4T]$ and we show it is indeed a minimizing solution over the entire interval $[0, 4T]$ without using numerical techniques. The extension is based on the symmetry properties of the system under consideration. It is fundamental to note that obtained solution is the same as Chen's and as such it corresponds to an $8T$-periodic solution in the ambient plane. An issue left open in Chen's analysis is about the interval on which the solution in the reduced space can be extended to a minimizing solution. In Chen's construction the solution is proved to be minimizing over $[0, T]$ using numerical integration. The aim of this paper is to extend Chen's solution to $[0, 2T]$ first and then to $[0, 4T]$ using one additional symmetry constaint $u(2T) \in S$ and to show that in the reduced space this indeed a minimizing solution. The proof provided does not involve numerical techniques.

The paper is organized as follows: Section 2 discusses the reduced space where Jacobi coordintes and the Hopf map are considered. Section 3 introduces the dynamical structures of the orbit and contains the constructions of the periodic solutions. Section 4 contains some of the discussion of existence of periodic solutions and the reduced variational problem for the equal mass parallelogram orbits, which were considered by Chen in 2001 [2]. This section also includes the discussion of the main result.

The following section will describe the reduced space and the construction of the shape sphere where these motions may be visualized.

2. The quotient map

As we have already remarked, the configuration space $\mathcal{M}$ is a four dimensional real manifold. It is convenient to parametrize $\mathcal{M}$ using complex Jacobi coordinates (z_1, z_2) as in [3, 9], which are identified as vectors in $\mathbb{R}^2$:

$$\mathcal{J} : \mathcal{M} \mapsto \mathbb{R}^4,$$

and defined by

$$\mathcal{J}(q_1, q_2, q_3, q_4) = (q_2 - q_1, q_4 - q_1) = (q_2 - q_1, -q_2 - q_1) =: (z_1, z_2) \in (\mathbb{R}^2)^2 \simeq \mathbb{C}^2.$$

Figure 2.1 shows how this construction can be visualized. It is easy to see that the inverse map $\mathcal{J}^{-1}$ is given by

$$\mathcal{J}^{-1}(z_1, z_2) := \left(-\frac{z_1 + z_2}{2}, \frac{z_1 - z_2}{2}, \frac{z_1 + z_2}{2}, \frac{z_2 - z_1}{2} \right) = (q_1, q_2, q_3, q_4).$$

Let us observe also that $\mathcal{J}$ is not onto $\mathbb{C}^2$. Indeed the hyperplane $z_1 = z_2$ is not in the image of $\mathcal{J}$. To see this, just observe that $z_1 = z_2$ implies $q_2 = 0$, but on $\mathcal{M}$ this gives $q_2 = q_4 = 0$ which does not belong to $\mathcal{M}$. Analogously, the hyperplane $z_1 = -z_2$ does not belong to the image of $\mathcal{J}$. Call $\Delta := \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 = \pm z_2\}$ and call $(\mathbb{C}^2)^\circ := \mathbb{C}^2 \setminus \Delta$. Then it is easy to see that not only $\mathcal{J}$ is onto $(\mathbb{C}^2)^\circ$, but it is actually a *diffeomorphism* between $\mathcal{M}$ and $(\mathbb{C}^2)^\circ$.

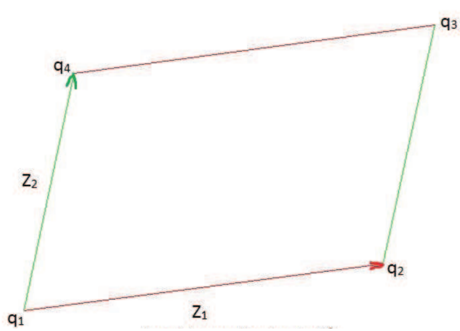

Figure 2.1: Jacobi coordinates in $\mathbb{R}^2$: $Z_1 = q_2 - q_2$, $Z_2 = q_4 - q_1 = -q_2 - q_1$.

The reduced configuration space $\mathcal{M}/\,\mathrm{SO}(2)$ is obtained by quotienting out from $\mathcal{M}$ the rotational symmetry given by the SO(2)-action: $e^{i\theta} \cdot (q_1, q_2, q_3, q_4) := (e^{i\theta} q_1, e^{i\theta} q_2, e^{i\theta} q_3,$

$e^{i\theta}q_4)$. Since the action is free and the stabilizers are all trivial, the quotient $\mathcal{M}/\operatorname{SO}(2)$ is a smooth three-dimensional real manifold.

The action of SO(2) corresponds to the diagonal action of the complex unit scalars $e^{i\theta}$ on Jacobi coordinates (z_1, z_2). Moreover, the diffeomorphism $\mathcal{J}$ is S^1-equivariant, i.e., $\mathcal{J}(e^{i\theta} \cdot (q_1, q_2, q_3, q_4)) = e^{i\theta} \cdot \mathcal{J}(q_1, q_2, q_3, q_4)$. This implies that $\mathcal{M}/\operatorname{SO}(2)$ can be identified with $(\mathbb{C}^2)^\circ/S^1$. Now, we can further identify the quotient manifold $(\mathbb{C}^2)^\circ/S^1$ with $(\mathbb{R}^3)^* := \mathbb{R}^3 \setminus \{(0,0,0)\}$ essentially using the Hopf map $\mathcal{F}\colon (\mathbb{C}^2)^\circ/S^1 \to (\mathbb{R}^3)^*$ as follows

$$\mathcal{F}([z_1],[z_2]) := (|z_1|^2 - |z_2|^2, 2\overline{z}_1 z_2),$$

where $[z_1]$ and $[z_2]$ denote the coset of z_1 and z_2 respectively, under the diagonal action of S^1. Notice that the map $\mathcal{F}$ is well-defined on these cosets, since the right hand-side does not depend on the choice of the representatives.

It is also easy to see that $\mathcal{F}$ is onto $(\mathbb{R}^3)^*$ and that there is no $([z_1],[z_2]) \in (\mathbb{C}^2)^\circ/S^1$ such that $\mathcal{F}([z_1],[z_2]) = (0,0,0)$. We can view $(\mathbb{R}^3)^*$ as an open submanifold of $\mathbb{R} \times \mathbb{C}$ with coordinates $(|z_1|^2 - |z_2|^2, 2\overline{z}_1 z_2) =: (u_1, u_2 + iu_3)$.

Each single point in $\mathcal{M}/\operatorname{SO}(2)$ represents a congruence class of configurations formed by the four mass points. Fixing a positive constant $c > 0$, the level set $I^{-1}(c)$, (here $I(q)$ is the moment of inertia $I(q) = q \cdot q$, recall that the masses have been normalized to 1) is a three sphere (in $\mathcal{M}$) and it is mapped onto the two sphere $S^2_{c^2} := \{(u_1,u_2,u_3) \mid u_1^2 + u_2^2 + u_3^2 = c^2\} \subset (\mathbb{R}^3)^* \subset \mathbb{R} \times \mathbb{C}$ via the composition of the mappings described above. In particular, each point on the unit sphere $S^2 := \{(u_1,u_2,u_3) \mid |u|^2 = 1\}$ called the unit shape sphere, represents a similarity class of configurations. We have seen that the reduced space $\mathcal{M}/\operatorname{SO}(2)$ is diffeomorphic to $(\mathbb{R}^3)^*$, which on the other hand is diffeomorphic to $\mathbb{R}^+ \times \mathcal{F}(I^{-1}(c))$, where we have indicated with $\mathbb{R}^+$ the set of positive real numbers and $c > 0$ is a fixed value of the constant of motion given by the moment of inertia. The projection of a path $q \in H^1([0,T],\mathcal{M})$ to a path in $H^1([0,T],\mathcal{M}/\operatorname{SO}(2))$ is called the *reduced path* of q. Observe that since the moment of inertia is conserved, the reduced path of q can be though to live on the shape sphere $S^2_{c^2} := \{(u_1,u_2,u_3) \mid u_1^2 + u_2^2 + u_3^2 = c^2\}$ for some fixed value of c.

Using spherical coordinates

$$(u_1, u_2, u_3) = (r^2 \cos\phi \cos\theta, r^2 \cos\phi \sin\theta, r^2 \sin\phi),$$

we can describe the following relations between points on the unit shape sphere $(r = 1)$ and the configuration of the four bodies (see Figure 2.2) as follows:

- The configuration is a collinear if and only if $u_3 = 0$ $(\phi = 0)$.
- The configuration is a square if and only if $u_1 = u_2 = 0$ $(\phi = \pi/2)$.

- The configuration is a rhombus if and only if $u_1 = 0$ ($\theta = \pi/2$ or $3\pi/2$).
- The configuration is a rectangle if and only if $u_2 = 0$ ($\theta = 0$ or π).

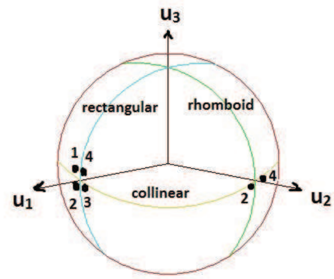

Figure 2.2: The unit shape sphere.

3. Dynamical structures of the orbit

In this section we need to construct the Hamiltonian equations in the reduced space, which we need for our purposes. Denote $p_i = \dot{q}_i$ as the momentum coordinates. Then we have

$$\begin{aligned} z_1 &= q_2 - q_1, & v_1 &= p_2 - p_1, \\ z_2 &= -q_2 - q_1, & v_2 &= -p_2 - p_1. \end{aligned} \tag{3.1}$$

The new symmetry constrained Hamiltonian has the form,

$$H(z_1, z_2, v_1, v_2) = \frac{1}{2}(|v_1|^2 + |v_2|^2) - U(z_1, z_2),$$

where $U(z_1, z_2)$ will be given in details in the following lemma.

Lemma 3.1. *The force function (negative of the potential energy) $U = U(z_1, z_2)$ has the following form:*

$$U(z_1, z_2) = \frac{2}{|z_1|} + \frac{2}{|z_2|} + \frac{1}{|z_1 + z_2|} + \frac{1}{|z_1 - z_2|}. \tag{3.2}$$

Proof. Since $m_i = m_j = 1$ for all indices i, j, the standard force function has the following form

$$U(q) = \frac{1}{|q_1 - q_2|} + \frac{1}{|q_1 - q_3|} + \frac{1}{|q_1 - q_4|} + \frac{1}{|q_2 - q_3|} + \frac{1}{|q_2 - q_4|} + \frac{1}{|q_3 - q_4|}.$$

Since $q_3 = -q_1$, $q_4 = -q_2$, we get,

$$U(q_1, q_2) = \frac{2}{|q_1 - q_2|} + \frac{2}{|q_1 + q_2|} + \frac{1}{2|q_1|} + \frac{1}{2|q_2|}. \tag{3.3}$$

From equation (3.1), we have $|z_1| = |q_1 - q_2|$, $|z_2| = |q_1 + q_2|$, $|z_1 + z_2| = 2|q_1|$ and $|z_1 - z_2| = 2|q_2|$. Therefore, the formula (3.2) follows by (3.3). $\square$

In these coordinates, it is easy to see that the moment of inertia becomes $I = |z_1|^2 + |z_2|^2$. The level set $I^{-1}(1)$ is a cross section of reduced configuration space. To obtain coordinates on $\mathcal{M}/\operatorname{SO}(2)$, we use the Hopf map to generate a point transformation $u = \mathcal{F}(z)$. For notational purposes, $z \cdot v$ refers to the usual dot product in $\mathbb{R}^2$. We treat $z \times v$ as a scalar obtained by taking the non-zero component of the cross product of two vectors in $\mathbb{R}^2$. Coordinates on the phase space are introduced by extending $\mathcal{F}$ to a symplectic transformation in the usual way, $(z, v) \mapsto (u, w) = \big(\mathcal{F}(z), (\frac{\partial \mathcal{F}}{\partial z})^{-T} v\big)$. Our new variables are:

$$
\begin{aligned}
u_1 &= |z_1|^2 - |z_2|^2, & w_1 &= \frac{1}{2I}(z_1 \cdot v_1 - z_2 \cdot v_2),\\
u_2 &= 2(z_1 \cdot z_2), & w_2 &= \frac{1}{2I}(\mu z_1 \cdot v_1 - \nu z_2 \times v_2 + z_1 \cdot v_2),\\
u_3 &= 2(z_1 \times z_2), & w_3 &= \frac{1}{2I}(\nu z_1 \cdot v_1 + \mu z_2 \times v_2 + z_1 \times v_2),\\
u_4 &= \arg(z_1), & w_4 &= z_1 \times v_1 + z_2 \times v_2,
\end{aligned}
\tag{3.4}
$$

where $\mu = z_1 \cdot z_2/|z_1|^2$, $\nu = z_1 \times z_2/|z_1|^2$. Notice also that $u_2 = \Re(2\overline{z}_1 z_2)$ and $u_3 = \Im(2\overline{z}_1 z_2)$ as given by the Hopf map.

The new Hamiltonian is independent of u_4, as it will be seen in Lemma 3.2, which in turn implies that w_4 is constant along solutions. Call d the constant value of w_4 along the solutions of the Hamiltonian system. Letting $u = (u_1, u_2, u_3)$ and $w = (w_1, w_2, w_3)$, the Hamiltonian function for the reduced problem is

$$
H(u, w) = 2K(w)I(u) + \frac{d(d + 2u_3 w_2 - 2u_2 w_3)}{u_1 + I(u)} - U(u),
$$

where $K(w) = w_1^2 + w_2^2 + w_3^2$ and $I(u) = (u_1^2 + u_2^2 + u_3^2)^{1/2}$.

The angular momentum $J(q, p) = w_4$ is conserved along solutions, since the angular variable u_4 does not appear in the Hamiltonian. We can then restrict our attention to the dynamics on the reduced space $J^{-1}(0)/\operatorname{SO}(2)$, defined by setting the angular momentum $w_4 = 0$, and projecting out the SO(2) symmetry. We identify the reduced phase space with $T^*(\mathcal{M}/\operatorname{SO}(2))$ with canonical cotangent coordinates (u, w), where the reduced configuration space has three degrees of freedom with coordinates (u_1, u_2, u_3). The new Hamiltonian in $J^{-1}(0)/\operatorname{SO}(2)$ reduces nicely to

$$
H(u, w) = 2K(w)I(u) - U(u),
\tag{3.5}
$$

where $U(u)$ will be given (in details) in the following lemma.

Lemma 3.2. *The force function $U(u_1, u_2, u_3)$ has the following form:*

$$
U(u) = \frac{2\sqrt{2}}{\sqrt{I + u_1}} + \frac{2\sqrt{2}}{\sqrt{I - u_1}} + \frac{1}{\sqrt{I + u_2}} + \frac{1}{\sqrt{I - u_2}}.
\tag{3.6}
$$

Proof. From (3.4), we have the following two equations:

$$I = |z_1|^2 + |z_2|^2 = (u_1^2 + u_2^2 + u_3^2)^{1/2}, \tag{3.7}$$

$$u_1 = |z_1|^2 - |z_2|^2. \tag{3.8}$$

Solving (3.7) and (3.8) for $|z_1|^2$, $|z_2|^2$ to obtain,

$$|z_1|^2 = \frac{1}{2}(u_1 + I), \quad |z_2|^2 = \frac{1}{2}(-u_1 + I). \tag{3.9}$$

Using (3.9) and the fact that u_2 is just the dot product of z_1, z_2, it follows

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2z_1 \cdot z_2 = \frac{1}{2}(u_2 + I) + \frac{1}{2}(-u_2 + I) + u_2 = I + u_2.$$

Similarly, for $|z_1 - z_2|^2$, we have

$$|z_1 - z_2|^2 = I - u_2.$$

Then formula (3.6) follows by (3.2). □

The dynamics on the reduced space are simply given by the equations of motion in Hamiltonian form:

$$\begin{aligned} \dot{u} &= \frac{\partial H}{\partial w} = 2K'(w)I(u), \\ \dot{w} &= -\frac{\partial H}{\partial u} = -2K(w)I'(u) + U'(u). \end{aligned} \tag{3.10}$$

The problem has been reduced to three degrees of freedom with coordinates (u, w). Using these coordinates, it is not difficult to understand the shape sphere $I = 1$. The cross product $z_1 \times z_2$ vanishes if and only if the four bodies are collinear. Therefore, the collinear configurations, denote by C, correspond to the equator $u_3 = 0$. The dot product $z_1 \cdot z_2$ vanishes if and only if z_1 is perpendicular to z_2 and this corresponds to $u_2 = 0$, namely to rectangular configurations. In particular the square configurations denoted by S, correspond to the u_3-axis, that is, $u_1 = u_2 = 0$.

The periodic solution we are interested in can be constructed by reflections in $\mathcal{M}/\operatorname{SO}(2)$. We consider two reflections on the phase space $T^*(\mathcal{M}/\operatorname{SO}(2))$,

$$\begin{aligned} \mathcal{R}_s &\colon (u_1, u_2, u_3, w_1, w_2, w_3) \mapsto (-u_1, -u_2, u_3, w_1, w_2, -w_3), \\ \mathcal{R}_c &\colon (u_1, u_2, u_3, w_1, w_2, w_3) \mapsto (u_1, u_2, -u_3, -w_1, -w_2, w_3). \end{aligned}$$

From now on for brevity, let $\mathcal{R}_{s,c}$ denote $\mathcal{R}_s$ or $\mathcal{R}_c$ respectively. The Hamiltonian (3.5) of the reduced system is invariant under these mappings, and they are anti-symplectic, that is to say $\mathcal{R}_{s,c}^*(\omega) = -\omega$, where ω is the standard symplectic form on $T^*(\mathcal{M}/\operatorname{SO}(2))$. We have the following:

Lemma 3.3. *The equations* (3.10) *of the reduced system are invariant under the two time-reversing symmetries given by*

$$(t, u_1, u_2, u_3, w_1, w_2, w_3) \mapsto (-t, \mathcal{R}_{s,c}(u_1, u_2, u_3, w_1, w_2, w_3)).$$

Proof. It is a straightforward computation left to the reader. □

A key property of these anti-symplectic symmetries is identified in the following proposition which can be found in [1]

Proposition 3.4. *Let $H\colon T^*\mathcal{M} \to \mathbb{R}$ be a Hamiltonian function, with corresponding Hamiltonian vector field X_H. Suppose $\mathcal{R}_{s,c}\colon T^*\mathcal{M} \to T^*\mathcal{M}$ is anti-symplectic mapping, and the Hamiltonian H is invariant under these reflections. Then $\mathcal{R}_{s,c}$ reverses the Hamiltonian vector field, $\mathcal{R}_{s,c} \circ X_H = -X_H \circ \mathcal{R}_{s,c}$, and the corresponding Hamiltonian flow ϕ_t satisfies the relation $\mathcal{R}_{s,c} \circ \phi_t = \phi_{-t} \circ \mathcal{R}_{s,c}$.*

4. Variational principle and the main result

In this section, we review some discussion about the existence of minimizing solutions for the equal mass parallelogram four body problem which was studied by Chen in [2]. We will also show that the minimizing solution can be extended to a symmetric $4T$-periodic solution in the reduced space (and not just $[0, T]$ as in the Chen's construction).

Chen's proof for Theorem 1.1 consists of a few parts. First he reduced the problem of minimizing $\mathcal{A}$ to the minimization of the reduced action function $\mathcal{A}_{\text{red}}$, where

$$\mathcal{A}_{\text{red}} = \int_0^T \left(\frac{1}{2}K_{\text{red}} + U(u)\right) dt,$$

and the reduced kinetic energy is defined explicitly (see Appendix C of [6]) as follows:

$$K_{\text{red}} = \frac{1}{8}\frac{|\dot{u}|^2}{I(u)}.$$

Define Λ_{pr} as the following space of paths:

$$\Lambda_{pr} = \{u \in H^1([0, T], \mathcal{M}/\operatorname{SO}(2)) \mid u(0) \in S^+, u(T) \in C\},$$

where S^+, S^- denote positively and negatively oriented square configurations, in the sense that z_2 located counter-clockwise or clockwise relative to z_1, respectively as shown in Figure 4.1.

Chen also showed that the minimizer of $\mathcal{A}$ on Λ_{pr} has angular momentum zero and therefore it also minimizes $\mathcal{A}_{\text{red}}$ on Λ_{pr}.

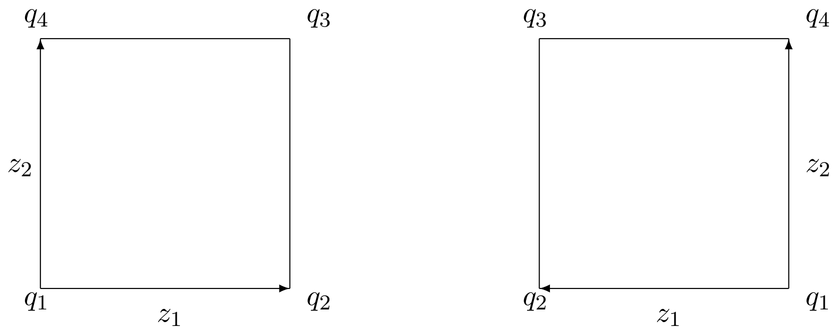

Figure 4.1: Positively and negatively oriented square configuration.

In the second part of the proof, the following inequality is proved

$$\inf_{u\in\Lambda_{pr}} \mathcal{A} < \inf_{q\in\Lambda} \mathcal{A},$$

where Λ is the collection of paths in Λ_{pr} which have a rhomboid configuration for all time. The upper bound he got for the left hand side is approximately $5.33T^{1/3}$, and the right-hand side has a lower bound $3\big(\frac{9+4\sqrt{2}}{2}\pi^2\big)^{1/3}T^{1/3} \approx 12.499456T^{1/3}$.

The last part of Chen's construction describes the shape of the orbit, including showing that each lobe of the curve is star-shaped.

In Section 3, we introduced the reduced space $J^{-1}(0)/\operatorname{SO}(2) \simeq T^*(\mathcal{M}/\operatorname{SO}(2))$ with coordinates $(u,w) = (u_1,u_2,u_3,w_1,w_2,w_3)$. The dynamics are described by a system with three degrees of freedom, governed by the reduced Hamiltonian described in (3.5).

Now we wish to introduce the reduced action symmetry group which is generated by two reflections θ_s, θ_c which are respectively reflections through square and collinear configurations. First, we define the symmetries θ_s, θ_c on $\mathcal{M}/\operatorname{SO}(2)$ by

$$\theta_s\colon (u_1,u_2,u_3) \to (-u_1,-u_2,u_3), \quad \theta_c\colon (u_1,u_2,u_3) \to (u_1,u_2,-u_3).$$

In addition, the antisymplectic symmetries, defined in Section 3, are

$$\mathcal{R}_s\colon (u,w) \mapsto (\theta_s u, -\theta_s w), \quad \mathcal{R}_c\colon (u,w) \mapsto (\theta_c u, -\theta_c w).$$

The variational problem that is introduced by Chen is

$$\mathcal{A}_{\text{red}}(\overline{u}) = \min_{\Lambda_{pr}} \mathcal{A}_{\text{red}}(u). \tag{4.1}$$

The functional $\mathcal{A}_{\text{red}}$ restricted on Λ_{pr} is coercive and a standard argument in the Calculus of Variations shows that the infimum of $\mathcal{A}_{\text{red}}$ on Λ_{pr} is attained [2]. Non-collision of parallelogram minimizing solutions was proven in [2] by studying the behaviour of the action of the minimizing orbits in the reduced space and comparing their action with the rhomboid motions.

In the reduced space, Chen's solution is defined only over the interval $[0, T]$ using the boundary conditions $u(0) \in S^+$, $u(T) \in C$. The following proposition shows how Chen's solution can be extended symmetrically to $[0, 4T]$. Figure 4.2 shows how this construction can be visualized.

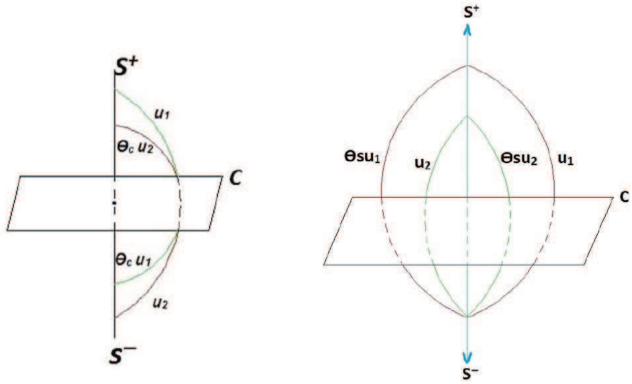

Figure 4.2: The symmetric extension of Chen's solution minimizes $\mathcal{A}_{\text{red}}(u)$ over $[0, 2T]$, $[0, 4T]$.

Proposition 4.1. *The solution $\overline{u}(t)$ to the variational problem* (4.1) *can be extended symmetrically to* $[0, 4T]$ *using the following symmetry constraints:*

(i) $u(t) = \theta_s \cdot u(-t)$,

(ii) $u(t + T) = \theta_c \cdot u(-t + T)$.

Proof. We first extend Chen's solution $\overline{u}(t)$ to $[T, 2T]$. Using condition (ii), applied to $\overline{u}$ on the interval $[0, T]$, we can define $u(t+T) = \theta_c \cdot \overline{u}(-t+T)$ on the interval $t \in [0, T]$. Setting $t = T$, we have $u(2T) = \theta_c \cdot \overline{u}(0)$. That in turn implies $u(t)$ is a symmetric extension of $\overline{u}(t)$ on the interval $[T, 2T]$. Secondly, we can extend the symmetric extension of $\overline{u}(t)$ to $[2T, 3T]$. Using condition (ii), applied to $u(t)$ on the interval $[T, 2T]$, we can define $u(t + T) = \theta_c \cdot u(-t + T)$ on the interval $t \in [T, 2T]$. Setting $t = 2T$, we have

$$u(3T) = \theta_c \cdot u(-T). \tag{4.2}$$

Using condition (i), setting $t = T$, we have $u(T) = \theta_s \cdot u(-T)$. Plugging this into (4.2), we get

$$u(3T) = \theta_s \cdot \theta_c \cdot u(T) = \theta_s \cdot u(T).$$

That in turn implies $u(t)$ is symmetrically extended to $[2T, 3T]$. Finally, we want to extend the symmetric extension of $u(t)$ to $[3T, 4T]$. Using condition (ii), applied to the symmetric extension of $u(t)$ on the interval $[2T, 3T]$, we can define $u(t+T) = \theta_c \cdot u(-t+T)$ on the interval $t \in [2T, 3T]$. Setting $t = 3T$, we have

$$u(4T) = \theta_c \cdot u(-2T) = u(2T) = u(0). \qquad \square$$

Let us point out that the extension so constructed is a periodic curve of period $4T$ in the reduced space. Also the extension procedure itself does not depend on the fact that $\overline{u}(t)$ is the solution of a variational problem, but it can be applied to any non-self-intersecting curve γ in the reduced phase space, satisfying the boundary conditions $\gamma(0) \in S^+$ and $\gamma(T) \in C$, namely the extension procedure is a purely geometrical fact that has nothing to do with the dynamics.

Now we show that the solution $\overline{u}(t)$ of the variational problem (4.1) can be immediately extended to a $4T$-periodic solution of the reduced Hamiltonian system that is $\overline{u}(t)$ can be extended not just as a closed non-self-intersecting geometric curve, but as a solution of the reduced Hamiltonian system.

Proposition 4.2. *The solution $\overline{u}(t)$ to the variational problem* (4.1) *may be extended so as to satisfy the relation $u(t+2T) = -u(t)$. The corresponding momentum $w(t)$ satisfies the same symmetry $w(t+2T) = -w(t)$. Together, the pair $(u(t), w(t))$ may be extended to a $4T$-periodic solution of the reduced system.*

Proof. We first show that the symmetric extension is smooth. Using the transversality conditions [5], $w(0) \perp S^+$, $w(T) \perp C$, we have $(u(0), w(0)) \in \mathrm{Fix}(\mathcal{R}_S)$ and $(u(T), w(T)) \in \mathrm{Fix}(\mathcal{R}_c)$, where $\mathrm{Fix}(T)$ denotes the set of fixed points for a transformation T. We will now show that $\overline{u}(t)$ can be extended so as to satisfy the relation $(u(t+2T), w(t+2T)) = -(u(t), w(t))$. We use the flow equivariance relation,

$$\mathcal{R}_s \circ \phi_t = \phi_{-t} \circ \mathcal{R}_s. \tag{4.3}$$

Applying the symmetry $\mathcal{R}_c$ to both sides of (4.3), we get

$$(\mathcal{R}_c\mathcal{R}_s) \circ \phi_t = \mathcal{R}_c \circ \phi_{-t} \circ \mathcal{R}_s = \phi_t \circ (\mathcal{R}_c\mathcal{R}_s). \tag{4.4}$$

Applying (4.4) to $(u(2T), w(2T))$, we get

$$(\mathcal{R}_c\mathcal{R}_s) \circ \phi_t \cdot (u(2T), w(2T)) = \phi_t \circ (\mathcal{R}_c\mathcal{R}_s) \cdot (u(2T), w(2T)),$$

and therefore,

$$(\mathcal{R}_c\mathcal{R}_s) \cdot (u(t+2T), w(t+2T)) = \phi_t \circ (\mathcal{R}_c\mathcal{R}_s) \cdot (u(2T), w(2T)). \tag{4.5}$$

Since $(u(2T), w(2T)) = \mathcal{R}_c \cdot (u(0), w(0))$, (4.5) becomes

$$(\mathcal{R}_c\mathcal{R}_s) \cdot (u(t+2T), w(t+2T)) = \phi_t \circ (\mathcal{R}_c\mathcal{R}_s)\mathcal{R}_c \cdot (u(0), w(0)). \tag{4.6}$$

Since $\mathcal{R}_s\mathcal{R}_c = -I$, $\mathcal{R}_c^2 = I$, it follows that $\mathcal{R}_s$, $\mathcal{R}_c$ commute. From (4.6), we have

$$\begin{aligned} -(u(t+2T), w(t+2T)) &= \phi_t \circ \mathcal{R}_s \cdot (u(0), w(0)) \\ &= (u(t), w(t)). \end{aligned}$$

Now iterating the symmetry establishes periodicity of the solution

$$(u(4T), w(4T)) = -(u(2T), w(2T)) = (u(0), w(0)).$$

Now recall that the curve $\overline{u}$ is a solution of the Hamiltonian system on $[0, T]$ and that the Lagrangian is invariant under the symmetries considered here. Therefore the symmetric extension u of $\overline{u}$ on $[0, 4T]$ just constructed is a solution of the Hamiltonian system, except possibly at the boundary points corresponding to times 0, T, $2T$, $4T$. □

The minimization over the interval $[0, T]$ was considered by Chen in (4.1) using numerical techniques. Now we show that the minimizing property can be extended to the interval $[0, 2T]$.

Proposition 4.3. *Let $\overline{u}(t)$ be the minimizing solution of the variational problem* (4.1), *defined on the interval $[0, T]$, then the symmetric extension of $\overline{u}$ introduced in Proposition* 4.1 *minimizes the action functional over the interval $[0, 2T]$ with respect to boundary conditions*

$$u(0) \in S^+, \quad u(2T) \in S^-, \quad u(T) \in C.$$

Proof. We construct a variation

$$\widetilde{u}(t) = \begin{cases} u_1 & \text{if } 0 \le t \le T, \\ u_2 & \text{if } T \le t \le 2T, \end{cases}$$

where the curve u_1 joins S^+ to C, u_2 joins C to S^-. We also construct $\theta_c u_1$ that joins S^- to C and $\theta_c u_2$ that joins C to S^+. We note that the actions of the curves $\theta_c u_1$, $\theta_c u_2$ are the same as those of u_1 and u_2 because the Lagrangian function is invariant under both reflections.

We can concatenate u_1 and $\theta_c u_1$ to give a curve from S^+ to S^- and obtain the estimate:

$$\mathcal{A}_T(u_1) + \mathcal{A}_T(\theta_c u_1) \ge 2\mathcal{A}_T(\overline{u}).$$

Analogously, we can concatenate u_2 and $\theta_c u_2$ and find the estimate:

$$\mathcal{A}_T(u_2) + \mathcal{A}_T(\theta_c u_2) \ge 2\mathcal{A}_T(\overline{u}).$$

Since $\mathcal{A}_T(\theta_c u_j) = \mathcal{A}_T(u_j)$, $j = 1, 2$, we get using the estimates above:

$$2\mathcal{A}_{2T}(\widetilde{u}) = 2[\mathcal{A}_T(u_1) + \mathcal{A}_T(u_2)] \ge 4\mathcal{A}_T(\overline{u}) = 2\mathcal{A}_{2T}(\overline{u}).$$ □

The following proposition shows that the minimization over the interval $[0, 2T]$ (see Proposition 4.3) can also be extended to $[0, 4T]$.

Proposition 4.4. *The symmetric extension of the solution $\overline{u}(t)$ of the variational problem* (4.1) *can also be minimized over the interval* $[0, 4T]$ *subject to the boundary constraints*

$$u(0) \in S^+, \quad u(2T) \in S^-, \quad u(T) \in C, \quad u(3T) \in C.$$

Proof. We construct a variation

$$\widetilde{u}(t) = \begin{cases} u_1 & \text{if } 0 \le t \le 2T, \\ u_2 & \text{if } 2T \le t \le 4T, \end{cases}$$

where $\widetilde{u}$ is not necessarily symmetric or even periodic. From Proposition 4.3, it is clear that

$$\mathcal{A}_{2T}(u_1) \ge \mathcal{A}_{2T}(\overline{u}), \quad \mathcal{A}_{2T}(u_2) \ge \mathcal{A}_{2T}(\overline{u}),$$

and therefore,

$$\mathcal{A}_{4T}(\widetilde{u}) = \mathcal{A}_{2T}(u_1) + \mathcal{A}_{2T}(u_2) \ge 2\mathcal{A}_{2T}(\overline{u}) = \mathcal{A}_{4T}(\overline{u}).$$ □

Proposition 4.5. *The symmetric extension constructed in Proposition* 4.2 *can also be shown to minimize the action functional on the interval* $[0, 4T]$ *subject to the following symmetry constraints:*

(i) $u(t) = \theta_s \cdot u(-t)$,

(ii) $u(t + T) = \theta_c \cdot u(-t + T)$,

which are equivalent to the boundary conditions in Proposition 4.4.

Proof. We only need to check that conditions (i) and (ii) are equivalent to that of Proposition 4.4. From condition (i), it is clear that at $t = 0$, $u(0)$ is fixed by the symmetry θ_s, that is to say, $u(0) = \theta_s \cdot u(0)$ or $u(0) \in S^+$. Similarly, using condition (ii) and at $t = 0$, $u(T)$ is fixed by the symmetry θ_c, that is to say, $u(T) = \theta_c \cdot u(T)$, or $u(T) \in C$. Setting $t = T$, using condition (ii), we have

$$u(2T) = \theta_c \cdot u(0). \tag{4.7}$$

Applying the symmetry θ_s to both sides of (4.7) and using the relation $u(t + 2T) = -u(t)$ for $t = 0$, we obtain

$$\theta_s \cdot u(2T) = -u(0) = u(2T).$$

That is to say that $u(2T)$ is fixed by the symmetry θ_s or $u(2T) \in S^-$. Similarly, at $t = 2T$, using condition (ii), we have

$$u(3T) = \theta_c \cdot u(-T). \tag{4.8}$$

Applying the symmetry θ_c to both sides of (4.8) and using the fact that the orbit is $4T$-periodic, we get

$$\theta_c \cdot u(3T) = u(-T) = u(3T).$$

That is to say that $u(3T)$ is fixed by θ_c or $u(3T) \in C$. Thus, we have shown that conditions (i), (ii) imply the conditions in Proposition 4.4.

Conversely, we show that the conditions in Proposition 4.4 imply conditions (i), (ii). First, apply the time reversing property $\mathcal{R}_s \circ \phi_t = \phi_{-t} \circ \mathcal{R}_s$ to $(u(0), w(0))$ to get

$$\begin{aligned} \mathcal{R}_s \circ \phi_t \cdot (u(0), w(0)) &= \phi_{-t} \circ \mathcal{R}_s \cdot (u(0), w(0)) \\ &= (u(-t), w(-t)). \end{aligned}$$

Projecting this into the configuration component, we get $\theta_s \cdot u(t) = u(-t)$.

Secondly, apply the relation $\mathcal{R}_c \circ \phi_t = \phi_{-t} \circ \mathcal{R}_c$ to $(u(T), w(T))$ to get

$$\begin{aligned} \mathcal{R}_c \cdot (u(t+T), w(t+T)) &= \phi_{-t} \circ \mathcal{R}_c \cdot (u(T), w(T)) \\ &= (u(-t+T), w(-t+T)). \end{aligned}$$

Projecting this into the configuration component, we get $\theta_c \cdot u(t+T) = u(-t+T)$. Thus, we have shown that conditions (i), (ii) are equivalent to that of proposition 4.4 and therefore the solution $\overline{u}(t)$ minimizes the action functional over the interval $[0, 4T]$ with respect to conditions (i), (ii). □

We already observed at the end of the proof of Proposition 4.2 that the symmetric extension u of $\overline{u}$ on $[0, 4T]$ is indeed a solution of the Hamiltonian system, except possibly at the boundary points that appear in the extension constructed with symmetries (namely the boundary points that corresponds to times 0, T, $2T$, $4T$). Using the Principle of Symmetric Criticality (see [10]), we show that the symmetric extension is indeed a solution of the equations of motions also at the boundary points, namely we obtain the following.

Proposition 4.6. *The symmetric extension of Proposition* 4.2 *is a solution of the equation of motions on the interval* $[0, 4T]$.

Proof. We consider the action function $\mathcal{A}_{4T}\colon \Sigma \to \mathbb{R}$, where

$$\Sigma := \{u \in H^1([0, 4T], \mathcal{M}/\operatorname{SO}(2)) \mid u(0) \in S^+, u(4T) \in S^+\}.$$

Any critical point of $\mathcal{A}_{4T}$ in Σ is a solution of the equations of motion with boundary values $u(0) \in S^+$, $u(4T) \in S^+$.

We consider the following transformations $\rho\colon \Sigma \to \Sigma$ given by $(\rho u)(t) := -u(t + 2T)$ (up to translation of time in order for $(\rho u)(t)$ to be belong to Σ). It is clear that $\rho \circ \rho$

is equal to the identity transformation and moreover the extension of ρ to the tangent bundle leaves the reduced action functional unchanged.

Consider now the subspace $\Sigma^\rho \subset \Sigma$ given by the paths in Σ which are left fixed by ρ, namely such that $(\rho u)(t) = u(t)$, i.e., the paths for which $u(t) = -u(t+2T)$. We know by the previous propositions that these paths satisfy the boundary constraints of the symmetric extension.

The Principle of Symmetric Criticality can be applied directly in this case, since the relevant symmetry group SO(2) is compact (see Theorem 5.4 in [10]). Therefore a minimizer u of $\mathcal{A}_{\text{red}}$ on Σ^ρ is also a critical point of the reduced action functional on Σ (hence a solution of the equations of motion satisfying the boundary values $u(0) \in S^+$, $u(4T) \in S^+$). On the other hand, we proved that the symmetric extension of a minimizer $\overline{u}$ of $\mathcal{A}_{\text{red}}$ on $[0,T]$ is actually a minimizer for A_{red} on Σ^ρ. Therefore we conclude that the symmetric extension of $\overline{u}$ is a critical point of A_{red} on Σ and as such it provides a $4T$-periodic solution of the equations of motion, with the right boundary constraints. $\square$

References

[1] R. Abraham and J. E. Marsden, *Foundations of Mechanics*, Second edition, Advanced Book Program, Reading, Mass., 1978.

[2] K.-C. Chen, *Action-minimizing orbits in the parallelogram four-body problem with equal masses*, Arch. Ration. Mech. Anal. **158** (2001), no. 4, 293–318.

[3] A. Chenciner and R. Montgomery, *A remarkable periodic solution of the three-body problem in the case of equal masses*, Ann. of Math. (2) **152** (2000), no. 3, 881–901.

[4] W. B. Gordon, *A minimizing property of Keplerian orbits*, Amer. J. Math. **99** (1977), no. 5, 961–971.

[5] M. R. Hestenes, *Calculus of Variations and Optimal Control Theory*, John Wiley & Sons, New York, 1966

[6] A. Mansur, *Instability of Periodic Solutions of Some Rhombus and the Parallelogram Four Body Problem*, Ph.D. thesis, Queen's University, 2012.

[7] A. Mansur and D. Offin, *A minimizing property of homographic solutions*, Acta Math. Sin. (Engl. Ser.) **30** (2014), no. 2, 353–360.

[8] A. Mansur, D. Offin and M. Lewis, *Instability for a family of homographic periodic solutions in the parallelogram four body problem*, Qual. Theory Dyn. Syst. **16** (2017), no. 3, 671–688.

[9] K. R. Meyer, G. R. Hall and D. Offin, *Introduction to Hamiltonian Dynamical Systems and the N-body Problem*, Second edition, Applied Mathematical Sciences **90**, Springer, New York, 2009.

[10] R. S. Palais, *The principle of symmetric criticality*, Comm. Math. Phys. **69** (1979), no. 1, 19–30

[11] S. Zhang and Q. Zhou, *A minimizing property of Lagrangian solutions*, Acta Math. Sin. (Engl. Ser.) **17** (2001), no. 3, 497–500.

[12] ———, *A minimizing property of Eulerian solutions*, Celestial Mech. Dynam. Astronom. **90** (2004), no. 3-4, 239–243.

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