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Minimal Ruled Submanifolds Associated with Gauss Map

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Abstract. We set up the new models of product manifolds, namely a generalized circular cylinder and a generalized hyperbolic cylinder as cylindrical types of ruled submanifold in Minkowski space. We also establish some characterizations of generalized circular cylinders and hyperbolic cylinders in Minkowski space with the Gauss map. We also show that there do not exist non-cylindrical marginally trapped ruled submanifolds with the pointwise 1-type Gauss map of the first kind, which gives a characterization of non-cylindrical minimal ruled submanifolds in Minkowski space.

1. Introduction

Ruled submanifolds in Euclidean space or Minkowski space are defined in such a way that they are foliated by totally geodesic submanifolds over a curve. By extending the classical results on minimal surfaces in 3-dimensional Euclidean space to ruled submanifolds in Euclidean space, minimal ruled submanifolds are proved to be generalized helicoids or planes [3]. Similarly we can consider minimal ruled submanifolds in the Minkowski space $\mathbb{L}^m$. However, because of the causal characters of generators, not many works on ruled submanifolds in the Minkowski space $\mathbb{L}^m$ including those of ruled surfaces in 3-dimensional Minkowski space have been made. Very recently, two of the authors characterized minimal ruled submanifolds of the Minkowski space $\mathbb{L}^m$ [20]. In [21, 22], some new examples of ruled submanifolds with degenerate rulings in $\mathbb{L}^m$ were introduced.

A submanifold M in an Euclidean space $\mathbb{E}^m$ or a pseudo-Euclidean space $\mathbb{E}^m_s$ is said to be of finite-type if its isometric immersion $x\colon M \to \mathbb{E}^m$ or $x\colon M \to \mathbb{E}^m_s$ can be represented as a sum of finitely many eigenvectors of Laplacian Δ. In [6] B.-Y. Chen et al. proved that a ruled surface of finite-type in an m-dimensional Euclidean space is an open part of either a cylinder over a curve of finite-type or a helicoid in $\mathbb{E}^3$. It follows that a ruled surface of

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finite-type in $\mathbb{E}^3$ is part of a plane, a circular cylinder or a helicoid. F. Dillen extended these results to ruled submanifolds in Euclidean space with finite-type immersion [14]. (For finite type immersions, see [4].) Of course, such a notion of finite-type can be extended to any smooth maps on submanifolds.

As is well known that Gauss map plays an important role in the theory of submanifolds. For oriented surfaces of 3-dimensional Euclidean space, it can be used to measure the total curvature of the Gauss curvature which gives some topological character. However, a right cone or a helicoid in Euclidean 3-space has its Gauss map G satisfying $\Delta G = f(G + C)$ for some non-zero function f and a constant vector C. Generalizing such a notion, one of the authors defined a notion of pointwise 1-type Gauss map: The Gauss map G on a submanifold M in $\mathbb{L}^m$ is said to be of pointwise 1-type if it satisfies

$$\Delta G = f(G + C)$$

for some non-zero smooth function f and a constant vector C [5]. More precisely speaking, if C is zero, it is said to be of pointwise 1-type of the first kind. Otherwise, it is said to be of pointwise 1-type of the second kind [5, 8–12, 26, 27]. Especially, submanifolds of Euclidean space or Minkowski space with pointwise 1-type Gauss map of the first kind have close relationship with those with constant mean curvature. In [27, 28], the authors proved that ruled surfaces in Minkowski space with pointwise 1-type of the first kind are minimal or of constant mean curvature depending on the dimension of the ambient space. Recently, U. Dursun showed that all oriented hypersurfaces in Minkowski space with constant mean curvature are characterized with pointwise 1-type Gauss map of the first kind [15]. In [16] all flat timelike rotational surface of elliptic and hyperbolic types with pointwise 1-type Gauss map of first and second kind are classified.

On the other hand, for the ruled surfaces with null rulings in Minkowski m-space, two of the present authors et al. defined the extended B-scroll and the generalized B-scroll which are generalizations of a usual B-scroll in 3-dimensional Minkowski space and they completely classified the family of ruled surfaces of Minkowski space with finite-type Gauss map [1, 2, 23, 24]. In [19, 25] the ruled surfaces and the ruled submanifolds of finite-type immersion in Minkowski space were studied and classification theorems of such ruled surfaces and ruled submanifolds were given.

Very recently the authors classified ruled submanifolds with harmonic Gauss map in Minkowski space and characterized minimal ruled submanifolds in Minkowski space with harmonic Gauss map [21].

A submanifold M in a pseudo Riemannian manifold N is said to be marginally trapped (or pseudo-minimal) if its mean curvature vector is null. In particular, marginally trapped surfaces in a space-time which play an important role in general relativity have been studied by many scientists [7, 13, 17, 30]. In [29], V. Milousheva and N. C. Turgay proved that a non-flat marginally trapped surface with flat normal connection has pointwise 1-type Gauss map if and only if it has constant mean curvature.

We now pose a natural question: *Can we completely classify ruled submanifolds in Minkowski space with pointwise* 1*-type Gauss map of the first kind?*

In this paper, we study ruled submanifolds of $\mathbb{L}^m$ with the notion of the Gauss map of pointwise 1-type of the first kind and examine an relationship regarding ruled submanifolds of $\mathbb{L}^m$ with constant mean curvature.

2. Preliminaries

Let $\mathbb{E}^m_s$ be an m-dimensional pseudo-Euclidean space of signature $(m-s, s)$. In particular, for $m \geq 2$, $\mathbb{E}^m_1$ is called a *Lorentz-Minkowski* m-space or simply *Minkowski* m-space, which is denoted by $\mathbb{L}^m$. A curve in $\mathbb{L}^m$ is said to be *space-like*, *time-like* or *null* if its tangent vector field is space-like, time-like or null, respectively.

Let $x\colon M \to \mathbb{E}^m_s$ be an isometric immersion of an n-dimensional pseudo-Riemannian manifold M into $\mathbb{E}^m_s$. From now on, a submanifold in $\mathbb{E}^m_s$ always means pseudo-Riemannian, that is, each tangent space of the submanifold M in $\mathbb{E}^m_s$ is non-degenerate.

Let $(x_1, x_2, \ldots, x_n)$ be a local coordinate system of M in $\mathbb{E}^m_s$. For the components g_{ij} of the pseudo-Riemannian metric $\langle \cdot, \cdot \rangle$ on M induced from that of $\mathbb{E}^m_s$, we denote by (g^{ij}) (respectively, $\mathcal{G}$) the inverse matrix (respectively, the determinant) of the matrix (g_{ij}) of the components of the induced metric $\langle \cdot, \cdot \rangle$. Then, the Laplacian Δ defined on M is given by

$$\Delta = -\frac{1}{\sqrt{|\mathcal{G}|}} \sum_{i,j} \frac{\partial}{\partial x_i} \left(\sqrt{|\mathcal{G}|} g^{ij} \frac{\partial}{\partial x_j} \right).$$

We now define the Gauss map G on M with the Grassmannian manifold. Consider the map $G\colon M \to G(n, m)$ of a point p of M mapped to the oriented tangent space at p, where $G(n, m)$ is the Grassmannian manifold consisting of all oriented n-planes passing through the origin. Roughly speaking, it can be achieved by parallel displacement of the oriented tangent space at p to the origin of $\mathbb{L}^m$. By an isomorphism, $G(n, m)$ can be identified with $G(m-n, m)$. Let us express the Gauss map rigorously. Choose an adapted local orthonormal frame $\{e_1, e_2, \ldots, e_m\}$ in $\mathbb{E}^m_s$ such that $e_1, e_2, \ldots, e_n$ are tangent to M and $e_{n+1}, e_{n+2}, \ldots, e_m$ normal to M. Define the map $G\colon M \to G(n, m) \subset \mathbb{E}^N$ $(N = {}_mC_n)$, $G(p) = (e_1 \wedge e_2 \wedge \cdots \wedge e_n)(p)$.

An indefinite scalar product $\langle\langle \cdot, \cdot \rangle\rangle$ on $G(n, m) \subset \mathbb{E}^N$ is defined by

$$\langle\langle e_{i_1} \wedge \cdots \wedge e_{i_n}, e_{j_1} \wedge \cdots \wedge e_{j_n} \rangle\rangle = \det(\langle e_{i_l}, e_{j_k} \rangle).$$

Then, $\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} \mid 1 \leq i_1 < \cdots < i_n \leq m\}$ is an orthonormal basis of $\mathbb{E}^N_k$ for some positive integer k.

Now, let us recall the notion of a ruled submanifold M in $\mathbb{L}^m$. A non-degenerate $(r+1)$-dimensional submanifold M in $\mathbb{L}^m$ is called a *ruled submanifold* if M is foliated by r-dimensional totally geodesic submanifolds $E(s,r)$ of $\mathbb{L}^m$ along a regular curve $\alpha = \alpha(s)$ on M defined on an open interval I. Thus, we can give a parametrization of a ruled submanifold M in $\mathbb{L}^m$ by

$$x = x(s, t_1, t_2, \ldots, t_r) = \alpha(s) + \sum_{i=1}^{r} t_i e_i(s), \quad s \in I,\ t_i \in I_i, \tag{2.1}$$

where I_i's are some open intervals for $i = 1, 2, \ldots, r$. Without loss of generality, we may assume that $0 \in I_i$ for all $i = 1, 2, \ldots, r$. For each s, $E(s,r)$ is open in $\mathrm{Span}\{e_1(s), e_2(s), \ldots, e_r(s)\}$, which is the linear span of linearly independent vector fields $e_1(s), e_2(s), \ldots, e_r(s)$ along the curve α. Here, we assume $E(s,r)$ are either non-degenerate or degenerate for all s along α. We call $E(s,r)$ the *rulings* and α the *base curve* of the ruled submanifold M. In particular, the ruled submanifold M is said to be *cylindrical* if $E(s,r)$ are parallel along α, or *non-cylindrical* otherwise.

Remark 2.1. [19, 20] (1) If the rulings of M are non-degenerate, then the base curve α can be chosen to be orthogonal to the rulings as follows: Let V be a unit vector field on M which is orthogonal to the rulings. Then α can be taken as an integral curve of V.

(2) If the rulings are degenerate, we can choose a null base curve which is transversal to the rulings: Let V be a null vector field on M which is not tangent to the rulings. An integral curve of V can be the base curve.

Definition 2.2. [7] A space-like submanifold M of the Minkowski space $\mathbb{L}^m$ is called marginally trapped or pseudo-minimal if the mean curvature vector field is null.

Definition 2.3. [18] An $(r+1)$-dimensional cylindrical ruled submanifold M is called a generalized circular cylinder $\Sigma \times \mathbb{E}^{r-1}$ if the base curve α is a circle and the generators of rulings are orthogonal to the plane containing the circle α, where Σ is a circular cylinder in $\mathbb{E}^3$.

Similarly, we can define the generalized hyperbolic cylinder.

Definition 2.4. An $(r+1)$-dimensional cylindrical ruled submanifold M is called a generalized hyperbolic cylinder $\Sigma_h \times \mathbb{E}^{r-1}$ if the base curve α is a hyperbola in $\mathbb{L}^2$ and the generators of rulings are orthogonal to the plane containing the hyperbola α, where Σ_h is a hyperbolic cylinder over α in $\mathbb{L}^3$.

By solving a system of ordinary differential equations similarly set up relative to a frame along a curve in $\mathbb{L}^m$ as given in [3], we have

Lemma 2.5. [20] *Let $V(s)$ be a smooth l-dimensional non-degenerate distribution in the Minkowski m-space $\mathbb{L}^m$ along a curve $\alpha = \alpha(s)$, where $l \geq 2$ and $m \geq 3$. Then, we can choose orthonormal vector fields $e_1(s), \ldots, e_{m-l}(s)$ along α which generate the orthogonal complement $V^{\perp}(s)$ satisfying $e_i'(s) \in V(s)$ for $1 \leq i \leq m-l$.*

Let M be a non-cylindrical ruled submanifold in $\mathbb{L}^m$ whose some of generating vector fields of rulings are constant vectors fields. By modifying a similar argument to get Proposition 3.3 in [18], we have

Proposition 2.6. *Let M be an $(r+1)$-dimensional non-cylindrical ruled submanifold of $\mathbb{L}^m$ parametrized by (2.1). Suppose that some of generators $e_{j_1}, e_{j_2}, \ldots, e_{j_k}$ $(1 \leq k < r)$ of the rulings are constant vectors along α. Then, M has pointwise 1-type Gauss map of the first kind if and only if the ruled submanifold M_1 has pointwise 1-type Gauss map of the first kind, where M_1 is non-cylindrical ruled submanifold defined over the base curve α with the rulings generated by e_j for $j \neq j_1, j_2, \ldots, j_k$.*

3. Marginally trapped ruled submanifolds in $\mathbb{L}^m$

Let M be a marginally trapped ruled submanifold in the Minkowski space $\mathbb{L}^m$ parameterized by

$$x = x(s, t_1, t_2, \ldots, t_r) = \alpha(s) + \sum_{i=1}^{r} t_i e_i(s), \quad s \in I,\ t_i \in I_i, \tag{3.1}$$

where I_i's are some open intervals for $i = 1, 2, \ldots, r$. Without loss of generality, we may assume that α is a unit speed curve, that is, $\langle \alpha'(s), \alpha'(s) \rangle = 1$, and $0 \in I_i$ for all $i = 1, 2, \ldots, r$. Also, by Remark 2.1 and Lemma 2.5, we may assume that orthonormal vector fields $e_1(s), \ldots, e_r(s)$ along α satisfy

$$\langle \alpha', e_i \rangle = 0 = \langle e_j', e_i \rangle \quad \text{and} \quad \langle e_i, e_j \rangle = \delta_{ij}$$

for $i, j = 1, 2, \ldots, r$. From now on, the prime $'$ stands for d/ds unless otherwise stated.

Then, the mean curvature vector field H of M is defined by

$$\begin{aligned} H &= \frac{1}{r+1} \left\{ h\left(\frac{x_s}{\|x_s\|}, \frac{x_s}{\|x_s\|} \right) + \sum_{i=1}^{r} h(x_{t_i}, x_{t_i}) \right\} \\ &= \frac{1}{r+1} \left\{ \frac{1}{q} h(x_s, x_s) + \sum_{i=1}^{r} h(x_{t_i}, x_{t_i}) \right\}, \end{aligned}$$

where h is the second fundamental form of M and q is the function of s and t_i defined by

$$q = \langle x_s, x_s \rangle = 1 + \sum_{i=1}^{r} 2u_i t_i + \sum_{i,j=1}^{r} w_{ij} t_i t_j,$$

where $u_i(s) = \langle \alpha', e_i' \rangle$ and $w_{ij}(s) = \langle e_i', e_j' \rangle$ for $i, j = 1, 2, \ldots, r$. Note that q is a polynomial in $t = (t_1, \ldots, t_r)$ with functions in s as coefficients. From now on, for a polynomial $F(t)$ in $t = (t_1, t_2, \ldots, t_r)$, $\deg F(t)$ denotes the degree of $F(t)$ in $t = (t_1, t_2, \ldots, t_r)$ unless otherwise stated.

Since M is marginally trapped and $x_{t_i t_i} = 0$,

$$H = \frac{1}{2q}\left(x_{ss} - \frac{1}{q}\langle x_{ss}, x_s \rangle x_s - \sum_{i=1}^{r} \langle x_{ss}, x_{t_i} \rangle x_{t_i} \right) \tag{3.2}$$

is null at each point of M. Therefore, we can consider a pseudo-orthonormal normal frame field $\{n_1, n_2, e_{r+3}, \ldots, e_{m-1}\}$ such that

$$n_1 = H, \quad \langle n_1, n_1 \rangle = 0 = \langle n_2, n_2 \rangle, \quad \langle n_1, n_2 \rangle = -1,$$
$$\langle n_1, e_a \rangle = 0 = \langle n_2, e_a \rangle \quad \text{and} \quad \langle e_a, e_b \rangle = \delta_{ab}$$

for $a, b = r+3, r+4, \ldots, m-1$.

Now, we will show that there do not exist marginally trapped ruled submanifolds in $\mathbb{L}^m$ with pointwise 1-type Gauss map of the first kind.

Suppose that a marginally trapped ruled submanifold M of $\mathbb{L}^m$ has pointwise 1-type Gauss map of the first kind.

First, we consider the case that e_i' are non-null for all $i = 1, 2, \ldots, r$. If $e_i' = \mathbf{0}$ for all i, that is, M is cylindrical, then

$$q = 1$$

and the mean curvature vector field H, the Gauss map G and the Laplacian Δ of M are respectively given by

$$H = \frac{1}{r+1}\alpha'', \quad G = \alpha' \wedge e_1 \wedge \cdots \wedge e_r \quad \text{and} \quad \Delta = -\frac{\partial^2}{\partial s^2} - \sum_{i=1}^{r} \frac{\partial^2}{\partial t_i^2},$$

where $\mathbf{0}$ denotes zero vector. Note that α'' is null for all s. Equation $\Delta G = fG$ provides that

$$\Delta' \alpha' \wedge e_1 \wedge \cdots \wedge e_r = f\alpha' \wedge e_1 \wedge \cdots \wedge e_r,$$

which gives

$$\alpha''' = -f\alpha',$$

where $\Delta' = -\partial^2/\partial s^2$ is the Laplacian of α. But,

$$0 = \langle \alpha'', \alpha'' \rangle = -\langle \alpha''', \alpha' \rangle = f\langle \alpha', \alpha' \rangle = f$$

for all s, a contradiction. Therefore, according to Proposition 2.6, we may assume that e_i' is non-zero for all i. In this case, $\deg q = 2$. As will be used the same argument to be developed in Section 4, we quote the result

$$\frac{\partial q}{\partial s} = 0$$

if $\Delta G = fG$. Thus, the mean curvature vector field H of (3.2) is expressed as

$$\begin{aligned} H = \frac{1}{(r+1)q^2} \Bigg\{ & \left(\alpha'' + \sum_{i=1}^{r} u_i e_i\right) + \sum_{i=1}^{r} \left(2u_i\alpha'' + e_i'' + 2u_i \sum_{j=1}^{r} u_j e_j + \sum_{j=1}^{r} w_{ij} e_j\right) t_i \\ & + \sum_{i,j=1}^{r} \left(w_{ij}\alpha'' + 2u_i e_j'' + 2u_i \sum_{k=1}^{r} w_{jk} e_k + w_{ij} \sum_{k=1}^{r} u_k e_k\right) t_i t_j \\ & + \sum_{i,j,k=1}^{r} \left(w_{ij} e_k'' + w_{ij} \sum_{l=1}^{r} w_{kl} e_l\right) t_i t_j t_k \Bigg\}. \end{aligned} \tag{3.3}$$

Also, the Gauss map G and the Laplacian Δ of M are given by respectively,

$$G = \frac{1}{|q|^{1/2}} \left(\Phi + \sum_{j=1}^{r} t_i \Psi_j\right),$$

$$\Delta = \frac{1}{2q^2}\frac{\partial q}{\partial s}\frac{\partial}{\partial s} - \frac{1}{q}\frac{\partial^2}{\partial s^2} - \frac{1}{2q}\sum_{i=1}^{r}\frac{\partial q}{\partial t_i}\frac{\partial}{\partial t_i} - \sum_{i=1}^{r}\frac{\partial^2}{\partial t_i^2},$$

where Φ and Ψ_i are the vectors defined by

$$\Phi = \alpha' \wedge e_1 \wedge \cdots \wedge e_r \quad \text{and} \quad \Psi_i = e_i' \wedge e_1 \wedge \cdots \wedge e_r.$$

Then, the equation $\Delta G = fG$ can be rewritten as

$$\begin{aligned} & q\left(\Phi'' + \sum_{j=1}^{r} t_j \Psi_j''\right) - \frac{1}{2} q \sum_{i=1}^{r} q_{t_i} \Psi_i \\ & + \left\{\frac{1}{2}\sum_{i=1}^{r}(q_{t_i}^2 - q q_{t_i t_i}) + f q^2\right\} \left(\Phi + \sum_{j=1}^{r} t_j \Psi_j\right) = \mathbf{0}. \end{aligned} \tag{3.4}$$

Note that the left-hand side of (3.4) is a polynomial of t with the functions of s as the coefficients.

Meanwhile, along the curve α, we may put

$$\alpha'' = -\sum_{i=1}^{r} u_i e_i + (\alpha'')^{\perp}, \tag{3.5}$$

where $\perp$ denotes the normal parts of the corresponding vector fields. Since the mean curvature vector field H is null along the base curve α, (3.3) and (3.5) tell us that $(\alpha'')^{\perp}$ has to be null for all s. Furthermore, we can see that

$$H = \frac{1}{r+1}(\alpha'')^{\perp}$$

along the curve α. Therefore, using the frame $\{\alpha', e_1, \ldots, e_r, n_1, n_2, e_{r+3}, \ldots, e_{m-1}\}$, along the curve α, we can put

$$
\begin{aligned}
\alpha'' &= -\sum u_i e_i - \langle \alpha'', n_2\rangle n_1,\\
\alpha''' &= -\left(\sum u_i^2\right)\alpha' + \sum \langle \alpha''', e_j\rangle e_j - \langle \alpha''', n_2\rangle n_1 - \langle \alpha''', n_1\rangle n_2 + \sum \langle \alpha''', e_a\rangle e_a,\\
e_i' &= u_i \alpha' - \langle e_i', n_2\rangle n_1 - \langle e_i', n_1\rangle n_2 + \sum \langle e_i', e_a\rangle e_a,\\
e_i'' &= \langle e_i'', \alpha'\rangle \alpha' - \sum w_{ij} e_j - \langle e_i'', n_2\rangle n_1 - \langle e_i'', n_1\rangle n_2 + \sum \langle e_i'', e_a\rangle e_a,\\
e_i''' &= \langle e_i''', \alpha'\rangle \alpha' + \sum \langle e_i''', e_j\rangle e_j - \langle e_i''', n_2\rangle n_1 - \langle e_i''', n_1\rangle n_2 + \sum \langle e_i''', e_a\rangle e_a
\end{aligned}
\tag{3.6}
$$

for $i, j = 1, 2, \ldots, r$ and $a = r+3, \ldots, m-1$. Here, $\langle \alpha'', n_2\rangle$ has to be non-zero for all s.

We now examine (3.4), from the definitions of Φ and Ψ_j, we obtain

$$
\begin{aligned}
\Phi'' &= \alpha''' \wedge e_1 \wedge \cdots \wedge e_r + 2\sum_{i=1}^{r} \alpha'' \wedge e_1 \wedge \cdots \wedge e_i' \wedge \cdots \wedge e_r\\
&\quad + 2\sum_{i<k}^{r} \alpha' \wedge e_1 \wedge \cdots \wedge e_i' \wedge \cdots \wedge e_k' \wedge \cdots \wedge e_r + \sum_{i=1}^{r} \alpha' \wedge e_1 \wedge \cdots \wedge e_i'' \wedge \cdots \wedge e_r,\\
\Psi_j'' &= e_j''' \wedge e_1 \wedge \cdots \wedge e_r + 2\sum_{i=1}^{r} e_j'' \wedge e_1 \wedge \cdots \wedge e_i' \wedge \cdots \wedge e_r\\
&\quad + 2\sum_{i<k}^{r} e_j' \wedge e_1 \wedge \cdots \wedge e_i' \wedge \cdots \wedge e_k' \wedge \cdots \wedge e_r + \sum_{i=1}^{r} e_j' \wedge e_1 \wedge \cdots \wedge e_i'' \wedge \cdots \wedge e_r.
\end{aligned}
\tag{3.7}
$$

From which, we see that equation (3.4) consists of ten different types of vectors formed with the wedge products of $(r+1)$ vectors. Putting (3.6) into Ψ_j and (3.7), we can decompose the left-hand side of (3.4) into the tangential and the normal parts. By using the orthogonality of α', e_i, n_1, n_2 and e_a, we have

$$
\begin{aligned}
\sum_{i<k}^{r} \alpha' \wedge e_1 \wedge \cdots \wedge e_i' \wedge \cdots \wedge e_k' \wedge \cdots \wedge e_r = \mathbf{0},\\
\sum_{i<k}^{r} e_j' \wedge e_1 \wedge \cdots \wedge e_i' \wedge \cdots \wedge e_k' \wedge \cdots \wedge e_r = \mathbf{0}
\end{aligned}
$$

as the vectors containing α' and two normal vectors for $i, j, k \in \{1, 2, \ldots, r\}$. Considering the normal components of vectors contained in Φ'' as part of the constant terms of the left-hand side of (3.4), we obtain

$$2u_j \langle \alpha'', n_2\rangle - \langle e_j'', n_2\rangle = 0, \tag{3.8}$$

$$\langle \alpha'', n_2\rangle \langle e_j', n_1\rangle = 0, \quad \langle \alpha'', n_2\rangle \langle e_j', e_a\rangle = 0, \tag{3.9}$$

$$\langle e_j'', n_1\rangle = 0 \quad \text{and} \quad \langle e_j'', e_a\rangle = 0 \tag{3.10}$$

as the coefficients of $\alpha'\wedge e_1\wedge\cdots\wedge e_{j-1}\wedge n_1\wedge e_{j+1}\wedge\cdots\wedge e_r$, $n_1\wedge e_1\wedge\cdots\wedge e_{j-1}\wedge n_2\wedge e_{j+1}\wedge\cdots\wedge e_r$, $n_1 \wedge e_1 \wedge \cdots \wedge e_{j-1} \wedge e_a \wedge e_{j+1} \wedge \cdots \wedge e_r$, $\alpha' \wedge e_1 \wedge \cdots \wedge e_{j-1} \wedge n_2 \wedge e_{j+1} \wedge \cdots \wedge e_r$ and $\alpha' \wedge e_1 \wedge \cdots \wedge e_{j-1} \wedge e_a \wedge e_{j+1} \wedge \cdots \wedge e_r$ for all $j = 1, \ldots, r$ and $a = r + 3, \ldots, m - 1$, respectively. Equations (3.8), (3.9) and (3.10) yield that

$$e_j' = u_j\alpha' - \langle e_j', n_2\rangle n_1, \tag{3.11}$$

$$e_j'' = \langle e_j'', \alpha'\rangle\alpha' + \sum_{i=1}^{r} w_{ji}e_i - \langle e_j'', n_2\rangle n_1.$$

Together with (3.6), (3.11) and $u_j' = 0$, $\langle e_j'', \alpha'\rangle + \langle e_j', \alpha''\rangle = 0$ implies that

$$e_j'' = \sum_{i=1}^{r} w_{ji}e_i - \langle e_j'', n_2\rangle n_1. \tag{3.12}$$

Note that $u_j(s) \neq 0$ because e_j' is non-null for all j and for $s \in I$. Similarly, with the help of (3.11) and (3.12), considering the normal parts of vector fields contained in Ψ_j'' of the left-hand side of (3.4), we get

$$2u_k\langle e_j'', n_2\rangle - u_j\langle e_k'', n_2\rangle = 0 \tag{3.13}$$

as the coefficients of $\alpha' \wedge e_1 \wedge \cdots \wedge e_{j-1} \wedge n_1 \wedge e_{j+1} \wedge \cdots \wedge e_r$ for all $j, k = 1, \ldots, r$. Using (3.8), (3.13) implies

$$2u_ju_k\langle\alpha'', n_2\rangle = 0$$

which is a contradiction.

Therefore, we can conclude that e_j' has to be null for some $j \in \{1, 2, \ldots, r\}$. So, we suppose that some generators $e_{j_1}, e_{j_2}, \ldots, e_{j_k}$ of the rulings have null derivatives along the base curve α for $j_1 < j_2 < \cdots < j_k \in \{1, 2, \ldots, r\}$. We can rewrite the parametrization (3.1) of M as

$$x(s, t_1, \cdots, t_r) = \alpha(s) + \sum_{i\neq j_1, j_2, \ldots, j_k} t_ie_i(s) + \sum_{i=1}^{k} t_{j_i}e_{j_i}(s).$$

Then, there are two possible cases such that either all of $e_{j_{k+1}}, \ldots, e_{j_r}$ generating the rulings except $e_{j_1}(s), e_{j_2}(s), \ldots, e_{j_r}(s)$ are constant vector fields or not.

Case 1. Suppose that $e_{j_{k+1}}, \ldots, e_{j_r}$ are constant vector fields. In this case, according to Proposition 2.6, we may assume that e_i' is null for all $i = 1, \ldots, r$, otherwise the ruled submanifold M is a cylinder over the ruled submanifold parameterized by the base curve α and the rulings generated by e_i's except those constant vector fields. We then have three possible subcases according to the degree of q.

Subcase 1.1. Let $\deg q(t) = 0$. In this case, e_i' are null with $e_i'(s) \wedge e_l'(s) = 0$ for $i, l = 1, 2, \ldots, r$ and $\langle \alpha'(s), e_j'(s) \rangle = 0$ for $j = 1, 2, \ldots, r$. The mean curvature vector field H is given by

$$H = \frac{1}{r+1}\left(\alpha'' + \sum_{i=1}^{r} t_i e_i''\right). \tag{3.14}$$

Clearly, H is null along the curve α which implies that α'' has to be null for all $s \in I$. Also, the nullity of H yields that

$$\langle \alpha'', \alpha'' \rangle = \langle \alpha'', e_i'' \rangle = \langle e_i'', e_i'' \rangle = 0$$

and hence

$$\alpha'' \wedge e_i'' = \mathbf{0}$$

for all i. Therefore, we can put

$$e_i''(s) = \phi_i(s)\alpha''(s) \tag{3.15}$$

for some functions ϕ_i of s and for all i. With the help of (3.14) and (3.15), we have

$$H = \frac{1}{r+1}\left(1 + \sum_{i=1}^{r} \phi_i t_i\right)\alpha''. \tag{3.16}$$

On the other hand, the Gauss map of M is given by

$$G = \Phi + \sum_{i=1}^{r} t_i \Psi_i$$

and $\Delta G = fG$ implies

$$(\alpha' \wedge e_1 \wedge \cdots \wedge e_r)'' = -f\alpha' \wedge e_1 \wedge \cdots \wedge e_r,$$
$$(e_i' \wedge e_1 \wedge \cdots \wedge e_r)'' = -fe_i' \wedge e_1 \wedge \cdots \wedge e_r,$$

or, equivalently,

$$\begin{aligned} &\alpha''' \wedge e_1 \wedge \cdots \wedge e_r + 2\sum_{j=1}^{r} \alpha'' \wedge e_1 \wedge \cdots \wedge e_j' \wedge \cdots \wedge e_r \\ &+ \sum_{j=1}^{r} \alpha' \wedge e_1 \wedge \cdots \wedge e_j'' \wedge \cdots \wedge e_r = -f\alpha' \wedge e_1 \wedge \cdots \wedge e_r, \end{aligned} \tag{3.17}$$

$$e_i''' \wedge e_1 \wedge \cdots \wedge e_r + \sum_{j=1}^{r} e_i'' \wedge e_1 \wedge \cdots \wedge e_j' \wedge \cdots \wedge e_r = -fe_i' \wedge e_1 \wedge \cdots \wedge e_r. \tag{3.18}$$

By (3.15) and (3.16), using the frame $\{\alpha', e_1, \ldots, e_r, n_1, n_2, e_{r+3}, \ldots, e_{m-1}\}$, along the curve α, we have

$$\begin{aligned}
\alpha'' &= -\langle \alpha'', n_2\rangle n_1,\\
\alpha''' &= \sum_j \langle \alpha''', e_j\rangle e_j - \langle \alpha''', n_2\rangle n_1 - \langle \alpha''', n_1\rangle n_2 + \sum_a \langle \alpha''', e_a\rangle e_a,\\
e_i' &= -\langle e_i', n_2\rangle n_1 - \langle e_i', n_1\rangle n_2 + \sum_a \langle e_i', e_a\rangle e_a,\\
e_i'' &= -\langle e_i'', n_2\rangle n_1,\\
e_i''' &= -\langle e_i''', n_2\rangle n_1 - \langle e_i''', n_1\rangle n_2 + \sum_a \langle e_i''', e_a\rangle e_a
\end{aligned} \tag{3.19}$$

for $i, j = 1, 2, \ldots, r$ and $a = r+3, \ldots, m-1$. Using (3.18) and (3.19), we repeat the same methods to get (3.8), (3.9), (3.10) and (3.13). Then, from (3.18), we get

$$\langle e_i''', n_2\rangle = -f\langle e_i', n_2\rangle, \tag{3.20}$$

$$\langle e_i''', n_1\rangle = -f\langle e_i', n_1\rangle, \tag{3.21}$$

$$\langle e_i''', e_a\rangle = -f\langle e_i', e_a\rangle, \tag{3.22}$$

$$\langle e_i'', n_2\rangle\langle e_j', n_1\rangle = \langle e_i'', n_2\rangle\langle e_j', e_a\rangle = 0 \tag{3.23}$$

as the coefficients of $n_1 \wedge e_1 \wedge \cdots \wedge e_r$, $n_2 \wedge e_1 \wedge \cdots \wedge e_r$, $e_a \wedge e_1 \wedge \cdots \wedge e_r$, $n_1 \wedge e_1 \wedge \cdots \wedge e_{j-1} \wedge n_2 \wedge e_{j+1} \wedge \cdots \wedge e_r$ and $n_1 \wedge e_1 \wedge \cdots \wedge e_{j-1} \wedge e_a \wedge e_{j+1} \wedge \cdots \wedge e_r$ for all $j = 1, \ldots, r$ and $a = r+3, \ldots, m-1$, respectively.

If $\langle e_i'', n_2\rangle \neq 0$, then

$$\langle e_j', n_1\rangle = \langle e_j', e_a\rangle = 0 \tag{3.24}$$

for all $j = 1, 2, \ldots, r$ by (3.23). Together with (3.19) and (3.24), equation (3.17) gives us the following

$$\begin{aligned}
&\alpha''' \wedge e_1 \wedge \cdots \wedge e_r - \sum_{j=1}^{r} \langle e_j'', n_2\rangle \alpha' \wedge e_1 \wedge \cdots \wedge e_{j-1} \wedge n_1 \wedge e_{j+1} \wedge \cdots \wedge e_r\\
&= -f\alpha' \wedge e_1 \wedge \cdots \wedge e_r,
\end{aligned}$$

which implies that $\langle e_j'', n_2\rangle = 0$, a contradiction. Therefore, we have

$$\langle e_j'', n_2\rangle = 0,$$

which, together with (3.19) implies $e_j'' = \mathbf{0}$ for all s and all j. In (3.20), (3.21) and (3.22), we get

$$f\langle e_j', n_2\rangle = f\langle e_j', n_1\rangle = f\langle e_j', e_a\rangle = 0$$

for all j. Since f is non-zero, e'_j of (3.19) have the value zero at some point $s_0 \in I$. This contradicts the character of e'_j. Thus, this case never occur.

Subcase 1.2. Let $\deg q(t) = 1$. In this case, $\langle \alpha'(s), e'_i(s)\rangle \neq 0$ for some i $(1 \leq i \leq r)$ and the null vector fields e'_i satisfy $e'_i \wedge e'_l = 0$ for $i, l = 1, 2, \ldots, r$. By the same reason to get $\partial q/\partial s = 0$ previously, we have $u'_i = 0$ for all i from $\Delta G = fG$. Thus, the mean curvature vector field H is given by

$$H = \frac{1}{(r+1)q}\left(\alpha'' + \sum_{i=1}^{r} t_i e''_i + \sum_{i=1}^{r} u_i e_i\right). \tag{3.25}$$

We now put

$$\alpha'' = -\sum_{i=1}^{r} u_i e_i + (\alpha'')^{\perp}. \tag{3.26}$$

The nullity of H of (3.25) guarantees that $(\alpha'')^{\perp}$ is null for all s. Furthermore, it gives us

$$\langle \alpha'', \alpha''\rangle = \sum_{i=1}^{r} u_i^2 \quad \text{and} \quad \langle \alpha'', e''_j\rangle = 0 = \langle e''_j, e''_j\rangle \tag{3.27}$$

for all j. Combining (3.26) and (3.27), we see that

$$e''_j = \phi_j(s)(\alpha'')^{\perp}$$

for some function ϕ_j of s. Therefore, we have

$$H = \frac{1}{(r+1)q}\left(1 + \sum_{i=1}^{r} \phi_i t_i\right)(\alpha'')^{\perp}$$

and hence

$$\begin{aligned}
\alpha'' &= -\sum_j u_j e_j - \langle \alpha'', n_2\rangle n_1,\\
\alpha''' &= -\sum_j u_j^2 \alpha' + \sum_j \langle \alpha''', e_j\rangle e_j - \langle \alpha''', n_2\rangle n_1 - \langle \alpha''', n_1\rangle n_2 + \sum_a \langle \alpha''', e_a\rangle e_a,\\
e'_i &= u_i \alpha' - \langle e'_i, n_2\rangle n_1 - \langle e'_i, n_1\rangle n_2 + \sum_a \langle e'_i, e_a\rangle e_a,\\
e''_i &= -\langle e''_i, n_2\rangle n_1,\\
e'''_i &= \langle e'''_i, \alpha'\rangle \alpha' + \sum_j \langle e'''_i, e_j\rangle e_j - \langle e'''_i, n_2\rangle n_1 - \langle e'''_i, n_1\rangle n_2 + \sum_a \langle e'''_i, e_a\rangle e_a
\end{aligned}$$

along the curve α for $i, j = 1, \ldots, r$ and $a = r+3, \ldots, m-1$. Then, equation $\Delta G = fG$ is rewritten as

$$q\left(\Phi'' + \sum_{j=1}^{r} t_j \Psi''_j\right) - \frac{1}{2} q \sum_{i=1}^{r} q_{t_i} \Psi_i + \left\{\frac{1}{2}\sum_{i=1}^{r} (q_{t_i})^2 + f q^2\right\}\left(\Phi + \sum_{j=1}^{r} t_j \Psi_j\right) = \mathbf{0},$$

which provides

$$2u_j\langle \alpha'', n_2\rangle = \langle e_j'', n_2\rangle \quad \text{and} \quad \langle \alpha'', n_2\rangle\langle e_j', n_1\rangle = 0 = \langle \alpha'', n_2\rangle\langle e_j', e_a\rangle \tag{3.28}$$

for all $j = 1, \ldots, r$ and $a = r+3, \ldots, m-1$ by applying the similar approaches as we did previously. Since $\langle \alpha'', n_2\rangle$ in (3.28) is non-vanishing for all s, we have

$$e_j' = u_j\alpha' - \langle e_j', n_2\rangle n_1$$

which implies that

$$u_j = 0$$

because of $w_{jj} = \langle e_j', e_j'\rangle = 0$ for all j. This is a contradiction to $\deg q = 1$. Therefore, this case also never occur.

Subcase 1.3. Let $\deg q(t) = 2$. In this case, by referring to the case that $e_1', e_2', \ldots, e_r'$ are non-null, we can get (3.11), that is,

$$e_j' = u_j\alpha' - \langle e_j', n_2\rangle n_1$$

for all $j = 1, \ldots, r$. The nullity of e_j' implies that

$$u_j = 0 \quad \text{and hence} \quad e_i' \wedge e_j' = \mathbf{0}$$

for all $i, j = 1, \ldots, r$. This contradicts $\deg q = 2$.

Consequently, we can see that in this case, there is no marginally trapped ruled submanifold in $\mathbb{L}^m$ with pointwise 1-type Gauss map of the first kind.

Case 2. Suppose that $e_i' \neq \mathbf{0}$ for some $i = j_{k+1}, \ldots, j_r$.

In this case, we may also assume that $e_i' \neq \mathbf{0}$ for all $i = j_{k+1}, \ldots, j_r$. Then, e_i' are non-null for all $i = j_{k+1}, \ldots, j_r$ and $\deg q = 2$. If we follow the similar argument for the case that $e_1', e_2', \ldots, e_r'$ are non-null, we obtain $e_j' = -\langle e_j', n_2\rangle n_1$ if e_j' is null, or, $e_j' = u_j\alpha' - \langle e_j', n_2\rangle n_1$ for some non-zero function u_j. Then, we have

$$w_{ij} = \langle e_i', e_j'\rangle = 0$$

which is a contradiction.

Consequently, we have

Theorem 3.1. *There do not exist marginally trapped ruled submanifolds in $\mathbb{L}^m$ with pointwise 1-type Gauss map of the first kind.*

4. Characterizations of generalized circular and hyperbolic cylinders

Let M be an $(r+1)$-dimensional ruled submanifold in $\mathbb{L}^m$ with non-degenerate rulings. Then, by Remark 2.1 and Lemma 2.5, we may assume that

$$\langle \alpha'(s), \alpha'(s)\rangle = \varepsilon\ (=\pm 1), \quad \langle \alpha'(s), e_i(s)\rangle = 0 \quad \text{and} \quad \langle e_i'(s), e_j(s)\rangle = 0 \tag{4.1}$$

for $i, j = 1, 2, \ldots, r$. A parametrization of M is given by

$$x = x(s, t_1, t_2, \ldots, t_r) = \alpha(s) + \sum_{i=1}^{r} t_i e_i(s). \tag{4.2}$$

In this section, we always assume that the parametrization (4.2) satisfies Condition (4.1). Then, the Gauss map G of M is given by

$$G = \frac{1}{\|x_s\|} x_s \wedge x_{t_1} \wedge \cdots \wedge x_{t_r},$$

or, equivalently

$$G = \frac{1}{|q|^{1/2}} \left(\Phi + \sum_{i=1}^{r} t_i \Psi_i \right).$$

First, we consider the case of cylindrical ruled submanifolds that are the ones of two typical types of ruled submanifolds. Before discussing cylindrical ruled submanifolds, we consider the following lemma.

Lemma 4.1. *Suppose that a unit speed curve $\alpha(s)$ in the m-dimensional Minkowski space $\mathbb{L}^m$ defined on an interval I satisfies*

$$\alpha'''(s) = f(s)(\alpha'(s) + C),$$

where f is a function and C is a constant vector in $\mathbb{L}^m$. Then, the curve α lies in a 3-dimensional space in $\mathbb{L}^m$. In particular, if the constant vector C is zero, we see that α is a plane curve.

Proof. See Lemma 3.1 of [18]. □

Let M be a cylindrical $(r+1)$-dimensional ruled submanifold in $\mathbb{L}^m$ generated by non-degenerate rulings, which is parameterized by (4.2). Without loss of generality, we may assume that $e_1, e_2, \ldots, e_r$ generating the rulings are constant vectors.

The Laplacian Δ of M is then naturally expressed by

$$\Delta = -\varepsilon \frac{\partial^2}{\partial s^2} - \sum_{i=1}^{r} \varepsilon_i \frac{\partial^2}{\partial t_i^2},$$

where $\varepsilon_i = \langle e_i(s), e_i(s)\rangle = \pm 1$ and the Gauss map G of M is given by

$$G = \alpha' \wedge e_1 \wedge \cdots \wedge e_r.$$

If we denote by Δ' the Laplacian of α, that is $\Delta' = -\varepsilon \frac{\partial^2}{\partial s^2}$, the Laplacian ΔG of the Gauss map becomes

$$\Delta G = \Delta' \alpha' \wedge e_1 \wedge \cdots \wedge e_r.$$

We now suppose that the Gauss map G is of pointwise 1-type of the first kind, that is, $\Delta G = fG$ for some non-zero smooth function f. Then we have

$$\Delta' \alpha' \wedge e_1 \wedge \cdots \wedge e_r = f\alpha' \wedge e_1 \wedge \cdots \wedge e_r$$

and hence

$$\Delta' \alpha' = f\alpha'. \tag{4.3}$$

From (4.3), we see that $f = \langle \alpha'', \alpha''\rangle$ is a non-zero constant by considering the Frenet equations in Minkowski space. Thus, the curvature of the non-null base curve is non-zero constant. Furthermore, Lemma 4.1 implies that the curve α is contained in the 2-dimensional subspace of $\mathbb{L}^m$. Therefore, we can see that the plane curve α is part of a circle or a hyperbola.

Conversely, it is easy to show that a generalized circular cylinder or a generalized hyperbolic cylinder has the Gauss map of pointwise 1-type of the first kind.

Therefore, we have

Theorem 4.2. *The cylindrical ruled submanifold M in $\mathbb{L}^m$ has pointwise 1-type Gauss map of the first kind if and only if M is part of a generalized circular cylinder or a generalized hyperbolic cylinder.*

Next, we consider the case of non-cylindrical ruled submanifolds. Let M be an $(r+1)$-dimensional non-cylindrical ruled submanifold parameterized by (4.2) in $\mathbb{L}^m$. Then, we have

$$x_s = \alpha'(s) + \sum_{j=1}^{r} t_j e_j'(s), \quad x_{t_i} = e_i(s)$$

for $i = 1, 2, \ldots, r$. As we introduced in Section 3, the function q is given by

$$q = \langle x_s, x_s\rangle = \varepsilon + \sum_{i=1}^{r} 2u_i t_i + \sum_{i,j=1}^{r} w_{ij} t_i t_j, \tag{4.4}$$

where $u_i(s) = \langle \alpha', e_i'\rangle$ and $w_{ij}(s) = \langle e_i', e_j'\rangle$ for $i, j = 1, \ldots, r$.

Based on Proposition 2.6, without loss of generality, we may assume that $e'_j \neq \mathbf{0}$ for all $j = 1, 2, \ldots, r$ on the domain I of α. Then, we get the components of the metric $\langle \cdot, \cdot \rangle$ on M

$$g_{11} = q, \quad g_{1j} = 0 \quad \text{and} \quad g_{ij} = \varepsilon_i \delta_{ij}$$

for $i, j = 2, 3, \ldots, r+1$. By definition of Δ, we have the Laplacian

$$\Delta = \frac{1}{2q^2}\frac{\partial q}{\partial s}\frac{\partial}{\partial s} - \frac{1}{q}\frac{\partial^2}{\partial s^2} - \frac{1}{2q}\sum_{i=1}^{r}\varepsilon_i \frac{\partial q}{\partial t_i}\frac{\partial}{\partial t_i} - \sum_{i=1}^{r}\varepsilon_i \frac{\partial^2}{\partial t_i^2}. \tag{4.5}$$

First, we suppose that $e'_1, e'_2, \ldots, e'_r$ are non-null. Then, using (4.5), $\Delta G = fG$ is rewritten as

$$\begin{aligned}
&\left(\frac{\partial q}{\partial s}\right)^2 \left(\Phi + \sum_{j=1}^{r}\Psi_j t_j\right) - \frac{3}{2}q\frac{\partial q}{\partial s}\left(\Phi' + \sum_{j=1}^{r}\Psi'_j t_j\right) - \frac{1}{2}q\frac{\partial^2 q}{\partial s^2}\left(\Phi + \sum_{j=1}^{r}\Psi_j t_j\right) \\
&+ q^2\left(\Phi'' + \sum_{j=1}^{r}\Psi''_j t_j\right) + \frac{1}{2}q\sum_{i=1}^{r}\varepsilon_i\left(\frac{\partial q}{\partial t_i}\right)^2\left(\Phi + \sum_{j=1}^{r}\Psi_j t_j\right) - \frac{1}{2}q^2\sum_{i=1}^{r}\varepsilon_i\frac{\partial q}{\partial t_i}\Psi_i \\
&- \frac{1}{2}q^2\sum_{i=1}^{r}\varepsilon_i\frac{\partial^2 q}{\partial t_i^2}\left(\Phi + \sum_{j=1}^{r}\Psi_j t_j\right) + fq^3\left(\Phi + \sum_{j=1}^{r}\Psi_j t_j\right) = \mathbf{0}.
\end{aligned} \tag{4.6}$$

To deal with the above equation (4.6), we use the indefinite scalar product $\langle\langle \cdot, \cdot \rangle\rangle$ on $G(r+1, m)$. We then have

$$\begin{gathered}
\langle\langle \Phi, \Phi \rangle\rangle = \widetilde{\varepsilon}, \quad \langle\langle \Phi, \Phi' \rangle\rangle = 0, \\
\langle\langle \Phi, \Phi'' \rangle\rangle = -\widetilde{\varepsilon}\varepsilon\mu + 2\sum_{k=1}^{r}\widetilde{\varepsilon}\varepsilon\varepsilon_k u_k^2 - \sum_{k=1}^{r}\widetilde{\varepsilon}\varepsilon_k w_{kk}, \\
\langle\langle \Phi, \Psi_i \rangle\rangle = \widetilde{\varepsilon}\varepsilon u_i, \quad \langle\langle \Phi, \Psi'_i \rangle\rangle = \widetilde{\varepsilon}\varepsilon p_i, \\
\langle\langle \Phi, \Psi''_i \rangle\rangle = \widetilde{\varepsilon}\varepsilon y_i + 2\sum_{k=1}^{r}\widetilde{\varepsilon}\varepsilon\varepsilon_k u_k w_{ik} - \sum_{k=1}^{r}\widetilde{\varepsilon}\varepsilon\varepsilon_k u_i w_{kk}, \\
\langle\langle \Psi_i, \Phi' \rangle\rangle = \widetilde{\varepsilon}\varepsilon z_i, \quad \langle\langle \Psi_i, \Psi_j \rangle\rangle = \widetilde{\varepsilon}\varepsilon w_{ij}, \quad \langle\langle \Psi_i, \Psi'_j \rangle\rangle = \widetilde{\varepsilon}\varepsilon \xi_{ij},
\end{gathered}$$

where we put $\widetilde{\varepsilon} = \varepsilon\varepsilon_1 \cdots \varepsilon_r$, $\mu = \langle \alpha'', \alpha'' \rangle$, $p_i = \langle \alpha', e''_i \rangle$, $y_i = \langle \alpha', e'''_i \rangle$, $z_i = \langle \alpha'', e'_i \rangle$ and $\xi_{ij} = \langle e'_i, e''_j \rangle$. For later use, we note that

$$u'_i(s) = p_i(s) + z_i(s) \quad \text{and} \quad w'_{ij} = \xi_{ij} + \xi_{ji} \tag{4.7}$$

for $i, j = 1, 2, \ldots, r$. By taking the indefinite scalar product with the vector Φ to the both sides of (4.6), we obtain

$$\begin{aligned}(4.8)\qquad &\left(\frac{\partial q}{\partial s}\right)^2\left(1+\sum_{j=1}^r\varepsilon u_jt_j\right)-\frac{3}{2}q\frac{\partial q}{\partial s}\sum_{j=1}^r\varepsilon p_jt_j-\frac{1}{2}q\frac{\partial^2 q}{\partial s^2}\left(1+\sum_{j=1}^r\varepsilon u_jt_j\right)\\ &+q^2\left(\widetilde{\varepsilon}\phi+\sum_{j=1}^r\widetilde{\varepsilon}\varphi_jt_j\right)+\frac{1}{2}q\sum_{i=1}^r\varepsilon_i\left(\frac{\partial q}{\partial t_i}\right)^2\left(1+\sum_{j=1}^r\varepsilon u_jt_j\right)-\frac{1}{2}q^2\sum_{i=1}^r\varepsilon_i\frac{\partial q}{\partial t_i}\varepsilon u_i\\ &-\frac{1}{2}q^2\sum_{i=1}^r\varepsilon_i\frac{\partial^2 q}{\partial t_i^2}\left(1+\sum_{j=1}^r\varepsilon u_jt_j\right)+fq^3\left(1+\sum_{j=1}^r\varepsilon u_jt_j\right)=0,\end{aligned}$$

where we have put

$$\phi=\langle\langle\Phi,\Phi''\rangle\rangle\quad\text{and}\quad\varphi_i=\langle\langle\Phi,\Psi_i''\rangle\rangle.$$

From (4.8), we can see that the function f is a rational function in t with functions in s as coefficients which is of the form

$$(4.9)\qquad f(t)=-\frac{P(t)}{q^3\left(1+\sum_{j=1}^r\varepsilon u_jt_j\right)},$$

where we put

$$\begin{aligned}P(t)=&\left(\frac{\partial q}{\partial s}\right)^2\left(1+\sum_{j=1}^r\varepsilon u_jt_j\right)-\frac{3}{2}q\frac{\partial q}{\partial s}\sum_{j=1}^r\varepsilon p_jt_j-\frac{1}{2}q\frac{\partial^2 q}{\partial s^2}\left(1+\sum_{j=1}^r\varepsilon u_jt_j\right)\\ &+q^2\left(\widetilde{\varepsilon}\phi+\sum_{j=1}^r\widetilde{\varepsilon}\varphi_jt_j\right)+\frac{1}{2}q\sum_{i=1}^r\varepsilon_i\left(\frac{\partial q}{\partial t_i}\right)^2\left(1+\sum_{j=1}^r\varepsilon u_jt_j\right)\\ &-\frac{1}{2}q^2\sum_{i=1}^r\varepsilon_i\frac{\partial q}{\partial t_i}\varepsilon u_i-\frac{1}{2}q^2\sum_{i=1}^r\varepsilon_i\frac{\partial^2 q}{\partial t_i^2}\left(1+\sum_{j=1}^r\varepsilon u_jt_j\right).\end{aligned}$$

Putting (4.9) into (4.6) and multiplying $\left(1+\sum_{j=1}^r\varepsilon u_jt_j\right)$ with the equation obtained in such a way, we get

(4.10)

$$\begin{aligned}&-\frac{3}{2}q\left(\frac{\partial q}{\partial s}\right)\left(\Phi'+\sum_{j=1}^r t_j\Psi_j'\right)\left(1+\sum_{k=1}^r\varepsilon u_kt_k\right)+\frac{3}{2}q\left(\frac{\partial q}{\partial s}\right)\left(\Phi+\sum_{j=1}^r t_j\Psi_j\right)\sum_{k=1}^r\varepsilon p_kt_k\\ &+q^2\left(\Phi''+\sum_{j=1}^r t_j\Psi_j''\right)\left(1+\sum_{k=1}^r\varepsilon u_kt_k\right)-q^2\left(\Phi+\sum_{j=1}^r t_j\Psi_j\right)\left(\widetilde{\varepsilon}\phi+\sum_{k=1}^r\widetilde{\varepsilon}\varphi_kt_k\right)\\ &-\frac{1}{2}q^2\sum_{i=1}^r\varepsilon_i\left(\frac{\partial q}{\partial t_i}\right)\Psi_i\left(1+\sum_{k=1}^r\varepsilon u_kt_k\right)+\frac{1}{2}q^2\sum_{i=1}^r\varepsilon_i\left(\frac{\partial q}{\partial t_i}\right)\varepsilon u_i\left(\Phi+\sum_{j=1}^r t_j\Psi_j\right)=\mathbf{0}.\end{aligned}$$

Lemma 4.3. *Let M be an $(r+1)$-dimensional non-cylindrical ruled submanifold parameterized by (4.2) in $\mathbb{L}^m$ with pointwise 1-type Gauss map of the first kind. Let $e_1, e_2, \ldots, e_r$ be the orthonormal generators of the rulings along the base curve α. If e_i' are non-null for $i = 1, 2, \ldots, r$, then the functions*

$$u_i(s) = \langle \alpha'(s), e_i'(s) \rangle \quad \textit{and} \quad w_{ij}(s) = \langle e_i'(s), e_j'(s) \rangle$$

are constant functions for all $i, j = 1, 2, \ldots, r$.

Proof. We suppose that $\partial q/\partial s \neq 0$ on some open interval I_1. Then, on I_1, since each term of the left-hand side in (4.10) involves $\partial q/\partial s$ or q^2, by rearranging (4.10), we get

$$-\frac{3}{2}\left(\frac{\partial q}{\partial s}\right) R(t) = qQ(t), \tag{4.11}$$

where

$$R(t) = \left(\Phi' + \sum_{j=1}^{r} t_j \Psi_j'\right)\left(1 + \sum_{k=1}^{r} \varepsilon u_k t_k\right) - \left(\Phi + \sum_{j=1}^{r} t_j \Psi_j\right)\left(\sum_{k=1}^{r} \varepsilon p_k t_k\right)$$

and

$$\begin{aligned} Q(t) = &-\left(\Phi'' + \sum_{j=1}^{r} t_j \Psi_j''\right)\left(1 + \sum_{k=1}^{r} \varepsilon u_k t_k\right) + \left(\Phi + \sum_{j=1}^{r} t_j \Psi_j\right)\left(\widetilde{\varepsilon}\phi + \sum_{k=1}^{r} \widetilde{\varepsilon}\varphi_k t_k\right) \\ &+ \frac{1}{2}\sum_{i=1}^{r} \varepsilon_i \left(\frac{\partial q}{\partial t_i}\right)\Psi_i\left(1 + \sum_{k=1}^{r} \varepsilon u_k t_k\right) - \frac{1}{2}\sum_{i=1}^{r} \varepsilon_i \left(\frac{\partial q}{\partial t_i}\right)\varepsilon u_i\left(\Phi + \sum_{j=1}^{r} t_j \Psi_j\right). \end{aligned}$$

Recall that q is a polynomial in t of degree 2 with functions in s as coefficients. Then, we have two cases whether the function q is of the form $q(t) = \varepsilon\big(1 + \sum_{i=1}^{r} \varepsilon u_i t_i\big)^2$ or not.

Case 1. Suppose that $q \neq \varepsilon\big(1 + \sum_{i=1}^{r} \varepsilon u_i t_i\big)^2$. If we use a similar argument to get Lemma 3.4 in [18], equation (4.11) with the aid of (4.4) yields that $R(t)$ has to be expressed as

$$R(t) = B(s)q$$

for some vector field $B(s)$ along α. Namely, we have

$$\begin{aligned} &\left(\Phi' + \sum_{j=1}^{r} t_j \Psi_j'\right)\left(1 + \sum_{k=1}^{r} \varepsilon u_k t_k\right) - \left(\Phi + \sum_{j=1}^{r} t_j \Psi_j\right)\sum_{k=1}^{r} \varepsilon p_k t_k \\ &= B(s)\left(\varepsilon + \sum_{i=1}^{r} 2u_i t_i + \sum_{i,j=1}^{r} w_{ij} t_i t_j\right). \end{aligned} \tag{4.12}$$

Considering the constant terms in (4.12) with respect to t, we see that the vector $B(s)$ is given by

$$B(s) = \varepsilon\Phi'(s). \tag{4.13}$$

Using (4.13), we compare the coefficients of the terms containing t_i and $t_i t_j$ for any $i, j = 1, 2, \ldots, r$ in (4.12). Then, we obtain the following two equations:

$$\Psi_i' = \varepsilon u_i \Phi' + \varepsilon p_i \Phi, \tag{4.14}$$

$$\varepsilon u_i \Psi_j' + \varepsilon u_j \Psi_i' - \varepsilon p_i \Psi_j - \varepsilon p_j \Psi_i = 2\varepsilon w_{ij} \Phi' \tag{4.15}$$

for $i, j = 1, 2, \ldots, r$. Taking the indefinite product with Ψ_j to the both sides of (4.14), we have

$$\xi_{ji} = \varepsilon u_i z_j + \varepsilon p_i u_j$$

for $i, j = 1, 2, \ldots, r$. So we get

$$\begin{aligned} \xi_{ji} + \xi_{ij} &= (\varepsilon u_i z_j + \varepsilon p_i u_j) + (\varepsilon u_j z_i + \varepsilon p_j u_i) \\ &= \varepsilon u_i (p_j + z_j) + \varepsilon u_j (p_i + z_i) \end{aligned} \tag{4.16}$$

for $i, j = 1, 2, \ldots, r$. Due to (4.7), (4.16) yields

$$w_{ij}' = \varepsilon u_i u_j' + \varepsilon u_j u_i'$$

which means that

$$w_{ij} = \varepsilon u_i u_j + c_{ij} \tag{4.17}$$

for some constants c_{ij} and $i, j = 1, 2, \ldots, r$.

Let $\overline{e}_{r+1}, \overline{e}_{r+2}, \ldots, \overline{e}_{m-1}$ be the orthogonal normal vector fields to M along α. If we apply Lemma 2.5 to the normal space $T_{\alpha(s)}N$ of M, then there exists an orthonormal frame $\{e_a\}_{a=r+1}^{m-1}$ of the normal space $T_{\alpha(s)}N$ satisfying

$$\langle e_a'(s), e_b(s)\rangle = 0$$

for all $a, b = r+1, \ldots, m-1$. Then we can put

$$e_i' = \varepsilon u_i \alpha' + \sum_{a=r+1}^{m-1} \varepsilon_a \langle e_i', e_a\rangle e_a, \tag{4.18}$$

where $\varepsilon_a = \langle e_a, e_a\rangle = \pm 1$ for $a = r+1, \ldots, m-1$. From (4.17) and the definitions of u_i and w_{ij}, the constants c_{ij} are given by

$$c_{ij} = \sum_{a=r+1}^{m-1} \varepsilon_a \langle e_i', e_a\rangle \langle e_j', e_a\rangle \tag{4.19}$$

for $i, j = 1, 2, \ldots, r$.

In (4.15), we replace i with j and then we have

$$u_j \Psi'_j - p_j \Psi_j = w_{jj} \Phi'. \tag{4.20}$$

Putting (4.14) into (4.20), we obtain

$$\varepsilon u_j^2 \Phi' + \varepsilon p_j u_j \Phi - p_j \Psi_j = w_{jj} \Phi',$$

or,

$$p_j(\varepsilon u_j \Phi - \Psi_j) = c_{jj} \Phi' \tag{4.21}$$

because of (4.17). Taking the indefinite product with Ψ_j to (4.21), we have

$$p_j(\varepsilon u_j^2 - w_{jj}) = c_{jj} z_j$$

which implies that

$$c_{jj}(z_j + p_j) = c_{jj} u'_j = 0$$

for $j = 1, 2, \ldots, r$.

If the constant $c_{jj} \neq 0$ for some $j \in \{1, 2, \ldots, r\}$, then

$$u'_j = 0.$$

We consider the case of $c_{j_0 j_0} = 0$ for some j_0. If M is Lorentzian, then the normal space of M at each point is space-like. From (4.19), we can see that

$$e'_{j_0} = \varepsilon u_{j_0} \alpha'.$$

We now suppose that M is space-like. Then $\varepsilon = 1$ and $\varepsilon_i = 1$ for $i = 1, 2, \ldots, r$. Since $w_{j_0 j_0} = u_{j_0}^2$, the vector field $\sum_a \varepsilon_a \langle e'_{j_0}, e_a \rangle e_a$ of (4.18) is vanishing or a null vector field along α.

Suppose that $\sum_a \varepsilon_a \langle e'_{j_0}, e_a \rangle e_a$ is a null vector field along α. Then, from $e'_{j_0} = u_{j_0} \alpha' + \sum_a \varepsilon_a \langle e'_{j_0}, e_a \rangle e_a$, we have

$$\Psi_{j_0} = u_{j_0} \Phi + \sum_{a=r+1}^{m-1} \varepsilon_a \langle e'_{j_0}, e_a \rangle \xi_a, \tag{4.22}$$

where $\xi_a = e_a \wedge e_1 \wedge \cdots \wedge e_r$ for $a = r+1, \ldots, m-1$. Substituting (4.22) into (4.21) and using $c_{j_0 j_0} = 0$, we get

$$p_{j_0} \sum_{a=r+1}^{m-1} \varepsilon_a \langle e'_{j_0}, e_a \rangle \xi_a = \mathbf{0}.$$

By the hypothesis, the vector field $\sum_a \varepsilon_a \langle e'_{j_0}, e_a\rangle \xi_a$ is non-vanishing for all s. Therefore, we see that the function

$$p_{j_0} \equiv 0 \tag{4.23}$$

on I. Equations (4.14), (4.22) and (4.23) yield

$$u'_{j_0}\Phi + \sum_{a=r+1}^{m-1} \varepsilon_a \langle e'_{j_0}, e_a\rangle' \xi_a + \sum_{a=r+1}^{m-1} \varepsilon_a \langle e'_{j_0}, e_a\rangle \xi'_a = \mathbf{0}. \tag{4.24}$$

Note that $\langle\langle \Phi, \xi_b\rangle\rangle = \langle\langle \xi'_a, \xi_b\rangle\rangle = 0$ and $\langle\langle \xi_a, \xi_b\rangle\rangle = \varepsilon_a\delta_{ab}$ for $a, b = r+1, \dots, m-1$. By taking the indefinite product with ξ_b to the both sides of (4.24) for $b = r+1, \dots, m-1$, we obtain

$$\langle e'_{j_0}, e_a\rangle' = 0$$

and hence

$$u'_{j_0}\Phi = -\sum_{a=r+1}^{m-1} \varepsilon_a \langle e'_{j_0}, e_a\rangle \xi'_a \tag{4.25}$$

for all $a = r+1, \dots, m-1$. By straightforward computation of ξ'_a, (4.25) takes the form

$$\begin{aligned} u'_{j_0}\Phi = &-\sum_{a=r+1}^{m-1} \varepsilon_a \langle e'_{j_0}, e_a\rangle u_a \Phi \\ &-\sum_{i=1}^{r}\sum_{a=r+1}^{m-1} \varepsilon_a u_i \langle e'_{j_0}, e_a\rangle e_a \wedge e_1 \wedge \cdots \wedge e_{i-1} \wedge \alpha' \wedge e_{i+1} \wedge \cdots \wedge e_r, \end{aligned}$$

where $u_a = \langle \alpha', e'_a\rangle$ for $a = r+1, \dots, m-1$. Since the vectors Φ and $e_a \wedge e_1 \wedge \cdots \wedge e_{i-1} \wedge \alpha' \wedge e_{i+1} \wedge \cdots \wedge e_r$ are linearly independent, we get

$$u_i \langle e'_{j_0}, e_a\rangle = 0$$

for all $i = 1, \dots, r$ and $a = r+1, \dots, m-1$. Note that $u_{j_0} \neq 0$ for all $s \in I$. Thus, we have $\langle e'_{j_0}, e_a\rangle = 0$ for all $a = r+1, \dots, m-1$, which means that the vector field $\sum_a \varepsilon_a \langle e'_{j_0}, e_a\rangle e_a$ is zero that is a contradiction.

Therefore, we can conclude that if $c_{jj} = 0$ for some $j = 1, 2, \dots, r$, then

$$e'_j = \varepsilon u_j \alpha'.$$

Now, we will show that $u'_j = 0$ when $c_{jj} = 0$ for $j = 1, 2, \dots, r$. To do that, we consider the set $\Lambda = \{i \mid c_{ii} = 0\} \subset \{1, 2, \dots, r\}$. Note that for $i \in \Lambda$, $w_{ik} = \varepsilon u_i u_k$ for all $k = 1, 2, \dots, r$. Then, the function $q = \varepsilon + \sum 2u_i t_i + \sum w_{ij} t_i t_j$ can be rewritten as

$$q = \varepsilon\left(1 + \sum_{i\in\Lambda} \varepsilon u_i t_i\right)^2 + 2\sum_{k\notin\Lambda} \varepsilon u_k t_k + \sum_{i\in\Lambda}\sum_{k\notin\Lambda} w_{ik} t_i t_k + \sum_{k,h\notin\Lambda} w_{kh} t_k t_h.$$

Since u_k and w_{kh} are constant for $k, h \notin \Lambda$,

$$\begin{aligned}\frac{\partial q}{\partial s} &= 2\varepsilon\left(1+\sum_{i\in\Lambda}\varepsilon u_i t_i\right)\sum_{i\in\Lambda}\varepsilon u_i' t_i + 2\sum_{i\in\Lambda}\sum_{k\notin\Lambda}\varepsilon u_k u_i' t_i t_k \\ &= 2\varepsilon\left(1+\sum_{j=1}^{r}\varepsilon u_j t_j\right)\sum_{i\in\Lambda}\varepsilon u_i' t_i.\end{aligned}$$

Then, (4.11) implies

$$\begin{aligned}&-3\left(1+\sum_{j=1}^{r}\varepsilon u_j t_j\right)\left(\sum_{i\in\Lambda}\varepsilon u_i' t_i\right)\Phi' \\ &= -\left(\Phi''+\sum_{j=1}^{r}t_j\Psi_j''-\sum_{j=1}^{r}\varepsilon_j u_j\Psi_j-\sum_{l=1}^{r}\left(\sum_{j=1}^{r}\varepsilon_j w_{jl}\Psi_j\right)t_l\right)\left(1+\sum_{j=1}^{r}\varepsilon_j u_j t_j\right) \\ &\quad+\left(\Phi+\sum_{j=1}^{r}t_j\Psi_j\right)\left(\widetilde{\varepsilon}\phi+\sum_{j=1}^{r}\widetilde{\varepsilon}\varphi_j t_j-\varepsilon\sum_{j=1}^{r}\varepsilon_j u_j^2-\varepsilon\sum_{l=1}^{r}\left(\sum_{j=1}^{r}\varepsilon_j u_j w_{jl}\right)t_l\right).\end{aligned} \tag{4.26}$$

In (4.26), considering the constant terms with respect to t and the coefficients of terms containing t_i for $i \in \Lambda$, we have the following equations

$$-\Phi''+\sum_{j=1}^{r}\varepsilon_j u_j\Psi_j+\widetilde{\varepsilon}\phi\Phi-\varepsilon\left(\sum_{j=1}^{r}\varepsilon_j u_j^2\right)\Phi=\mathbf{0}, \tag{4.27}$$

$$\begin{aligned}-3\varepsilon u_i'\Phi' &= -\varepsilon u_i\Phi''+\varepsilon\left(\sum_{j=1}^{r}\varepsilon_j u_j\Psi_j\right)u_i-\Psi_i''+\sum_{j=1}^{r}\varepsilon_j w_{ij}\Psi_j \\ &\quad+\widetilde{\varepsilon}\varphi_i\Phi-\varepsilon\left(\sum_{j=1}^{r}\varepsilon_j u_j w_{ij}\right)\Phi+\widetilde{\varepsilon}\phi\Psi_i-\varepsilon\left(\sum_{j=1}^{r}\varepsilon_j u_j^2\right)\Psi_i.\end{aligned} \tag{4.28}$$

Putting (4.27) into (4.28) and using the fact that $\Psi_i = \varepsilon u_i\Phi$ for $i \in \Lambda$, we get

$$-3\varepsilon u_i'\Phi' = -\Psi_i''+\sum_{j=1}^{r}\varepsilon_j w_{ij}\Psi_j+\widetilde{\varepsilon}\varphi_i\Phi-\varepsilon\left(\sum_{j=1}^{r}\varepsilon_j u_j w_{ij}\right)\Phi. \tag{4.29}$$

From $\Psi_i = \varepsilon u_i\Phi$, we have

$$\Psi_i'' = \varepsilon u_i''\Phi+2\varepsilon u_i'\Phi'+\varepsilon u_i\Phi'' \quad\text{and}\quad \varphi_i=\widetilde{\varepsilon}\varepsilon u_i''+\varepsilon u_i\phi \tag{4.30}$$

for $i \in \Lambda$. By (4.27) and (4.30), equation (4.29) yields that

$$u_i'\Phi' = \mathbf{0}$$

for $i \in \Lambda$.

If $\Phi' \equiv \mathbf{0}$, then, by definition,

$$\Phi' = \alpha'' \wedge e_1 \wedge \cdots \wedge e_r + \sum_{k \notin \Lambda} \alpha' \wedge e_1 \wedge \cdots \wedge e_k' \wedge \cdots \wedge e_r \equiv \mathbf{0}.$$

This implies that

$$\alpha' \wedge e_1 \wedge \cdots \wedge e_k' \wedge \cdots \wedge e_r \wedge e_k = \mathbf{0}$$

for $k \notin \Lambda$. Thus, the vector fields $\alpha', e_1, \ldots, e_r, e_k'$ are linearly dependent for all s which means that $e_k' = \varepsilon u_k \alpha'$ for $k \notin \Lambda$. But it contradicts the definition of q, which is not of the form of completing the square. Therefore, we have

$$u_i' = 0$$

for $i \in \Lambda$.

So, for the case of $q \neq \varepsilon\big(1 + \sum_{i=1}^r \varepsilon u_i t_i\big)^2$ we can see that u_i are constant functions for $i = 1, 2, \ldots, r$ and hence the functions w_{ij} are constant for all $i, j = 1, 2, \ldots, r$ because of (4.17). Therefore, we can conclude that

$$\frac{\partial q}{\partial s} = 0$$

for all s, which contradicts $\partial q/\partial s \neq 0$ on the open interval I_1.

Case 2. Suppose that $q = \varepsilon\big(1 + \sum_{i=1}^r \varepsilon u_i t_i\big)^2$. Then $w_{ij} = \varepsilon u_i u_j$ and the constants c_{ij} defined in (4.17) are zero for all $i, j = 1, 2, \ldots, r$. In Case 1, we showed that if $c_{jj} = 0$ for some j, then $e_j' = \varepsilon u_j \alpha'$.

So, we easily see that $\Psi_i = \varepsilon u_i \Phi$ for all $i = 1, 2, \ldots, r$. Therefore, $G = \Phi$ and hence $\Delta G = fG$ is rewritten as

$$\frac{\varepsilon\left(\sum_{i=1}^r \varepsilon u_i' t_i\right)}{\left(1 + \sum_{i=1}^r \varepsilon u_i t_i\right)^3}\Phi' - \frac{\varepsilon}{\left(1 + \sum_{i=1}^r \varepsilon u_i t_i\right)^2}\Phi'' = f\Phi. \tag{4.31}$$

Taking the indefinite scalar product with Φ to the both sides of (4.31), we have

$$f = -\frac{\varepsilon\widetilde{\varepsilon}}{\left(1 + \sum_{i=1}^r \varepsilon u_i t_i\right)^2}\phi. \tag{4.32}$$

Then, equation (4.31) with the help of (4.32) implies

$$\left(\sum_{i=1}^r \varepsilon u_i' t_i\right)\Phi' - \left(1 + \sum_{i=1}^r \varepsilon u_i t_i\right)\Phi'' = -\widetilde{\varepsilon}\phi\left(1 + \sum_{i=1}^r \varepsilon u_i t_i\right)\Phi. \tag{4.33}$$

From (4.33), we can see that

$$\Phi'' = \widetilde{\varepsilon}\phi\Phi \quad \text{and hence} \quad u_i'\Phi' = \mathbf{0} \tag{4.34}$$

for all $i = 1, 2, \dots, r$.

If $\Phi' \equiv \mathbf{0}$, (4.32) and (4.34) yield that the function f is identically zero because Φ is non-zero vector field for all $s \in I$. It is a contradiction. Thus, we have $u_i' = 0$ on some open interval $I_2 \subset I_1$ for all $i = 1, 2, \dots, r$ and hence $\partial q/\partial s = 0$ on $I_2 \subset I_1$, a contradiction. Therefore, we can conclude that

$$\frac{\partial q}{\partial s} = 0$$

for all $s \in I$. This is a contradiction.

According to Cases 1 and 2, we conclude from equation (4.6) that

$$\frac{\partial q}{\partial s} = 0$$

for all $s \in I$. □

The following lemma helps us examine the mean curvature of the ruled submanifold on $\mathbb{L}^m$ with pointwise 1-type Gauss map of the first kind:

Lemma 4.4. [28] *Let M be an n-dimensional submanifold of a pseudo-Euclidean space $\mathbb{E}^m_s$ with pointwise* 1*-type Gauss map G of the first kind. Then, the mean curvature vector field H is parallel in the normal bundle.*

For a ruled submanifold M in $\mathbb{L}^m$, the mean curvature vector field H is defined by

$$\begin{aligned} H &= \frac{1}{r+1}\left\{\varepsilon h\left(\frac{x_s}{\|x_s\|}, \frac{x_s}{\|x_s\|}\right) + \sum_{i=1}^r \varepsilon_i h(x_{t_i}, x_{t_i})\right\} \\ &= \frac{\varepsilon}{(r+1)q}\left(x_{ss} - \frac{\varepsilon}{q}\langle x_{ss}, x_s\rangle x_s - \sum_{i=1}^r \varepsilon_i \langle x_{ss}, e_i\rangle e_i\right) \end{aligned}$$

by virtue of $x_{t_i t_i} = 0$ for all i. By computation, we can see easily

$$\langle x_{ss}, x_s\rangle = \sum_{i,j} \xi_{ij} t_i t_j = 0 \quad \text{and} \quad \langle x_{ss}, e_i\rangle = -u_i - \sum_j w_{ij} t_j.$$

So, we have

$$H = \frac{\varepsilon}{(r+1)q}\left\{\alpha''(s) + \sum_{i=1}^r t_i e_i''(s) + \sum_{i=1}^r \varepsilon_i u_i e_i(s) + \sum_{j=1}^r \left(\sum_{i=1}^r \varepsilon_i w_{ij} e_i(s)\right) t_j\right\},$$

which yields

$$\begin{aligned} \langle H, H\rangle = \frac{1}{(r+1)^2 q^2}\Bigg\{&\langle \alpha'', \alpha''\rangle - \sum_{k=1}^r \varepsilon_k u_k^2 + 2\sum_{i=1}^r \langle \alpha'', e_i''\rangle t_i - 2\sum_{i,j=1}^r \varepsilon_j u_j w_{ij} t_i \\ &+ \sum_{i,j=1}^r \langle e_i'', e_j''\rangle t_i t_j - \sum_{i,j=1}^r \left(\sum_{k=1}^r \varepsilon_k w_{ik} w_{jk}\right) t_i t_j\Bigg\}. \end{aligned} \tag{4.35}$$

Now, we show that $\langle H, H\rangle = 0$ on M.

Differentiating (4.35) with respect to t_{i_0} for some i_0 and using Lemma 4.4, we have

$$0 = \frac{-2}{(r+1)^2q^3}\left(\frac{\partial q}{\partial t_{i_0}}\right)\Bigg\{\langle\alpha'',\alpha''\rangle - \sum_{k=1}^{r}\varepsilon_k u_k^2 + 2\sum_{i=1}^{r}\langle\alpha'',e_i''\rangle t_i - 2\sum_{k,i=1}^{r}\varepsilon_k u_k w_{ki}t_i + \sum_{i,j=1}^{r}\langle e_i'',e_j''\rangle t_it_j + \sum_{i,j=1}^{r}\left(\sum_{k=1}^{r}\varepsilon_k w_{ik}w_{jk}\right)t_it_j\Bigg\} + \frac{2}{(r+1)^2q^2}\Bigg\{\langle\alpha'',e_{i_0}''\rangle - \sum_{k=1}^{r}\varepsilon_k u_k w_{i_0k} + \sum_{j=1}^{r}\langle e_{i_0}'',e_j''\rangle t_j - \sum_{j=1}^{r}\left(\sum_{k=1}^{r}\varepsilon_k w_{i_0k}w_{jk}\right)t_j\Bigg\},$$

or, equivalently,

(4.36)
$$0 = -2\left(u_{i_0} + \sum_{j=1}^{r} w_{i_0j}t_j\right)\Bigg\{\langle\alpha'',\alpha''\rangle - \sum_{k=1}^{r}\varepsilon_k u_k^2 + 2\sum_{i=1}^{r}\langle\alpha'',e_i''\rangle t_i - 2\sum_{k,i=1}^{r}\varepsilon_k u_k w_{ki}t_i + \sum_{i,j=1}^{r}\langle e_i'',e_j''\rangle t_it_j + \sum_{i,j=1}^{r}\left(\sum_{k=1}^{r}\varepsilon_k w_{ik}w_{jk}\right)t_it_j\Bigg\} + \left(\varepsilon + \sum_{i=1}^{r}2u_it_i + \sum_{i,j=1}^{r}w_{ij}t_it_j\right) \times\Bigg\{\langle\alpha'',e_{i_0}''\rangle - \sum_{k=1}^{r}\varepsilon_k u_k w_{i_0k} + \sum_{j=1}^{r}\langle e_{i_0}'',e_j''\rangle t_j - \sum_{j=1}^{r}\left(\sum_{k=1}^{r}\varepsilon_k w_{i_0k}w_{jk}\right)t_j\Bigg\}.$$

Considering the coefficients of terms containing t_j, t_j^2 and t_j^3 for some $j = 1, 2, \ldots, r$ in (4.36), we obtain

(4.37)
$$-4u_{i_0}\langle\alpha'',e_j''\rangle + 4u_{i_0}\sum_{k=1}^{r}\varepsilon_k u_k w_{kj} - 2w_{i_0j}\langle\alpha'',\alpha''\rangle + 2w_{i_0j}\sum_{k=1}^{r}\varepsilon_k u_k^2 + \varepsilon\langle e_{i_0}'',e_j''\rangle - \varepsilon\sum_{k=1}^{r}\varepsilon_k w_{i_0k}w_{jk} + 2u_j\langle\alpha'',e_{i_0}''\rangle - 2u_j\sum_{k=1}^{r}\varepsilon_k u_k w_{i_0k} = 0,$$

(4.38)
$$-2u_{i_0}\langle e_j'',e_j''\rangle + 2u_{i_0}\sum_{k=1}^{r}\varepsilon_k w_{jk}^2 - 4w_{i_0j}\langle\alpha'',e_j''\rangle + 4w_{i_0j}\sum_{k=1}^{r}\varepsilon_k u_k w_{kj} + 2u_j\langle e_{i_0}'',e_j''\rangle - 2u_j\sum_{k=1}^{r}\varepsilon_k w_{i_0k}w_{jk} + w_{jj}\langle\alpha'',e_{i_0}''\rangle - w_{jj}\sum_{k=1}^{r}\varepsilon_k u_k w_{i_0k} = 0,$$

(4.39)
$$-2w_{i_0j}\langle e_j'',e_j''\rangle + 2w_{i_0j}\sum_{k=1}^{r}\varepsilon_k w_{jk}^2 + w_{jj}\langle e_{i_0}'',e_j''\rangle - w_{jj}\sum_{k=1}^{r}\varepsilon_k w_{i_0k}w_{jk} = 0.$$

Without loss of generality, we may assume that $w_{i_0i_0} \neq 0$. By replacing j with i_0 in (4.37), (4.38) and (4.39), we can get easily

$$\langle \alpha'', \alpha'' \rangle = \sum_{k=1}^{r} \varepsilon_k u_k^2, \quad \langle \alpha'', e''_{i_0} \rangle = \sum_{k=1}^{r} \varepsilon_k u_k w_{i_0k} \quad \text{and} \quad \langle e''_{i_0}, e''_{i_0} \rangle = \sum_{k=1}^{r} \varepsilon_k w_{i_0k}^2. \tag{4.40}$$

Equation (4.37) with the help of (4.40) yields

$$\langle e''_i, e''_j \rangle = \sum_{k=1}^{r} \varepsilon_k w_{ik} w_{jk} \tag{4.41}$$

and hence

$$\langle \alpha'', e''_i \rangle = \sum_{k=1}^{r} \varepsilon_k u_k w_{ik} \tag{4.42}$$

for all $i, j = 1, 2, \dots, r$. Together with equations (4.35), (4.40), (4.41) and (4.42), we can conclude that on M,

$$\langle H, H \rangle = 0.$$

According to Theorem 3.1, we see that there does not exist an non-empty open subset $\Theta = \{p \in M \mid H \neq \mathbf{0} \text{ and } \langle H, H \rangle = 0\}$ of a ruled submanifold M in $\mathbb{L}^m$ with pointwise 1-type Gauss map of the first kind.

Therefore, we have

Theorem 4.5. *Let M be an $(r+1)$-dimensional non-cylindrical ruled submanifold parameterized by* (4.2) *in $\mathbb{L}^m$. Let $e_1, e_2, \dots, e_r$ be the orthonormal generators of the rulings along the base curve α such that e'_i are non-null for all $i = 1, 2, \dots, r$. If M has pointwise* 1*-type Gauss map of the first kind, then M is minimal.*

We now deal with the case that some of generators of rulings have null derivatives. Let M be an $(r+1)$-dimensional non-cylindrical ruled submanifold parameterized by (4.2) in $\mathbb{L}^m$. We suppose that some generators $e_{j_1}, e_{j_2}, \dots, e_{j_k}$ of the rulings have null derivatives along the base curve α for $j_1 < j_2 < \cdots < j_k \in \{1, 2, \dots, r\}$. We can rewrite the parametrization (4.2) of M as

$$x(s, t_1, \dots, t_r) = \alpha(s) + \sum_{i \neq j_1, j_2, \dots, j_k} t_i e_i(s) + \sum_{i=1}^{k} t_{j_i} e_{j_i}(s)$$

and its Laplace operator is given by (4.5)

$$\Delta = \frac{1}{2q^2}\frac{\partial q}{\partial s}\frac{\partial}{\partial s} - \frac{1}{q}\frac{\partial^2}{\partial s^2} - \frac{1}{2q}\sum_{i=1}^{r} \varepsilon_i \frac{\partial q}{\partial t_i}\frac{\partial}{\partial t_i} - \sum_{i=1}^{r} \varepsilon_i \frac{\partial^2}{\partial t_i^2}.$$

Then, there are two possible cases such that either all of $e_{j_{k+1}}, \dots, e_{j_r}$ generating the rulings except $e_{j_1}(s), e_{j_2}(s), \dots, e_{j_r}(s)$ are constant vector fields or not.

Case 1. Suppose that $e_{j_{k+1}}, \dots, e_{j_r}$ are constant vector fields. In this case, we may assume that e_i' is null for all $i = 1, \dots, r$, otherwise the ruled submanifold M is a cylinder defined over the ruled submanifold parameterized by the base curve α and the rulings generated by e_i's except those constant vector fields. We then have three possible cases according to the degree of q.

Subcase 1.1. Let $\deg q(t) = 0$, that is, e_i' are null with $e_i'(s) \wedge e_l'(s) = 0$ for $i, l = 1, 2, \dots, r$ and $\langle \alpha'(s), e_j'(s) \rangle = 0$ for $j = 1, 2, \dots, r$. Note that $\varepsilon = 1$ and $\varepsilon_i = 1$ for all $i = 1, 2, \dots, r$. Then M has the Gauss map

$$G = \Phi + \sum_{i=1}^{r} t_i \Psi_i$$

and $\Delta G = fG$ implies

$$\Phi'' = -f\Phi \quad \text{and} \quad \Psi_i'' = -f\Psi_i \tag{4.43}$$

for all $i = 1, 2, \dots, r$. The mean curvature vector field H is given by

$$H = \frac{1}{r+1}\left(\alpha'' + \sum_{i=1}^{r} t_i e_i''\right), \tag{4.44}$$

from which,

$$\langle H, H \rangle = \frac{1}{(r+1)^2}\left(\langle \alpha'', \alpha'' \rangle + 2\sum_{i=1}^{r} \langle \alpha'', e_i'' \rangle t_i + \sum_{i,j=1}^{r} \langle e_i'', e_j'' \rangle t_i t_j\right). \tag{4.45}$$

Differentiating (4.45) with respect to t_{i_0} for some i_0 and using Lemma 4.4, we have

$$0 = \frac{1}{(r+1)^2}\left(2\langle \alpha'', e_{i_0}'' \rangle + 2\sum_{j=1}^{r} \langle e_{i_0}'', e_j'' \rangle t_j\right),$$

which gives

$$\langle \alpha'', e_i'' \rangle = 0 = \langle e_i'', e_j'' \rangle$$

for all $i, j = 1, 2, \dots, r$.

On the other hands, Lemma 4.4 tells us that the derivatives of the mean curvature vector H with respect to t_i are tangent to M for all $i \in \{1, 2, \dots, r\}$. Together with this fact and (4.44), we can see that

$$e_i'' = \langle e_i'', \alpha' \rangle \alpha' \tag{4.46}$$

for all $i = 1, 2, \ldots, r$. Since $\langle e_i'', e_j'' \rangle = 0$, taking the scalar product with e_i'' to (4.46) implies that

$$\langle e_i'', \alpha' \rangle = 0 \quad \text{and hence} \quad e_i'' = \mathbf{0}$$

for all $i = 1, 2, \ldots, r$. From $\Psi_i = e_i' \wedge e_1 \wedge \cdots \wedge e_r$ and the above equation, we obtain

$$\Psi_i'' = \mathbf{0}$$

for all $i = 1, 2, \ldots, r$. Thus, we can see that the function f is identically zero by virtue of (4.43) because Ψ_i is a non-zero vector field for all $s \in I$, which is a contradiction. Therefore, no ruled submanifold with $\deg q(t) = 0$ has pointwise 1-type Gauss map of the first kind.

Subcase 1.2. Let $\deg q(t) = 1$. In this case, $\langle \alpha'(s), e_i'(s) \rangle \neq 0$ for some i $(1 \le i \le r)$ and the null vector fields e_i' satisfy $e_i' \wedge e_l' = 0$ for $i, l = 1, 2, \ldots, r$. Then, $\Delta G = fG$ implies that

(4.47)

$$\left(\frac{\partial q}{\partial s}\right)^2 \left(1 + \sum_{i=1}^r \varepsilon u_i t_i\right) - \frac{3}{2} q \frac{\partial q}{\partial s} \sum_{i=1}^r \varepsilon p_i t_i - \frac{1}{2} q \frac{\partial^2 q}{\partial s^2} \left(1 + \sum_{i=1}^r \varepsilon u_i t_i\right) + q^2 \left(\widetilde{\varepsilon}\phi + \sum_{i=1}^r \widetilde{\varepsilon}\varphi_i t_i\right)$$
$$+ \frac{1}{2} q \sum_{j=1}^r \varepsilon_j \left(\frac{\partial q}{\partial t_j}\right)^2 \left(1 + \sum_{i=1}^r \varepsilon u_i t_i\right) - \frac{1}{2} q^2 \sum_{j=1}^r \varepsilon_j \frac{\partial q}{\partial t_j} \varepsilon u_j + f q^3 \left(1 + \sum_{i=1}^r \varepsilon u_i t_i\right) = 0$$

with the help of (4.8). Using the function f which is obtained from (4.47), we repeat the same process to get (4.10). Then, we have the following equation

$$\begin{aligned} & -\frac{3}{2}\frac{\partial q}{\partial s} \left\{ \left(\Phi' + \sum_{j=1}^r t_j \Psi_j'\right)\left(1 + \sum_{i=1}^r \varepsilon u_i t_i\right) - \left(\sum_{i=1}^r \varepsilon p_i t_i\right)\left(\Phi + \sum_{j=1}^r t_j \Psi_j\right) \right\} \\ & = -q \left\{ \left(\Phi'' + \sum_{j=1}^r t_j \Psi_j''\right)\left(1 + \sum_{i=1}^r \varepsilon u_i t_i\right) - \left(\widetilde{\varepsilon}\phi + \sum_{i=1}^r \widetilde{\varepsilon}\varphi_i t_i\right)\left(\Phi + \sum_{j=1}^r t_j \Psi_j\right) \right. \\ & \qquad \left. - \left(\sum_{j=1}^r \varepsilon_j u_j \Psi_j\right)\left(1 + \sum_{i=1}^r \varepsilon u_i t_i\right) + \left(\sum_{i=1}^r \varepsilon \varepsilon_i u_i^2\right)\left(\Phi + \sum_{j=1}^r t_j \Psi_j\right) \right\}. \end{aligned} \tag{4.48}$$

From $q = \varepsilon + \sum_i 2u_i t_i$ and $\partial q/\partial s = \sum_i 2u_i' t_i$, equation (4.48) implies

$$\left(\Phi' + \sum_{j=1}^r t_j \Psi_j'\right)\left(1 + \sum_{i=1}^r \varepsilon u_i t_i\right) - \left(\sum_{i=1}^r \varepsilon p_i t_i\right)\left(\Phi + \sum_{j=1}^r t_j \Psi_j\right) = qW(t) \tag{4.49}$$

for some vector $W(t)$. Considering the degree of (4.49) and comparing the constant terms of both sides with respect to t in (4.49), we can put

$$W(t) = \varepsilon \Phi' + \sum_{i=1}^r \Upsilon_i t_i \tag{4.50}$$

for some vector fields Υ_i along α. Using (4.50) and considering the coefficients of the terms containing t_{i_0} for some i_0 with $u_{i_0} \neq 0$, we have

$$\varepsilon \Upsilon_{i_0} + 2\varepsilon u_{i_0} \Phi' = \varepsilon u_{i_0} \Phi' + \Psi'_{i_0} - \varepsilon p_{i_0} \Phi,$$

or,

$$\Upsilon_{i_0} = -u_{i_0} \Phi' + \varepsilon \Psi'_{i_0} - p_{i_0} \Phi. \tag{4.51}$$

Putting (4.51) into (4.49) and comparing the coefficients of the terms containing $t_{i_0}^2$, we get

$$-2u_{i_0}^2 \Phi' - 2u_{i_0} p_{i_0} \Phi + \varepsilon u_{i_0} \Psi'_{i_0} + \varepsilon p_{i_0} \Psi_{i_0} = \mathbf{0}. \tag{4.52}$$

Taking the indefinite product with Ψ_{i_0} to (4.52), we obtain

$$-2u_{i_0}^2 z_{i_0} - 2u_{i_0} p_{i_0} u_{i_0} + \varepsilon u_{i_0} \xi_{i_0 i_0} + \varepsilon p_{i_0} w_{i_0 i_0} = 0. \tag{4.53}$$

In this case, $w_{jk} = 0$ and $\xi_{jj} = 0$ for all $j, k = 1, 2, \dots, r$, so (4.53) becomes

$$-2u_{i_0}^2 z_{i_0} - 2u_{i_0}^2 p_{i_0} = 0$$

which yields that

$$u_{i_0}^2 u'_{i_0} = 0.$$

Thus, u_{i_0} $(i_0 = 1, 2, \dots, r)$ is a non-zero constant function on I and hence

$$\frac{\partial q}{\partial s} = 0$$

for all $s \in I$.

On the other hand, the mean curvature vector field H is given by

$$H = \frac{\varepsilon}{(r+1)q} \left(\alpha'' + \sum_{i=1}^{r} t_i e''_i + \sum_{j=1}^{r} \varepsilon_j u_j e_j \right). \tag{4.54}$$

Note that $\varepsilon_j = 1$ for all $j = 1, 2, \dots, r$. By straightforward computation, we have

(4.55)

$$\begin{aligned} H_{t_{i_0}} &= \frac{-\varepsilon}{(r+1)q^2} \left(\frac{\partial q}{\partial t_{i_0}} \right) \left(\alpha'' + \sum_{i=1}^{r} t_i e''_i + \sum_{j=1}^{r} \varepsilon_j u_j e_j \right) + \frac{\varepsilon}{(r+1)q} e''_{i_0} \\ &= \frac{\varepsilon}{(r+1)q^2} \left\{ -2u_{i_0} \alpha'' - 2u_{i_0} \sum_{i=1}^{r} t_i e''_i - 2u_{i_0} \sum_{j=1}^{r} \varepsilon_j u_j e_j + \left(\varepsilon + \sum_{i=1}^{r} 2u_i t_i \right) e''_{i_0} \right\} \end{aligned}$$

for some $i_0 \in \{1, 2, \ldots, r\}$ with $u_{i_0} \neq 0$. From Lemma 4.4, the partial derivative of the mean curvature vector H with respect to t_{i_0}, $H_{t_{i_0}}$, is tangent to M. That is, the vector in (4.55) of the form

$$-2u_{i_0}\alpha'' - 2u_{i_0}\sum_{i=1}^{r} t_i e_i'' + \left(\varepsilon + \sum_{i=1}^{r} 2u_i t_i\right) e_{i_0}'' \tag{4.56}$$

has to be tangent to M for all s and $t = (t_1, t_2, \ldots, t_r)$. Recall that the vector α'' is expressed as

$$\alpha'' = -\sum_{i=1}^{r} u_i e_i - \sum_{a=r+1}^{m-1} \varepsilon_a u_a e_a. \tag{4.57}$$

By differentiating (4.18) with respect to s, we have

$$e_j'' = \varepsilon\left(\sum_{a=r+1}^{m-1} \varepsilon_a u_a \langle e_j', e_a\rangle\right)\alpha' + \sum_{a=r+1}^{m-1} \varepsilon_a \langle e_j'', e_a\rangle e_a \tag{4.58}$$

with the aid of (4.17),(4.19), (4.57) and the fact $w_{ij} = 0$ for all $i, j = 1, 2, \ldots, r$. Putting (4.57) and (4.58) into (4.56) and arranging the equation obtained in such a way, we obtain the normal part of the vector given in (4.56) as follows:

$$\sum_{a=r+1}^{m-1} \varepsilon_a \left\{ 2u_{i_0} u_a - \sum_{j=1}^{r} (2u_{i_0}\langle e_j'', e_a\rangle) t_j + \varepsilon\langle e_{i_0}'', e_a\rangle + \sum_{j=1}^{r} (2u_j \langle e_{i_0}'', e_a\rangle) t_j \right\} e_a \tag{4.59}$$

which becomes identically zero. It means that the coefficients of e_a are vanishing for all $a = r+1, \ldots, m-1$. Therefore, by considering the constant terms of the coefficients of e_a with respect to t in (4.59), we get

$$2u_{i_0} u_a = -\varepsilon\langle e_{i_0}'', e_a\rangle$$

and hence

$$2u_j u_a = -\varepsilon\langle e_j'', e_a\rangle \tag{4.60}$$

for all $j = 1, 2, \ldots, r$ and $a = r+1, \ldots, m-1$. By (4.60), (4.58) is rewritten as

$$e_j'' = \varepsilon\left(\sum_{a=r+1}^{m-1} \varepsilon_a u_a \langle e_j', e_a\rangle\right)\alpha' - 2\varepsilon u_j \sum_{a=r+1}^{m-1} \varepsilon_a u_a e_a. \tag{4.61}$$

On the other hand, from $e_{i_0}' \wedge e_j' = \mathbf{0}$, we can put

$$e_{i_0}' = f_j^{i_0} e_j'$$

for all $j = 1, 2, \ldots, r$, where $f_j^{i_0}$ are non-vanishing functions for all s. By the definition of u_j, we have that

$$u_{i_0} = f_j^{i_0} u_j.$$

Since $u_{i_0} \neq 0$ and u_j are constant, $f_j^{i_0}$ are also non-zero constant and hence $u_j \neq 0$ for all $j = 1, 2, \ldots, r$. Equations (4.18), (4.61) and $\xi_{jj} = \langle e_j', e_j'' \rangle = 0$ yield that

$$-\varepsilon u_j \left(\sum_{a=r+1}^{m-1} \varepsilon_a u_a \langle e_j', e_a \rangle \right) = 0, \quad j = 1, 2, \ldots, r$$

and hence

$$\sum_{a=r+1}^{m-1} \varepsilon_a u_a \langle e_j', e_a \rangle = 0. \tag{4.62}$$

By (4.61) and (4.62), we see that

$$e_j'' = -2\varepsilon u_j \sum_{a=r+1}^{m-1} \varepsilon_a u_a e_a \tag{4.63}$$

for all $j = 1, 2, \ldots, r$. Together with (4.57) and (4.63), the mean curvature vector field H given in (4.54) is rewritten as

$$H = \frac{-1}{r+1} \sum_{a=r+1}^{m-1} \varepsilon_a u_a e_a \tag{4.64}$$

and hence we have

$$u_a' = 0$$

by virtue of Lemma 4.4 and the fact that e_a' are tangent to M for all $a = r+1, \ldots, m-1$.

Equation $\langle e_j', e_j'' \rangle = 0$ also tells us that

$$\langle e_j'', e_j'' \rangle = -\langle e_j', e_j''' \rangle. \tag{4.65}$$

With the help of (4.18), (4.63) and the fact that u_a are constant, equation (4.65) can be rewritten as

$$4u_j^2 \sum_{a=r+1}^{m-1} \varepsilon_a u_a^2 = 2u_j^2 \sum_{a=r+1}^{m-1} \varepsilon_a u_a^2$$

which yields

$$\sum_{a=r+1}^{m-1} \varepsilon_a u_a^2 = 0. \tag{4.66}$$

Therefore, from (4.66) we can see that

$$\langle e_j'', e_j'' \rangle = 0 \quad \text{and} \quad \langle H, H \rangle = 0$$

by virtue of (4.63) and (4.64) for all $j = 1, 2, \dots, r$.

If $e_j'' = \mathbf{0}$ for all $j = 1, 2, \dots, r$, since $u_j \neq 0$, (4.63) and (4.64) yield

$$H = \mathbf{0}.$$

If e_j'' is null for some $j \in \{1, \dots, r\}$, then the mean curvature vector field H given in (4.64) is also null for all s because of the continuity of u_a

Therefore, together with Theorem 3.1 we can conclude that if a ruled submanifold M with $\deg q(t) = 1$ has pointwise 1-type Gauss map of the first kind, then M is minimal.

Subcase 1.3. Let $\deg q(t) = 2$. In this case, we can easily see that if a ruled submanifold M with $\deg q(t) = 2$ has pointwise 1-type Gauss map of the first kind, then M is minimal by referring to the case that $e_1', e_2', \dots, e_r'$ are non-null and Theorem 3.1.

Case 2. Suppose that $e_i' \neq \mathbf{0}$ for some $i = j_{k+1}, \dots, j_r$.

In this case, we may also assume that $e_i' \neq \mathbf{0}$ for all $i = j_{k+1}, \dots, j_r$ by virtue of Proposition 2.6. Then, e_i' are non-null for all $i = j_{k+1}, \dots, j_r$ and $\deg q = 2$. If we follow the similar argument for the case that $e_1', e_2', \dots, e_r'$ are non-null, then we can obtain the sufficient condition of the minimality of M by means of the Gauss map with the Laplace operator together with Theorem 3.1.

Conversely, suppose that a non-cylindrical ruled submanifold with non-degenerate rulings in $\mathbb{L}^m$ is minimal. Let M be an $(r+1)$-dimensional non-cylindrical ruled submanifold parameterized by (4.2) in $\mathbb{L}^m$ and let $e_1, e_2, \dots, e_r$ be orthonormal generators of the rulings along the base curve α.

For the case that $e_1', e_2', \dots, e_r'$ are non-null, it is sufficient to refer to Theorem 3.6 of [18]. To deal with the case that some of generators of rulings have null derivatives, as we see in the above cases according to the degree of q, it is enough to consider the subcase of $\deg q(t) = 1$. So, we suppose that M is minimal with $\deg q(t) = 1$. Since the mean curvature vector field H is given by

$$\begin{aligned} H &= \frac{1}{(r+1)q} \left\{ x_{ss} - \langle x_{ss}, x_s \rangle x_s - \sum_{i=1}^{r} \langle x_{ss}, e_i \rangle e_i \right\} \\ &= \frac{1}{(r+1)q} \left\{ \alpha'' + \sum_{i=1}^{r} t_i e_i'' + \sum_{i=1}^{r} u_i e_i \right\}, \end{aligned}$$

the minimality of M implies that

$$\alpha'' = -\sum_{i=1}^{r} u_i e_i \quad \text{and} \quad e_i'' = \mathbf{0}$$

and hence

$$u_i' = \langle \alpha'', e_i' \rangle + \langle \alpha', e_i'' \rangle = 0$$

which means that u_i are constant functions for all $i = 1, 2, \ldots, r$.

By direct computation, we have

$$\Phi'' = \frac{1}{2} \sum_{k=1}^{r} \frac{\partial q}{\partial t_k} \Psi_k \quad \text{and} \quad \Psi_i'' = \mathbf{0}$$

for all $i = 1, 2, \ldots, r$. Since $\partial q/\partial s = \partial^2 q/\partial s^2 = \partial^2 q/\partial t_i^2 = 0$ on M for all $i = 1, 2, \ldots, r$, by using terms in (4.6), we obtain

$$\Delta G = -\frac{1}{2q^{5/2}} \sum_{k=1}^{r} \left(\frac{\partial q}{\partial t_k} \right)^2 \left(\Phi + \sum_{i=1}^{r} t_i \Psi_i \right)$$

which means that the Gauss map G of M is of pointwise 1-type of the first kind. That is,

$$\Delta G = fG$$

for some function

$$f = -\frac{1}{2q^2} \sum_{k=1}^{r} \left(\frac{\partial q}{\partial t_k} \right)^2 .$$

Therefore, together with Theorem 4.5 we have

Theorem 4.6. *Let M be an $(r+1)$-dimensional non-cylindrical ruled submanifold with non-degenerate rulings in the Minkowski m-space $\mathbb{L}^m$. Then, M has pointwise 1-type Gauss map of the first kind if and only if M is minimal.*

Let us consider an example of a marginally trapped ruled submanifold M whose Gauss map is not of pointwise 1-type of the first kind.

Example 4.7. Let $\mathbf{N}$ be a null constant vector in the Minkowski m-space $\mathbb{L}^m$. We consider the ruled submanifold M parameterized by

$$x(s, t_1, \ldots, t_r) = s^2 \mathbf{N} + \mathbf{F}s + \sum_{j=1}^{r} t_j (\mathbf{N}s + \mathbf{D}_j) \tag{4.67}$$

for some constant vectors $\mathbf{F}$, $\mathbf{D}_j$ with $\langle \mathbf{N}, \mathbf{F} \rangle = \langle \mathbf{N}, \mathbf{D}_j \rangle = \langle \mathbf{F}, \mathbf{D}_j \rangle = 0$, $\langle \mathbf{F}, \mathbf{F} \rangle = 1$ and $\langle \mathbf{D}_j, \mathbf{D}_i \rangle = \delta_{ji}$. Then, the mean curvature vector field H of M is given by

$$H = \frac{2}{1+r} \mathbf{N}$$

which is null for all s. That is, M is a marginally trapped ruled submanifold with $\deg q = 0$. In this case, the straightforward computation provides that the vectors ΔG and G defined on (4.67) are not parallel, i.e., the Gauss map G is not of pointwise 1-type of the first kind.

5. Ruled submanifolds with degenerate rulings in $\mathbb{L}^m$

Let M be an $(r+1)$-dimensional ruled submanifold in $\mathbb{L}^m$ with degenerate rulings $E(s,r)$ along a regular curve and let its parametrization be given by $\widetilde{x}(s,t)$ where $t=(t_1,t_2,\dots,t_r)$. Since $E(s,r)$ is degenerate, it can be spanned by a degenerate frame $\{B(s)=e_1(s),e_2(s),\dots,e_r(s)\}$ such that

$$\langle B(s),B(s)\rangle=\langle B(s),e_i(s)\rangle=0,\quad \langle e_i(s),e_j(s)\rangle=\delta_{ij},\quad i,j=2,3,\dots,r.$$

Without loss of generality as Lemma 2.5, we may assume that

$$\langle e_i'(s),e_j(s)\rangle=0,\quad i,j=2,3,\dots,r.$$

Since the tangent space of M at $\widetilde{x}(s,t)$ is non-degenerate and contains the degenerate ruling $E(s,r)$, there exists a tangent vector field A to M which satisfies

$$\langle A(s,t),A(s,t)\rangle=0,\quad \langle A(s,t),B(s)\rangle=-1,\quad \langle A(s,t),e_i(s)\rangle=0,\quad i=2,3,\dots,r$$

at $\widetilde{x}(s,t)$.

Let $\alpha(s)$ be an integral curve of the vector field A on M. Then we can define another parametrization x of M as follows:

$$x(s,t_1,t_2,\dots,t_r)=\alpha(s)+\sum_{i=1}^{r}t_ie_i(s),$$

where $\alpha'(s)=A(s)$.

Lemma 5.1. [20] *We may assume that* $\langle A(s),B'(s)\rangle=0$ *for all* s.

If we put $P=\langle x_s,x_s\rangle$ and $Q=-\langle x_s,x_{t_1}\rangle$, Lemma 5.1 implies

$$P(s,t)=2\sum_{i=2}^{r}u_i(s)t_i+\sum_{i,j=1}^{r}w_{ij}(s)t_it_j\quad\text{and}\quad Q(s,t)=1+\sum_{i=2}^{r}v_i(s)t_i,$$

where $v_i(s)=\langle B'(s),e_i(s)\rangle$, $u_i(s)=\langle A(s),e_i'(s)\rangle$ and $w_{ij}(s)=\langle e_i'(s),e_j'(s)\rangle$ for $i,j=1,2,\dots,r$. Note that P and Q are polynomials in $t=(t_1,t_2,\dots,t_r)$ with functions in s as coefficients. Then the Laplacian Δ of M can be expressed as follows:

$$\Delta=\frac{1}{Q^2}\left\{\frac{\partial\overline{P}}{\partial t_1}\frac{\partial}{\partial t_1}-2Q\sum_{i=2}^{r}v_i\frac{\partial}{\partial t_i}+2Q\frac{\partial^2}{\partial s\partial t_1}+\overline{P}\frac{\partial^2}{\partial t_1^2}-2Q\sum_{i=2}^{r}v_it_1\frac{\partial^2}{\partial t_1\partial t_i}-Q^2\sum_{i=2}^{r}\frac{\partial^2}{\partial t_i^2}\right\},$$

where $\overline{P}=P-t_1^2\sum_{i=2}^{r}v_i^2$.

By definition of the indefinite scalar product $\langle\langle\cdot,\cdot\rangle\rangle$ on $G(r+1,m)$, we may put

$$\langle\langle x_s\wedge x_{t_1}\wedge x_{t_2}\wedge\cdots\wedge x_{t_r},x_s\wedge x_{t_1}\wedge x_{t_2}\wedge\cdots\wedge x_{t_r}\rangle\rangle=-Q^2.$$

Let $\bar{\varepsilon} = \operatorname{sign} Q(t)$. Then the Gauss map G is given by

$$\begin{aligned} G &= \frac{1}{\bar{\varepsilon}Q} x_s \wedge x_{t_1} \wedge x_{t_2} \wedge \cdots \wedge x_{t_r} \\ &= \frac{1}{\bar{\varepsilon}Q} \left\{ A \wedge B \wedge e_2 \wedge \cdots \wedge e_r + t_1 B' \wedge B \wedge e_2 \wedge \cdots \wedge e_r + \sum_{i=2}^{r} t_i e_i' \wedge B \wedge e_2 \wedge \cdots \wedge e_r \right\}. \end{aligned}$$

Suppose that the Gauss map G is of pointiwse 1-type of the first kind, i.e., $\Delta G = fG$ for some non-zero smooth function f. The straightforward computation provides

$$\begin{aligned} &\frac{2\bar{\varepsilon}}{Q^3} \sum_{h=r+1}^{m-1} \left\{ \left(\sum_{i=1}^{r} \langle B', e_i' \rangle t_i - \sum_{i=2}^{r} v_i' t_i \right) v_h + v_h' Q \right\} e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_r \\ &+ \frac{2\bar{\varepsilon}}{Q^2} \sum_{h=r+1}^{m-1} v_h^2 A \wedge B \wedge e_2 \wedge \cdots \wedge e_r \\ &+ \frac{2\bar{\varepsilon}}{Q^2} \sum_{i=2}^{r} \sum_{h=r+1}^{m-1} v_i v_h e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_{i-1} \wedge A \wedge e_{i+1} \wedge \cdots \wedge e_r \\ &- \frac{2\bar{\varepsilon}}{Q^2} \sum_{i=2}^{r} \sum_{h,l=r+1}^{m-1} v_h z_{i,l} e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_{i-1} \wedge e_l \wedge e_{i+1} \wedge \cdots \wedge e_r \\ &= f \frac{\bar{\varepsilon}}{Q} \Bigg\{ \left(1 + \sum_{i=2}^{r} t_i v_i \right) A \wedge B \wedge e_2 \wedge \cdots \wedge e_r \\ &\quad + \sum_{h=r+1}^{m-1} \left(t_1 v_h - \sum_{i=2}^{r} z_{i,h} t_i \right) e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_r \Bigg\} \\ &= \bar{\varepsilon} f A \wedge B \wedge e_2 \wedge \cdots \wedge e_r + \frac{\bar{\varepsilon} f}{Q} \sum_{h=r+1}^{m-1} \left(t_1 v_h - \sum_{i=2}^{r} z_{i,h} t_i \right) e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_r, \end{aligned} \tag{5.1}$$

where we have put

$$B' = \sum_{i=2}^{m-1} v_i e_i \quad \text{and} \quad e_j' = v_j A - u_j B + \sum_{l=r+1}^{m-1} (-z_{j,l}) e_l$$

for $j = 2, \ldots, r$ and $l = r+1, \ldots, m-1$. Considering the orthogonality of the vectors in (5.1), we can see that

$$v_i v_h = 0 \quad \text{and} \quad v_h z_{i,l} = 0$$

for $i = 2, 3, \ldots, r$ and $h, l = r+1, \ldots, m-1$.

If the function $v_h(s) \equiv 0$ for all $s \in I$ and $h = r+1, \ldots, m-1$, then the coefficient of the vector $A \wedge B \wedge e_2 \wedge \cdots \wedge e_r$ on the left-hand side of (5.1) vanishes. Thus the function f becomes identically zero because of the orthogonality of the vectors on the right-hand side of (5.1), a contradiction.

Therefore, $v_h \neq 0$ for some $h \in \{r+1, \dots, m-1\}$, say v_{h_0}. Then, we have

$$v_i = 0, \quad z_{i,l} = 0 \quad \text{and hence} \quad Q = 1$$

for $i = 2, 3, \dots, r$ and $l = r+1, \dots, m-1$. So, equation (5.1) implies

$$\begin{aligned} &2\bar{\varepsilon} \sum_{h=r+1}^{m-1} \left\{ \left(\sum_{i=1}^{r} \langle B', e_i' \rangle t_i \right) v_h + v_h' \right\} e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_r \\ &+ 2\bar{\varepsilon} \left(\sum_{h=r+1}^{m-1} v_h^2 \right) A \wedge B \wedge e_2 \wedge \cdots \wedge e_r \\ &= \bar{\varepsilon} f A \wedge B \wedge e_2 \wedge \cdots \wedge e_r + \bar{\varepsilon} f \sum_{h=r+1}^{m-1} (t_1 v_h) e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_r. \end{aligned} \tag{5.2}$$

Equation (5.2) yields

$$f = 2 \sum_{h=r+1}^{m-1} v_h^2 = 2\langle B', B' \rangle \quad \text{and} \quad 2 \sum_{i=2}^{r} (\langle B', e_i' \rangle v_h) t_i + v_h' = 0 \tag{5.3}$$

for $i = 2, 3, \dots, r$ and $h = r+1, \dots, m-1$. From (5.3), we can see that v_{h_0} is a non-zero constant function which means that the function f is also non-zero constant, that is, the Gauss map G of M is of usual 1-type.

Thus, we have

Theorem 5.2. *Let M be a ruled submanifold in the Lorentz-Minkowski m-space $\mathbb{L}^m$ with degenerate rulings. Then, M has pointwise 1-type Gauss map of the first kind if and only if the Gauss map is of non-null 1-type in usual sense.*

6. Minimal ruled submanifolds in $\mathbb{L}^m$

In Section 3, we characterized minimal ruled submanifolds with non-degenerate rulings in terms of pointwise 1-type Gauss map of the first kind in the Lorentz-Minkowski space $\mathbb{L}^m$. In [21], the authors defined a minimal ruled submanifold with degenerate rulings called a G-kind ruled submanifold in $\mathbb{L}^m$.

Therefore, considering Theorem 4.5 of [21] and Theorem 4.6, we have

Theorem 6.1. *Let M be a non-cylindrical ruled submanifold in a Lorentz-Minkowski m-space $\mathbb{L}^m$. Then, M is minimal if and only if, according to the character of the base curve, M is one of the followings:*

(1) *The Gauss map of M is of pointwise 1-type of the first kind if the base curve is non-null.*

(2) *M is an open portion of a G-kind ruled submanifold if the base curve is null.*

Remark 6.2. We would like to correct the authors' statement of Theorem 4.3 in [22]. In their proof of Case 1 of the theorem, they accidently dropped one trivial case of Gauss map which is of 1-type. The statement of the theorem should be "Let M be a ruled submanifold in $\mathbb{L}^m$ with degenerate rulings. If M has finite-type Gauss map G, G is of either 1-type or null 2-type."

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