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A Critical Nonlinear Elliptic Equation with Nonlocal Regional Diffusion

César E. Torres Ledesma

Abstract. In this article we are interested in the nonlocal regional Schrödinger equation with critical exponent

$$\epsilon^{2\alpha}(-\Delta)^{\alpha}_{\rho}u + u = \lambda u^q + u^{2^*_\alpha - 1} \quad \text{in } \mathbb{R}^n, \quad u \in H^\alpha(\mathbb{R}^n),$$

where ϵ is a small positive parameter, $\alpha \in (0,1)$, $q \in (1, 2^*_\alpha - 1)$, $2^*_\alpha = 2n/(n-2\alpha)$ is the critical Sobolev exponent, $\lambda > 0$ is a parameter and $(-\Delta)^\alpha_\rho$ is a variational version of the regional Laplacian, whose range of scope is a ball with radius $\rho(x) > 0$. We study the existence of a ground state and we analyze the behavior of semi-classical solutions as $\varepsilon \to 0$.

1. Introduction

In the present paper, we consider the existence and concentration phenomena of solutions to the nonlinear Schrödinger equation with nonlocal regional diffusion and critical exponent given by

$$\epsilon^{2\alpha}(-\Delta)^{\alpha}_{\rho}u + u = \lambda u^q + u^{2^*_\alpha - 1} \quad \text{in } \mathbb{R}^n, \quad u \in H^\alpha(\mathbb{R}^n), \tag{1.1}$$

where ϵ is a small positive parameter, $\alpha \in (0,1)$, $q \in (1, 2^*_\alpha - 1)$, $2^*_\alpha = 2n/(n-2\alpha)$ is the critical Sobolev exponent, $\lambda > 0$ is a parameter and the operator $(-\Delta)^\alpha_\rho$ is a variational version of the non-local regional Laplacian, with range of scope determined by the positive function $\rho \in C(\mathbb{R}^n, \mathbb{R}^+)$, which is defined as

$$\int_{\mathbb{R}^n} (-\Delta)^\alpha_\rho u(x)\varphi(x)\,dx = \int_{\mathbb{R}^n}\int_{B(0,\rho(x))} \frac{[u(x+z) - u(z)][\varphi(x+z) - \varphi(x)]}{|z|^{n+2\alpha}}\,dzdx.$$

In recent years, equations involving the fractional Laplacian have attracted much attention since its appear in several areas such as optimization, finance, phase transitions, stratified materials, crystal dislocation, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, materials science, and water waves, see for instance [8, 13], and [10, 11, 15] for an introduction to these topics and their applications.

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In the context of fractional quantum mechanics, non-linear fractional Schrödinger equation has been proposed by Laskin [25, 26] as a result of expanding the Feynman path integral, from the Brownian-like to the Lévy-like quantum mechanical paths. Motivated by these physical aspects, non-linear fractional Schrödinger equation has attracted a lot of attention of many researchers and some existence and multiplicity results have been obtained, for example see [7, 9, 12, 16, 17, 20, 27, 36].

On the other hand, research has been done regarding regional fractional Laplacian, where the scope of the operator is restricted to a variable region near each point. We mention the work by Guan [21] and Guan and Ma [22] where they study these operators, their relation with stochastic processes and they develop integration by parts formula, and the work by Ishii and Nakamura [24], where the authors studied the Dirichlet problem for regional fractional Laplacian modeled on the p-Laplacian.

Very recently Felmer and Torres [18, 19], considered positive solutions of nonlinear Schrödinger equation with non-local regional diffusion

$$\epsilon^{2\alpha}(-\Delta)_\rho^\alpha u + u = f(u) \quad \text{in } \mathbb{R}^n, \quad u \in H^\alpha(\mathbb{R}^n). \tag{1.2}$$

The operator $(-\Delta)_\rho^\alpha$ is a variational version of the non-local regional Laplacian, defined by

$$\int_{\mathbb{R}^n} (-\Delta)_\rho^\alpha uv\, dx = \int_{\mathbb{R}^n}\int_{B(0,\rho(x))} \frac{[u(x+z)-u(x)][v(x+z)-v(x)]}{|z|^{n+2\alpha}}\, dzdx.$$

Under suitable assumptions on the nonlinearity f and the range of scope ρ, they obtained the existence of a ground state by mountain pass argument and a comparison method. Furthermore, they analyzed symmetry properties and concentration phenomena of these solutions. These regional operators present various interesting characteristics that make them very attractive from the point of view of mathematical theory of non-local operators. We also mention the recent works by Alves and Torres [3, 4] and Torres [37–39], where existence, multiplicity and symmetry results are considered in bounded domain and $\mathbb{R}^n$.

Recently, some papers which analyzed fractional elliptic equations involving the critical Sobolev exponent, see [28–32]. By using variational methods, Shang and Zhang [33] studied the existence and the multiplicity of nonnegative solutions for

$$\epsilon^{2s}(-\Delta)^s u + V(x)u = |u|^{2_s^*-2}u + \lambda f(u) \quad \text{in } \mathbb{R}^n \tag{1.3}$$

where $\epsilon, \lambda > 0$, V is a positive continuous function and f has subcritical growth. Teng and He [35] combined the s-harmonic extension method of Caffarelli and Silvestre, the concentration-compactness principle of Lions and methods of Brezis and Nirenberg to prove the existence of ground state solutions for

$$(-\Delta)^s u + V(x)u = P(x)|u|^{p-2}u + Q(x)|u|^{2_s^*-2}u \quad \text{in } \mathbb{R}^n$$

where $p \in (2, 2_s^*)$ and P, Q are continuous functions satisfying suitable assumptions. In [23], by using penalization technique and Ljusternik-Schnirelmann theory, He and Zou obtained the existence and concentration results for the problem (1.3) with $\lambda = 1$, under local condition imposed on V and f is a subcritical nonlinearity. Further results concerning the fractional Schrödinger equations involving critical and subcritical nonlinearities can be found in [1, 2, 5, 6, 33].

Inspired by the above works, in the present work we aim to investigate the existence and concentration phenomena of least energy solution for the equation (1.1). For that purpose, we suppose that the scope function $\rho \in C(\mathbb{R}^n, \mathbb{R}^+)$ satisfies the following hypotheses:

(ρ_1) There are numbers $0 < \rho_0 < \rho_\infty \leq \infty$ such that

$$\rho_0 \leq \rho(x) < \rho_\infty, \quad \forall\, x \in \mathbb{R}^n \quad \text{and} \quad \lim_{|x|\to\infty} \rho(x) = \rho_\infty.$$

(ρ_2) In case $\rho_\infty = \infty$ we further assume that there exists $a \in (0, 1)$ such that

$$\limsup_{|x|\to\infty} \frac{\rho(x)}{|x|} \leq a.$$

(ρ_3) For any $x_0 \in \mathbb{R}^n$, the equation

$$|x| = \rho(x + x_0), \quad x \in \mathbb{R}^n,$$

defines an $(n-1)$-dimensional surface of class C^1 in $\mathbb{R}^n$.

Now we are in a position to state our main existence theorem.

Theorem 1.1. *Suppose that $\epsilon, \lambda > 0$, $q \in (1, 2_\alpha^* - 1)$, ρ verifies (ρ_1)–(ρ_2). Then, there exists $\epsilon_0 > 0$ such that* (1.1) *has a weak solution for all $\epsilon \in (0, \epsilon_0)$.*

In our second main theorem we are interested in the concentration behavior of ground states for the equation (1.1) when the positive parameter ϵ approaches zero. In the light of [19], the scope function ρ, that describes the size of the ball of the influential region of the non-local operator, plays a key role in deciding the concentration point of ground states of the equation. Even though, at a first sight, the minimum point of ρ seems to be the concentration point, there is a non-local effect that needs to be taken in account. We define the concentration function

$$\mathcal{H}(x) = -\frac{|S^{n-1}|}{2\alpha}\left(\frac{1}{\rho(x)^{2\alpha}} - \frac{1}{\rho_\infty^{2\alpha}}\right) + \frac{1}{2}\int_{\mathcal{C}^+(x)} \frac{dy}{|y|^{n+2\alpha}} - \frac{1}{2}\int_{\mathcal{C}^-(x)} \frac{dy}{|y|^{n+2\alpha}},$$

where the sets $\mathcal{C}^+(x)$ and $\mathcal{C}^-(x)$ are defined as follows

$$\mathcal{C}^-(x) = \{y \in \mathbb{R}^n : \rho(x + y) < |y| < \rho(x)\}$$

and

$$\mathcal{C}^{+}(x) = \{y \in \mathbb{R}^n : \rho(x) < |y| < \rho(x+y)\}.$$

Here we interpret the quotient $1/\rho_\infty^{2\alpha}$ as zero, when $\rho_\infty = \infty$. Now we state our second theorem.

Theorem 1.2. *Suppose that $\epsilon, \lambda > 0$, $q \in (1, 2_\alpha^* - 1)$, ρ verifies (ρ_1)–(ρ_3). Then for each sequence $\epsilon_m \to 0$, there exists a subsequence such that for every m, there is a non-negative solution $u_m = u_{\epsilon_m}$ of (1.1) that concentrates around a global minimum point x_0 of $\mathcal{H}$, as $\epsilon_m \to 0$. In more precise terms, for every $\delta > 0$ there exists $R > 0$ and $\epsilon_0 > 0$ such that if $\epsilon < \epsilon_0$ we have*

$$\int_{B^c(x_0,\epsilon_m R)} u_m^2(x)\,dx \le \epsilon_m^n \delta \quad \textit{and} \quad \int_{B(x_0,\epsilon_m R)} u_m^2(x)\,dx \ge \epsilon_m^n C, \quad \forall\, \epsilon_m \le \epsilon_0,$$

with C a constant independent of δ and m.

This article is organized as follows. In Section 2 we present preliminaries with the main tools and the functional setting of the problem. In Section 3 we prove the Theorem 1.1. In Section 4 we complete the study of the semi-classical limit, proving Theorem 1.2.

2. Preliminaries

For any $\alpha \in (0,1)$, the fractional Sobolev space $H^\alpha(\mathbb{R}^n)$ is defined by

$$H^\alpha(\mathbb{R}^n) = \left\{u \in L^2(\mathbb{R}^n) : \frac{|u(x)-u(z)|}{|x-z|^{(n+2\alpha)/2}} \in L^2(\mathbb{R}^n \times \mathbb{R}^n)\right\}$$

endowed with the norm

$$\|u\|_\alpha = \left(\int_{\mathbb{R}^n} |u(x)|^2\,dx + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(z)|^2}{|x-z|^{n+2\alpha}}\,dzdx\right)^{1/2}.$$

For the reader's convenience, we review the main embedding result for $H^\alpha(\mathbb{R}^n)$.

Lemma 2.1. [15] *Let $\alpha \in (0,1)$ such that $2\alpha < n$. Then there exist a constant $\mathfrak{C} = \mathfrak{C}(n,\alpha) > 0$, such that*

$$\|u\|_{L^{2_\alpha^*}} \le \mathfrak{C}\|u\|_\alpha$$

for every $u \in H^\alpha(\mathbb{R}^n)$. Moreover, the embedding $H^\alpha(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ is continuous for any $p \in [2, 2_\alpha^*]$ and is locally compact whenever $p \in [2, 2_\alpha^*)$.

Furthermore, we introduce the homogeneous fractional Sobolev space

$$\begin{aligned} H_0^\alpha(\mathbb{R}^n) &= \left\{u \in L^{2_\alpha^*}(\mathbb{R}^n) : |\xi|^\alpha \widehat{u} \in L^2(\mathbb{R}^n)\right\} \\ &= \overline{C_0^\infty(\mathbb{R}^n)}^{\|\cdot\|_0}, \end{aligned}$$

where

$$\|u\|_0^2 = \int_{\mathbb{R}^n} |\xi|^{2\alpha} |\widehat{u}(\xi)|^2 \, d\xi.$$

Now we consider the best Sobolev constant S as follows:

$$S = \inf_{u \in H_0^\alpha(\mathbb{R}^n) \setminus \{0\}} \frac{\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)-u(z)|^2}{|x-z|^{n+2\alpha}} \, dz dx}{\left(\int_{\mathbb{R}^n} |u(x)|^{2_\alpha^*} \, dx \right)^{2/2_\alpha^*}}. \tag{2.1}$$

According to [14], S is attained by the function $u_0(x)$ given by

$$u_0(x) = \frac{c}{(\theta^2 + |x - x_0|^2)^{(N-2\alpha)/2}}, \quad x \in \mathbb{R}^n, \tag{2.2}$$

where $c \in \mathbb{R} \setminus \{0\}$, $\theta > 0$ and $x_0 \in \mathbb{R}^N$ are fixed constants. For any $\epsilon > 0$ and $x \in \mathbb{R}^n$, let

$$U_\epsilon(x) = \epsilon^{-(N-2\alpha)/2} \widetilde{u} \left(\frac{x}{\epsilon S^{1/(2\alpha)}} \right), \quad \widetilde{u}(x) = \frac{u_0(x)}{\|u_0\|_{L^{2_\alpha^*}}}$$

which is solution of the problem

$$(-\Delta)^\alpha u = |u|^{2_\alpha^* - 2} u, \quad x \in \mathbb{R}^n.$$

Given a function ρ as above, we define

$$\|u\|_\rho^2 = \int_{\mathbb{R}^n} \int_{B(0,\rho(x))} \frac{|u(x) - u(z)|^2}{|x - z|^{n+2\alpha}} \, dz dx + \int_{\mathbb{R}^n} u(x)^2 \, dx \tag{2.3}$$

and the space

$$H_\rho^\alpha(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : \|u\|_\rho^2 < \infty\}.$$

This space is very natural for the study of our problem. Furthermore, we have the following result.

Proposition 2.2. [19] *If ρ satisfies (ρ_1) there exists a constant $C = C(n, \alpha, \rho_0) > 0$ such that*

$$\|u\|_\alpha \leq C \|u\|_\rho.$$

Remark 2.3. By Proposition 2.2 we have that $H_\rho^\alpha(\mathbb{R}^n) \hookrightarrow H^\alpha(\mathbb{R}^n)$ is continuous and then, by Lemma 2.1, we have that $H_\rho^\alpha(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ is continuous for any $q \in [2, 2_\alpha^*]$, and there exists $\mathfrak{C}_q > 0$ such that

$$\|u\|_{L^q} \leq \mathfrak{C}_q \|u\|_\rho \quad \text{for every } u \in H_\rho^\alpha(\mathbb{R}^n) \text{ and } q \in [2, 2_\alpha^*].$$

Furthermore $H_\rho^\alpha(\mathbb{R}^n) \hookrightarrow L_{\mathrm{loc}}^q(\mathbb{R}^n)$ is compact for any $q \in [2, 2_\alpha^*)$.

Remark 2.4. Since $\|u\|_\rho \leq \|u\|_\alpha$, under the condition (ρ_1) Proposition 2.2 implies $\|\cdot\|_\rho$ and $\|\cdot\|$ are equivalent norms in $H^\alpha(\mathbb{R}^n)$.

Lemma 2.5. [19] *Let $n \geq 2$. Assume that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $H^\alpha_\rho(\mathbb{R}^n)$ and it satisfies*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^n} \int_{B(y,R)} |u_n(x)|^2\,dx = 0,$$

where $R > 0$. Then $u_n \to 0$ in $L^q(\mathbb{R}^n)$ for $q \in (2, 2^*_\alpha)$.

Now, we consider the limit equations, namely

$$(-\Delta)^\alpha u + u = \lambda u^q + u^{2^*_\alpha - 1} \quad \text{in } \mathbb{R}^n, \qquad u \in H^\alpha(\mathbb{R}^n). \tag{2.4}$$

This equation was studied by Shang, Zhang and Yang in [34]. The solution of problem (2.4) are the critical point of the functional

$$I(u) = \frac{1}{2}\|u\|^2_\alpha - \frac{\lambda}{q+1}\int_{\mathbb{R}^n} u_+^{q+1}\,dx - \frac{1}{2^*_\alpha}\int_{\mathbb{R}^n} u_+^{2^*_\alpha}\,dx.$$

Furthermore, they studied the existence of ground state solutions to (2.4), namely, function in $\mathcal{N} = \{u \in H^\alpha(\mathbb{R}^n)\setminus\{0\} : I'(u)u = 0\}$ such that

$$C^* = \inf_{u\in\mathcal{N}} I(u)$$

is achieved and they got the following characterization:

$$C^* = \inf_{u\in H^\alpha(\mathbb{R}^n)\setminus\{0\}} \sup_{t\geq 0} I(tu) = C,$$

where $C = \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} I(\gamma(t)) > 0$ is the mountain pass critical value.

On the other hand, if $q \in (1, 2^*_\alpha - 1)$, they have proved that there exists $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$, the critical value satisfies

$$0 < C < \frac{\alpha}{n} S^{n/(2\alpha)}, \tag{2.5}$$

where S is the best Sobolev constant given by (2.1) and problem (2.4) has a nontrivial ground state solution.

3. Ground state

Let $\epsilon = 1$ and consider the following problem

$$(-\Delta)^\alpha_\rho u + u = \lambda u^q + u^{2^*_\alpha - 1} \quad \text{in } \mathbb{R}^N, \qquad u \in H^\alpha(\mathbb{R}^N). \tag{3.1}$$

We recall that $u \in H^\alpha_\rho(\mathbb{R}^n)\setminus\{0\}$ is a solution of (1.1) if $u(x) \geq 0$ and

$$\langle u, \varphi\rangle_\rho = \lambda\int_{\mathbb{R}^n} u_+^q \varphi\,dx + \int_{\mathbb{R}^n} u_+^{2^*_\alpha - 1}\varphi\,dx, \quad \forall\,\varphi \in H^\alpha_\rho(\mathbb{R}^n),$$

where

$$\langle u, \varphi \rangle_\rho = \int_{\mathbb{R}^n}\int_{B(0,\rho(x))} \frac{[u(x+z)-u(x)][\varphi(x+z)-\varphi(x)]}{|z|^{n+2\alpha}}\, dzdx + \int_{\mathbb{R}^n} u\varphi\, dx$$

and $u_+ = \max\{u, 0\}$.

In order to find solution for problem (3.1), we consider the functional $I_\rho\colon H^\alpha_\rho(\mathbb{R}^n) \to \mathbb{R}$ defined as

$$I_\rho(u) = \frac{1}{2}\|u\|^2_\rho - \frac{\lambda}{q+1}\int_{\mathbb{R}^n} u_+^{q+1}\, dx - \frac{1}{2^*_\alpha}\int_{\mathbb{R}^n} u_+^{2^*_\alpha}\, dx,$$

which is well defined and belongs to $C^1(H^\alpha_\rho(\mathbb{R}), \mathbb{R})$ with Fréchet derivative

$$I'_\rho(u)v = \langle u, v\rangle_\rho - \lambda\int_{\mathbb{R}^n} u_+^q v\, dx - \int_{\mathbb{R}^n} u_+^{2^*_\alpha - 1} v\, dx.$$

Now, we start recalling that the functional I_ρ satisfies the mountain pass geometry conditions

Lemma 3.1. *The functional I_ρ satisfies the following conditions:*

(1) *There exist $\beta, \delta > 0$, such that $I_\rho(u) \ge \beta$ if $\|u\|_\rho = \delta$.*

(2) *There exists an $e \in H^\alpha_\rho(\mathbb{R}^n)$ with $\|e\|_\rho > \delta$ such that $I_\rho(e) < 0$.*

From the previous lemma, by using the mountain pass theorem without $(\mathrm{PS})_c$ condition (see [40]) it follows that there exists a $(\mathrm{PS})_{C_\rho}$ sequence $\{u_k\} \subset H^\alpha_\rho(\mathbb{R}^n)$ such that

$$I_\rho(u_k) \to C_\rho \quad \text{and} \quad I'_\rho(u_k) \to 0, \tag{3.2}$$

where

$$C_\rho = \inf_{\gamma\in\Gamma_\rho} \sup_{t\in[0,1]} I_\rho(\gamma(t)) > 0, \tag{3.3}$$

and $\Gamma_\rho = \{\gamma \in C([0,1], H^\alpha_\rho(\mathbb{R}^n)) : \gamma(0) = 0, I_\rho(\gamma(1)) < 0\}$.

Also, we define

$$C^* = \inf_{u\in\mathcal{N}_\rho} I_\rho(u)$$

where

$$\mathcal{N}_\rho = \{u \in H^\alpha_\rho(\mathbb{R}^n) \setminus \{0\} : I'_\rho(u)u = 0\}.$$

Lemma 3.2. $C^* > 0$.

Proof. Suppose by contradiction that $C^* = 0$. Then there exists $u_k \in \mathcal{N}_\rho$ such that

$$I_\rho(u_k) \to C^* = 0 \quad \text{as } k \to \infty.$$

But, since $u_k \in \mathcal{N}_\rho$ we have

$$\begin{aligned} I_\rho(u_k) &= I_\rho(u_k) - \frac{1}{2} I'_\rho(u_k)u_k \\ &= \lambda \left(\frac{1}{2} - \frac{1}{q+1} \right) \|u_{k+}\|_{L^{q+1}}^{q+1} + \frac{\alpha}{n} \|u_{k+}\|_{L^{2^*_\alpha}}^{2^*_\alpha} \to 0 \quad \text{as } k \to \infty. \end{aligned}$$

So

$$\|u_{k+}\|_{L^{2^*_\alpha}} \to 0 \quad \text{and} \quad \|u_{k+}\|_{L^{q+1}} \to 0 \quad \text{as } k \to \infty.$$

Therefore

$$\|u_k\|_\rho \to 0 \quad \text{as } k \to \infty. \tag{3.4}$$

On the other hand, since $0 \neq u_k \in \mathcal{N}_\rho$, then by Remark 2.3 we have

$$1 \leq \lambda C_{q+1} \|u_k\|_\rho^{q-1} + C_{2^*_\alpha} \|u_k\|_\rho^{2^*_\alpha - 2}, \quad \forall\, k.$$

Combining this inequality with (3.4) we get a contradiction. This proves the lemma. □

Lemma 3.3. *Let* C_ρ *given by* (3.2) *and* (3.3). *Then*

$$C^* = \inf_{u \in H^\alpha_\rho(\mathbb{R}^n) \setminus \{0\}} \max_{t \geq 0} I_\rho(tu) = C_\rho.$$

Proof. We note that our nonlinearity $f(t) = \lambda t^q + t^{2^*_\alpha - 1}$, $t > 0$, is a C^1 function and $f(t)/t$ is a strictly increasing function. Let $u \in \mathcal{N}_\rho$, then we can show that the function $h(t) = I_\rho(tu)$, $t \neq 0$ has a unique maximum point t_u such that

$$I_\rho(t_u u) = \max_{t \geq 0} I_\rho(tu). \tag{3.5}$$

Furthermore, we can show that $t_u = 1$. Now choose $t_0 \in \mathbb{R}$ and $\widetilde{u} = t_0 u$ such that $I_\rho(\widetilde{u}) < 0$. Then $\gamma(t) = t\widetilde{u} \in \Gamma_\rho$ then $I_\rho(u) \geq C_\rho$, that is,

$$C^* \geq C_\rho. \tag{3.6}$$

On the other hand, let $\{u_k\}$ be the $(\mathrm{PS})_{C_\rho}$ sequence satisfying (3.2) and (3.3). Since $\{u_k\}$ is bounded, then $I'_\rho(u_k)u_k \to 0$ as $k \to \infty$, moreover, from (3.5) for each k, there is a unique $t_k \in \mathbb{R}^+$ such that

$$I'_\rho(t_k u_k) t_k u_k = 0, \quad \forall\, k. \tag{3.7}$$

Hence $t_k u_k \in \mathcal{N}_\rho$.

Now we note that by (3.7), we have

$$\|u_k\|_\rho^2 = \lambda t_k^{q-1} \|u_{k+}\|_{L^{q+1}}^{q+1} + t_k^{2^*_\alpha - 2} \|u_{k+}\|_{L^{2^*_\alpha}}^{2^*_\alpha}, \quad \forall\, k. \tag{3.8}$$

So t_k does not converge to 0; otherwise, since $\{u_k\}$ is bounded in $H^\alpha_\rho(\mathbb{R}^n)$, using (3.8) we obtain

$$\|u_k\|_\rho \to 0 \quad \text{as } k \to \infty,$$

which is impossible since $C_\rho > 0$. Also, t_k does not go to infinity. In fact, by (3.8) we get

$$\frac{\|u_k\|_\rho^2}{t_k^{2^*_\alpha - 2}} = \lambda t_k^{q-1-2^*_\alpha} \|u_{k+}\|^{q+1}_{L^{q+1}} + \|u_{k+}\|^{2^*_\alpha}_{L^{2^*_\alpha}}, \quad \forall\, k. \tag{3.9}$$

So, assuming that $t_k \to \infty$ as $k \to \infty$, by (3.9) we get that

$$u_k \to 0 \quad \text{in } L^{2^*_\alpha}(\mathbb{R}^n) \text{ as } k \to \infty.$$

Then using interpolation inequality it follows that

$$\|u_{k+}\|^{q+1}_{L^{q+1}} \to 0 \quad \text{as } k \to \infty. \tag{3.10}$$

Moreover, since $I'_\rho(u_k)u_k \to 0$ as $k \to \infty$, we obtain

$$\|u_k\|_\rho^2 = \lambda \|u_{k+}\|^{q+1}_{L^{q+1}} + \|u_k\|^{2^*_\alpha}_{L^{2^*_\alpha}} + o(1) \quad \text{as } k \to \infty. \tag{3.11}$$

So, by (3.10) and (3.11), we conclude that $\|u_k\|_\rho^2 \to 0$ as $k \to \infty$, contradicting $C_\rho > 0$. Hence, the sequence $\{t_k\}$ is bounded and there exists $t_0 \in (0, \infty)$ such that (up to subsequence) $t_k \to t_0$ as $k \to \infty$.

Now, from (3.8) and (3.11) we obtain

$$o(1) = \lambda (t_k^{q-1} - 1) \|u_k\|^{q+1}_{L^{q+1}} + (t_k^{2^*_\alpha - 2} 1) \|u_k\|^{2^*_\alpha}_{L^{2^*_\alpha}} \quad \text{as } k \to \infty. \tag{3.12}$$

From where $t_0 = 1$, namely

$$t_k \to 1 \quad \text{as } k \to \infty. \tag{3.13}$$

Therefore, by (3.13) and recalling that $t_k u_k \in \mathcal{N}_\rho$, we get

$$\begin{aligned}
C^* &\le I_\rho(t_k u_k) \\
&= t_k^2 \left[I_\rho(u_k) + \frac{\lambda}{q+1}(1 - t_k^{q-1}) \|u_{k+}\|^{q+1}_{L^{q+1}} + \frac{1}{2^*_\alpha}(1 - t_k^{2^*_\alpha - 2}) \|u_{k+}\|^{2^*_\alpha}_{L^{2^*_\alpha}} \right] \\
&= t_k^2 I_\rho(u_k) + o(1) \\
&= (t_k^2 - 1) I_\rho(u_k) + I_{\rho(u_k)} + o(1).
\end{aligned}$$

Passing to the limit we obtain $C^* \le C_\rho$.

On the other hand, by the previous comments, for any $u \in H^\alpha_\rho(\mathbb{R}^n) \setminus \{0\}$ there is a unique $t_u = t(u) > 0$ such that $t_u u \in \mathcal{N}_\rho$, then

$$C^* \le \inf_{u \in H^\alpha_\rho(\mathbb{R}^n) \setminus \{0\}} \max_{t \ge 0} I_\rho(tu).$$

Moreover, for any $u \in \mathcal{N}_\rho$, we have

$$I_\rho(u) = \max_{t \geq 0} I_\rho(tu) \geq \inf_{u \in H_\rho^\alpha(\mathbb{R}^n)\setminus\{0\}} \max_{t \geq 0} I_\rho(tu),$$

so

$$C^* = \inf_{\mathcal{N}_\rho} I_\rho(u) \geq \inf_{u \in H_\rho^\alpha(\mathbb{R}^n)\setminus\{0\}} \max_{t \geq 0} I_\rho(tu). \qquad \square$$

Remark 3.4. Suppose that (ρ_1) holds and without loss of generality take $\epsilon = 1$, then

$$C_\rho < C.$$

In fact, let u be a critical point of I with critical value C and for any $y \in \mathbb{R}^n$, define $u_y(x) = u(x + y)$. Then for any $t > 0$ we have

$$C = I(u_y) \geq I(tu_y) > I_\rho(tu_y).$$

By Lemma 3.2, we can take $t^* > 0$ such that $t^* u_y \in \mathcal{N}_\rho$ and

$$I_\rho(t^* u_y) = \sup_{t > 0} I_\rho(tu_y),$$

consequently $C > I_\rho(t^* u_y) \geq C_\rho$. In the same way we can show that

$$C_\rho < C_{\rho_\infty} < C.$$

Remark 3.5. According to (2.5) and by Remark 3.4 we have

$$0 < C_\rho < \frac{\alpha}{n} S^{n/(2\alpha)}.$$

Lemma 3.6. *Suppose that $C_\rho < C$. Then there are $\nu, R > 0$ such that*

$$\int_{B(0,R)} u_k^2(x)\, dx \geq \nu \quad \textit{for all } k \in \mathbb{N}.$$

Proof. By Lemma 3.3, for each $k \in \mathcal{N}$, there exist $t_k \subset \mathbb{R}$ such that $t_k \to 1$ and

$$I_\rho(t_k u_k) = \max_{t \geq 0} I_\rho(tu_k).$$

Furthermore

$$I_\rho(u_k) = I_\rho(t_k u_k) + o(1) \geq I_\rho(tu_k) + o(1) \quad \text{for all } t > 0.$$

Now we consider two cases, namely, when $\rho_\infty = \infty$ and $\rho_\infty < \infty$. In the first case, by (ρ_1), for every $\epsilon > 0$ there exist $R > 0$ such that

$$\mathbb{R}^n \setminus B(0, \rho(x)) \subset \mathbb{R}^n \setminus B(0, 1/\epsilon) \quad \text{whenever } |x| > R.$$

Then

$$
\begin{aligned}
I_\rho(tu_k) &= I(tu_k) - \frac{t^2}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n\setminus B(0,\rho(x))} \frac{|u_k(x+z)-u_k(x)|^2}{|z|^{n+2\alpha}}\,dzdx \\
&\geq I(tu_k) - \frac{|S^{n-1}|t^2}{\alpha\rho_0^{2\alpha}}\|u_k\|^2_{L^2(B(0,R))} + \frac{|S^{n-1}|t^2\epsilon^{2\alpha}}{\alpha}\|u_k\|^2_{L^2(\mathbb{R}^n)} \quad \text{for all } t>0.
\end{aligned} \tag{3.14}
$$

If $\{\tau_k\}$ is the bounded real sequence given by

$$I(\tau_k u_k) = \max_{t\geq 0} I(tu_k),$$

we obtain

$$I_\rho(t_k u_k) \geq I(\tau_k u_k) - \frac{|S^{n-1}|\tau_k^2}{\alpha\rho_0^{2\alpha}}\|u_k\|^2_{L^2(B(0,R))} + \frac{|S^{n-1}|\tau_k^2\epsilon^{2\alpha}}{\alpha}\|u_k\|^2_{L^2(\mathbb{R}^n)}. \tag{3.15}$$

If

$$\int_{B(0,R)} u_k^2(x)\,dx \to 0 \quad \text{as } k\to\infty, \tag{3.16}$$

from (3.14), (3.15) and (3.16) yield

$$C_\rho \geq C,$$

which is a contradiction with Remark 3.4.

Now we analyze the case $\rho_\infty < +\infty$. In this case we compare the functionals I_ρ and I_{ρ_∞} writing

$$I_\rho(u) = I_{\rho_\infty}(u) - \frac{1}{2}\int_{\mathbb{R}^n}\int_{B(0,\rho_\infty)\setminus B(0,\rho(x))} \frac{|u(x+z)-u(x)|^2}{|z|^{n+2\alpha}}\,dzdx. \tag{3.17}$$

By hypothesis (ρ_1), for any $\epsilon>0$ there is $R>0$ such that

$$0 < \rho_\infty - \rho(x) < \epsilon \quad \text{whenever } |x|>R.$$

Then, we obtain

$$I_\rho(tu_k) \geq I_{\rho_\infty}(tu_k) - C(\epsilon)t^2\|u_k\|^2_{L^2} - Ct^2\|u_k\|^2_{L^2(B(0,R))},$$

where

$$C(\epsilon) = \frac{2|S^{n-1}|}{\alpha}\left(\frac{1}{(\rho_\infty-\epsilon)^{2\alpha}} - \frac{1}{\rho_\infty^{2\alpha}}\right) \quad \text{and} \quad C = \frac{2|S^{n-1}|}{\alpha}\left(\frac{1}{\rho_0^{2\alpha}} - \frac{1}{\rho_\infty^{2\alpha}}\right).$$

Proceeding as before, by (3.16) we get $C_\rho \geq C_{\rho_\infty}$, which is a contradiction with Remark 3.4. □

The next result shows the existence of positive solution to (1.1) with $\epsilon = 1$.

Theorem 3.7. *Suppose that $\lambda > 0$, $q > 1$ and (ρ_1) hold. Then, problem* (1.1) *with $\epsilon = 1$ possesses a positive ground state solution.*

Proof. Using (3.2), Lemma 3.6 and the Sobolev embedding we have

$$\int_{B(0,R)} u^2(x)\,dx \geq \nu > 0,$$

which proves that $u \neq 0$. Furthermore, by standard arguments we have

$$I'_\rho(u)\varphi = 0 \quad \text{for all } \varphi \in H^\alpha_\rho(\mathbb{R}^n),$$

so choosing $\varphi = u_-(x) = \max\{-u(x), 0\}$ and noting that for $x, z \in \mathbb{R}^n$ we have

$$\begin{aligned}&(u(x+z) - u(x))(u_-(x+z) - u_-(x))\\ &= -u_+(x+z)u_-(x) - u_+(x)u_-(x+z) - (u_-(x+z) - u_-(x))^2 \leq 0,\end{aligned}$$

we can conclude that $\|u_-\|_\rho = 0$, thus $u(x) \geq 0$ a.e. $x \in \mathbb{R}^n$.

Moreover, from Lemma 3.3

$$C_\rho \leq \max_{t\geq 0} I_\rho(tu) = I_\rho(u). \tag{3.18}$$

On the other hand, we have

$$\begin{aligned}C_\rho &= I_\rho(u_k) - \frac{1}{2}I'_\rho(u_k)u_k + o_k(1)\\ &= \lambda\left(\frac{1}{2} - \frac{1}{q+1}\right)\int_{\mathbb{R}^n} u_{k+}^{q+1}\,dx + \frac{1}{n}\int_{\mathbb{R}^n} u_{k+}^{2^*_\alpha}\,dx + o_k(1).\end{aligned}$$

Applying Fatou's Lemma to the last inequality, we obtain

$$\begin{aligned}C_\rho &\geq \lambda\left(\frac{1}{2} - \frac{1}{q+1}\right)\int_{\mathbb{R}^n} u^{q+1}\,dx + \frac{1}{n}\int_{\mathbb{R}^n} u^{2^*_\alpha}\,dx\\ &= I_\rho(u) - \frac{1}{2}I'_\rho(u)u = I_\rho(u).\end{aligned} \tag{3.19}$$

From (3.18) and (3.19) we obtain

$$I_\rho(u) = C_\rho,$$

and hence u is a least energy solution and the proof is finished. □

Proof of Theorem 1.1. In what follows, we denote by $\{u_k\} \subset H^\alpha_{\rho_\epsilon}(\mathbb{R}^n)$ a sequence satisfying

$$I_{\rho_\epsilon}(u_k) \to C_{\rho_\epsilon} \quad \text{and} \quad I'_{\rho_\epsilon}(u_k) \to 0.$$

If $u_k \rightharpoonup 0$ in $H^\alpha_{\rho_\epsilon}(\mathbb{R}^n)$, then

$$u_k \to 0 \quad \text{in } L^p_{\text{loc}}(\mathbb{R}^n) \text{ for } p \in [2, 2^*_\alpha). \tag{3.20}$$

By (ρ_1), we obtain

$$I_{\rho_\epsilon}(tu_k) = I_{\rho_\infty/\epsilon}(tu_k) - \frac{t^2}{2}\int_{\mathbb{R}^n}\int_{B(0,\rho_\infty/\epsilon)\setminus B(0,\rho(\epsilon x)/\epsilon)} \frac{|u_k(x+z)-u_k(x)|^2}{|z|^{n+2\alpha}}\,dzdx$$

where

$$\begin{aligned} I_{\rho_\infty/\epsilon}(u) = {} & \frac{1}{2}\left(\int_{\mathbb{R}^n}\int_{B(0,\rho_\infty/\epsilon)} \frac{|u(x+z)-u(x)|^2}{|z|^{n+2\alpha}}\,dxdx + \int_{\mathbb{R}^n} u^2\,dx\right) \\ & - \frac{\lambda}{q+1}\int_{\mathbb{R}^n} u_+^{q+1}\,dx - \frac{1}{2_\alpha^*}\int_{\mathbb{R}^n} u_+^{2_\alpha^*}\,dx. \end{aligned}$$

Now we know that there exists a bounded sequence $\{\tau_k\}$ such that

$$I_{\rho_\infty/\epsilon}(\tau_k u_k) \geq C(\rho_\infty/\epsilon),$$

where

$$C(\rho_\infty/\epsilon) = \inf_{v\in H^\alpha(\mathbb{R})\setminus\{0\}} \sup_{t\geq 0} I_{\rho_\infty/\epsilon}(tv).$$

Thus,

$$\begin{aligned} C_{\rho_\epsilon} \geq {} & C(\rho_\infty/\epsilon) - \frac{|S^{n-1}|}{\alpha}\left(\frac{1}{\rho_0^{2\alpha}} - \frac{1}{\rho_\infty^{2\alpha}}\right)\tau_k^2\epsilon^{2\alpha}\|u_k\|^2_{L^2(B(0,R/\epsilon))} \\ & - \frac{|S^{n-1}|}{\alpha}\left(\frac{1}{(\rho_\infty-\delta)^{2\alpha}} - \frac{1}{\rho_\infty^{2\alpha}}\right)\tau_k^2\epsilon^{2\alpha}\|u_k\|^2_{L^2(\mathbb{R}^n)}. \end{aligned}$$

Taking the limit as $k\to\infty$, and after $\delta\to 0$, we find

$$C_{\rho_\epsilon} \geq C(\rho_\infty/\epsilon). \tag{3.21}$$

A standard argument shows that

$$\liminf_{\epsilon\to 0} C(\rho_\infty/\epsilon) \geq C.$$

Therefore, if there is $\epsilon_k\to 0$ such that the $(\mathrm{PS})_{C_{\rho_{\epsilon_k}}}$ sequence has weak limit equal to zero, we must have

$$C_{\rho_{\epsilon_k}} \geq C(\rho_\infty/\epsilon_k), \quad \forall\, k\in\mathbb{N},$$

leading to

$$\liminf_{n\to+\infty} C_{\rho_{\epsilon_n}} \geq C,$$

which contradicts Remark 3.4. This proves that the weak limit is non trivial for $\epsilon > 0$ small enough and standard arguments show that its energy is equal to C_{ρ_ϵ}, showing the desired result. □

4. Concentration behaviour

In this section we make a preliminary analysis of the asymptotic behavior of the functional associated to equation (1.1) when $\epsilon \to 0$. As is pointed up in [19], the scope function ρ, that describes the size of the ball of the influential region of the non-local operator, plays a key role in deciding the concentration point of ground states of the equation. Even though, at a first sight, the minimum point of ρ seems to be the concentration point, there is a non-local effect that needs to be taken in account. We define the concentration function

$$\mathcal{H}(x) = -\frac{|S^{n-1}|}{2\alpha}\left(\frac{1}{\rho(x)^{2\alpha}} - \frac{1}{\rho_\infty^{2\alpha}}\right) + \frac{1}{2}\int_{\mathcal{C}^+(x)} \frac{dy}{|y|^{n+2\alpha}} - \frac{1}{2}\int_{\mathcal{C}^-} \frac{dy}{|y|^{n+2\alpha}}, \tag{4.1}$$

where the sets $\mathcal{C}^+(x)$ and $\mathcal{C}^-(x)$ are defined as follows

$$\mathcal{C}^-(x) = \{y \in \mathbb{R}^n : \rho(x+y) < |y| < \rho(x)\}$$

and

$$\mathcal{C}^+(x) = \{y \in \mathbb{R}^n : \rho(x) < |y| < \rho(x+y)\}.$$

We start with some basic properties of the function $\mathcal{H}$.

Lemma 4.1. [19] *Assuming ρ satisfies (ρ_1)–(ρ_3), the function $\mathcal{H}$ is continuous and*

$$\lim_{|x|\to\infty} \mathcal{H}(x) = 0. \tag{4.2}$$

Moreover, there exists $x_0 \in \mathbb{R}^n$ such that

$$\inf_{x\in\mathbb{R}^n} \mathcal{H}(x) = \mathcal{H}(x_0) < 0. \tag{4.3}$$

Along this section we will consider a sequence of functions $\{w_m\} \subset H^\alpha(\mathbb{R}^n)$ such that $\|w_m - w\|_{L^2(\mathbb{R}^2)} \to 0$, where $w \in H^\alpha(\mathbb{R}^n)$. We will also consider sequences $\{z_m\} \subset \mathbb{R}^n$ and $\{\epsilon_m\} \subset \mathbb{R}$ and assume that $\epsilon_m \to 0$ as $m \to \infty$. We define $\overline{\rho}_m$ as

$$\overline{\rho}_m(x) = \frac{1}{\epsilon_m}\rho(\epsilon_m x + \epsilon_m z_m), \tag{4.4}$$

and the functional $I_{\overline{\rho}_m}$ defined as

$$\begin{aligned} I_{\overline{\rho}_m}(u) = \frac{1}{2}&\left(\int_{\mathbb{R}^n}\int_{B(0,\overline{\rho}_m(x))} \frac{|u(x+z)-u(x)|^2}{|z|^{n+2\alpha}}\,dzdx + \int_{\mathbb{R}^n} u^2\,dx\right) \\ &- \frac{\lambda}{q+1}\int_{\mathbb{R}^n} u_+^{q+1}\,dx - \frac{1}{2_\alpha^*}\int_{\mathbb{R}^n} u_+^{2_\alpha^*}\,dx. \end{aligned} \tag{4.5}$$

We will be considering the functionals

$$I_{\rho_\infty/\epsilon_m}, \quad I_{\rho(\overline{x})/\epsilon_m} \quad \text{and the functional } I \text{ in } \mathbb{R}^n \text{ with } \rho \equiv \infty.$$

As in [19] we have the following key theorem.

Theorem 4.2. *Under hypotheses* (ρ_1)*–*(ρ_3)*, we assume as above that* $w_m, w \in H^\alpha(\mathbb{R}^n)$ *are such that* $\|w_m - w\|_{L^2(\mathbb{R}^2)} \to 0$ *and* $\epsilon_m \to 0$ *as* $m \to \infty$*. Then we have:*

(i) *If* $\epsilon_m z_m \to \overline{x}$ *then*

$$\lim_{m\to\infty} \frac{I_{\overline{\rho}_m}(w_m) - I_{\rho_\infty/\epsilon_m}(w_m)}{\epsilon_m^{2\alpha}} = \|w\|_{L^2}^2 \mathcal{H}(\overline{x}). \tag{4.6}$$

(ii) *If* $|\epsilon_m| z_m \to \infty$ *then*

$$\lim_{m\to\infty} \frac{I_{\overline{\rho}_m}(w_m) - I_{\rho_\infty/\epsilon_m}(w_m)}{\epsilon_m^{2\alpha}} = 0. \tag{4.7}$$

Now, we rescaling equation (1.1), for this purpose we define $\rho_\epsilon(x) = \frac{1}{\epsilon}\rho(\epsilon x)$ and consider the rescaled equation

$$(-\Delta)^\alpha_{\rho_\epsilon} v + v = \lambda v^q + u^{2^*_\alpha - 1} \quad \text{in } \mathbb{R}^n \tag{4.8}$$

and we see that u is a weak solution of (1.1) if and only if $v_\epsilon(x) = u(\epsilon x)$ is a weak solution of (4.8).

In order to study equations (1.1) and (4.8), we consider the functional I_{ρ_ϵ} on the ϵ-dependent Hilbert space $H^\alpha_{\rho_\epsilon}(\mathbb{R}^n)$ with inner product $\langle \cdot , \cdot \rangle_{\rho_\epsilon}$. The functional I_{ρ_ϵ} is of class C^1 in $H^\alpha_{\rho_\epsilon}(\mathbb{R}^n)$ and the critical points of I_{ρ_ϵ} are the weak solutions of (4.8). We further introduce

$$\mathcal{N}_{\rho_\epsilon} = \{v \in H^\alpha_{\rho_\epsilon}(\mathbb{R}^n) \setminus \{0\} : I'_{\rho_\epsilon}(v)v = 0\},$$
$$\Gamma_{\rho_\epsilon} = \{\gamma \in C([0,1], H^\alpha_{\rho_\epsilon}(\mathbb{R}^n)) : \gamma(0) = 0, I_{\rho_\epsilon}(\gamma(1)) < 0\}$$

and the mountain pass minimax value

$$C_{\rho_\epsilon} = \inf_{\gamma \in \Gamma_{\rho_\epsilon}} \max_{t\in[0,1]} I_{\rho_\epsilon}(\gamma(t)).$$

From Lemma 3.3 we also have

$$0 < C_{\rho_\epsilon} = \inf_{v \in \mathcal{N}_{\rho_\epsilon}} I_{\rho_\epsilon}(v) = \inf_{v \in H^\alpha_{\rho_\epsilon}(\mathbb{R}^n)\setminus\{0\}} \max_{t\geq 0} I_{\rho_\epsilon}(tv). \tag{4.9}$$

For comparison purposes we consider the functional I, whose critical points are the solutions of (2.4). We also consider the critical value C that satisfies

$$C = \inf_{u \in H^\alpha(\mathbb{R}^n)\setminus\{0\}} \max_{t\geq 0} I(tu).$$

Now we start the proof of Theorem 1.2 with some preliminary lemmas.

Lemma 4.3. *Suppose* (ρ_1) *holds. Then*

$$\lim_{\epsilon\to 0^+} C_{\rho_\epsilon} = C. \tag{4.10}$$

Proof. Since we obviously have

$$\int_{\mathbb{R}^n}\int_{B(0,\rho_\epsilon(x))} \frac{|u(x+z)-u(x)|^2}{|z|^{n+2\alpha}}\,dzdx \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x+z)-u(x)|^2}{|z|^{n+2\alpha}}\,dzdx$$

for all $u \in H^\alpha_{\rho_\epsilon}(\mathbb{R}^n)$, then we have $I_{\rho_\epsilon}(u) \le I(u)$ and therefore

$$\limsup_{\epsilon\to 0^+} C_{\rho_\epsilon} \le C. \tag{4.11}$$

On the other hand, by (ρ_1) we have $\rho(\epsilon x) \ge \rho_0$ for all $x \in \mathbb{R}^n$ then

$$C_{\rho_\epsilon} \ge C_{\rho_0/\epsilon}.$$

By standard arguments we can show that

$$\lim_{\epsilon\to 0} C_{\rho_0/\epsilon} = C.$$

Thus

$$\liminf_{\epsilon\to 0} C_{\rho_\epsilon} \ge C. \tag{4.12}$$

Therefore, by (4.11) and (4.12) we obtain (4.10). □

Lemma 4.4. *There are* $\epsilon_0 > 0$, *a family* $y_\epsilon \subset \mathbb{R}^n$, *constants* $\beta, R > 0$ *such that*

$$\int_{B(y_\epsilon,R)} v_\epsilon^2\,dx \ge \beta \quad \textit{for all } \epsilon \in [0,\epsilon_0]. \tag{4.13}$$

Proof. By contradiction, there is a sequence $\epsilon_m \to 0$ such that for all $R > 0$

$$\lim_{m\to\infty} \sup_{y\in\mathbb{R}^n} \int_{B(y,R)} v_{\epsilon_m}^2\,dx = 0.$$

Using the following notation $v_m = v_{\epsilon_m}$ and $C_{\rho_m} = C_{\rho_{\epsilon_m}}$, by Lemma 2.5

$$\int_{\mathbb{R}^n} v_m^{q+1}\,dx = o_m(1).$$

Furthermore, since $I'_{\rho_m}(v_m)v_m = 0$ then

$$\int_{\mathbb{R}^n}\int_{B(0,\rho_m(x))} \frac{|v_m(x+z)-v_m(x)|^2}{|z|^{n+2\alpha}}\,dzdx + \int_{\mathbb{R}^n} v_m^2\,dx = \int_{\mathbb{R}^n} v_m^{2^*_\alpha}\,dx + o_m(1).$$

Let $l \geq 0$ be such that

$$\int_{\mathbb{R}^n}\int_{B(0,\rho_m(x))} \frac{|v_m(x+z)-v_m(x)|^2}{|z|^{n+2\alpha}}\,dzdx + \int_{\mathbb{R}^n} v_m^2\,dx \to l.$$

Now, since $I_{\rho_m}(v_m) = C_{\rho_m}$, we obtain

$$C_{\rho_m} = \frac{\alpha}{n}\left(\int_{\mathbb{R}^n}\int_{B(0,\rho_m(x))} \frac{|v_m(x+z)-v_m(x)|^2}{|z|^{n+2\alpha}}\,dzdx + \int_{\mathbb{R}^n} v_m^2\,dx\right) + o_m(1),$$

then by Lemma 4.3

$$l = \frac{n}{\alpha}C \tag{4.14}$$

hence $l > 0$. Now, using the definition of the Sobolev constant S and Remark 2.3, we have

$$\left(\int_{\mathbb{R}^n} v_m^{2_\alpha^*}\,dx\right)^{2/2_\alpha^*} S \leq \int_{\mathbb{R}^n}\int_{B(0,\rho_m(x))} \frac{|v_m(x+z)-v_m(x)|^2}{|z|^{n+2\alpha}}\,dzdx + \int_{\mathbb{R}^n} v_m^2\,dx.$$

Therefore, by (4.14) and taking the limit in the above inequality as $m \to \infty$ we achieved that

$$C \geq \frac{\alpha}{n}S^{n/(2\alpha)}$$

which is a contradiction with (2.5). □

Now let

$$w_\epsilon(x) = v_\epsilon(x+y_\epsilon) = u_\epsilon(\epsilon x + \epsilon y_\epsilon), \tag{4.15}$$

then by (4.13),

$$\liminf_{\epsilon\to 0^+} \int_{B(0,R)} w_\epsilon^2(x)\,dx \geq \beta > 0. \tag{4.16}$$

To continue, we consider the rescaled scope function $\overline{\rho}_\epsilon$, as defined in (4.4),

$$\overline{\rho}_\epsilon(x) = \frac{1}{\epsilon}\rho(\epsilon x + \epsilon y_\epsilon)$$

and then w_ϵ satisfies the equation

$$(-\Delta)^{\alpha}_{\overline{\rho}_\epsilon} w_\epsilon(x) + w_\epsilon(x) = w_\epsilon^p(x) \quad \text{in } \mathbb{R}^n. \tag{4.17}$$

Now we prove the convergence of w_ϵ as $\epsilon \to 0$.

Lemma 4.5. *For every sequence $\{\epsilon_m\}$ there is a subsequence, we keep calling the same, so that $w_{\epsilon_m} = w_m \to w$ in $H^\alpha(\mathbb{R}^n)$, when $m \to \infty$, where w is a solution of (2.4).*

Proof. Note that

$$
\begin{aligned}
C_{\rho_m} &= I_{\rho_m}(v_m) - \frac{1}{q+1} I'_{\rho_m}(v_m)v_m \\
&= \left(\frac{1}{2} - \frac{1}{q+1}\right)\left(\int_{\mathbb{R}^n}\int_{B(0,\frac{1}{\epsilon_m}\rho(\epsilon_m x))} \frac{|v_m(x+z)-v_m(x)|^2}{|z|^{n+2\alpha}}\,dzdx + \int_{\mathbb{R}^n} v_m^2(x)\,dx\right) \\
&\quad + \left(\frac{1}{q+1} - \frac{1}{2_\alpha^*}\right)\int_{\mathbb{R}^n} v_m^{2_\alpha^*}(x)\,dx \\
&\geq \left(\frac{1}{2} - \frac{1}{q+1}\right)\left(\int_{\mathbb{R}^n}\int_{B(0,\rho_0/\epsilon_m)} \frac{|v_m(x+z)-v_m(x)|^2}{|z|^{n+2\alpha}}\,dzdx + \int_{\mathbb{R}^n} v_m^2(x)\,dx\right) \\
&\quad + \left(\frac{1}{q+1} - \frac{1}{2_\alpha^*}\right)\int_{\mathbb{R}^n} v_m^{2_\alpha^*}(x)\,dx = \Lambda_m.
\end{aligned}
$$

By Lemma 4.3 we obtain that

$$
\limsup_{m\to\infty} \Lambda_m \leq C. \tag{4.18}
$$

On the other hand, by Fatou's Lemma and the weak convergence of $\{w_m\}$, we get

(4.19)

$$
\begin{aligned}
C &\leq I(w) \\
&= \left(\frac{1}{2} - \frac{1}{q+1}\right)\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|w(x+z)-w(x)|^2}{|z|^{n+2\alpha}}\,dzdx + \int_{\mathbb{R}^n} w^2\,dx\right) \\
&\quad + \left(\frac{1}{q+1} - \frac{1}{2_\alpha^*}\right)\int_{\mathbb{R}^n} w^{2_\alpha^*}\,dx \\
&\leq \liminf_{m\to\infty} \Lambda_m + \liminf_{m\to\infty}\left(\frac{1}{2} - \frac{1}{q+1}\right)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n\setminus B(0,\rho_0/\epsilon_m)} \frac{|w_m(x+z)-w_m(x)|^2}{|z|^{n+2\alpha}}\,dzdx \\
&= \liminf_{m\to\infty} \Lambda_m.
\end{aligned}
$$

So, by (4.18) and (4.19), $\lim_{m\to\infty} \Lambda_m = C$, from where we get

$$
\lim_{m\to\infty} \|w_m - w\|_\alpha = 0. \qquad \square
$$

We are now in a position to complete the proof of our second main theorem.

Proof of Theorem 1.2. We first obtain an upper bound for the critical values $C_{\rho_{\epsilon_m}} = C_m$, for the sequence $\{\epsilon_m\}$ given in Lemma 4.5. Next we consider the scope function

$$
\widetilde{\rho}_m(x) = \frac{1}{\epsilon_m}\rho(\epsilon_m x + x_0),
$$

where x_0 is a global minimum point of $\mathcal{H}$, see Lemma 4.1. To continue, we consider the function $w_m = w_{\epsilon_m}$ as given in (4.15) and let $t_m > 0$ such that $t_m w_m \in \mathcal{N}_{\widetilde{\rho}_m}$. According to Lemma 4.5, $\{w_m\}$ converges to $w \in \mathcal{N}$, then $t_m \to 1$ and $t_m w_m \to w$.

Now we apply Theorem 4.2 to obtain that

$$C_m \leq I_{\widetilde{\rho}_m}(t_m w_m) = I_{\rho_\infty/\epsilon_m}(t_m w_m) + \epsilon_m^{2\alpha} \left(\|w\|_{L^2}^2 \mathcal{H}(x_0) + o(1) \right). \tag{4.20}$$

We have used Theorem 4.2(i) with $z_m = x_0/\epsilon_m$.

On the other hand, since $w_m \in H^\alpha(\mathbb{R}^n)$ is a critical point of $I_{\overline{\rho}_m}$, we have that

$$C_m = I_{\overline{\rho}_m}(w_m) \geq I_{\overline{\rho}_m}(t_m w_m). \tag{4.21}$$

We write $y_m = y_{\epsilon_m}$. If $\epsilon_m |y_m| \to \infty$, then we may apply Theorem 4.2(ii) with $z_m = y_m$ in (4.21) and obtain that

$$C_m \geq I_{\rho_\infty/\epsilon_m}(t_m w_m) + \epsilon_m^{2\alpha} o(1),$$

which contradicts (4.20). We conclude then, that $\{\epsilon_m y_m\}$ is bounded and that, for a subsequence, $\epsilon_m y_m \to \overline{x}$, for some $\overline{x} \in \mathbb{R}^n$. Now we apply Theorem 4.2 again, but now part (i) with $z_m = y_m$ in (4.21), and we obtain that

$$C_m \geq I_{\rho_\infty/\epsilon_m}(t_m w_m) + \epsilon_m^{2\alpha} \left(\|w\|_{L^2}^2 \mathcal{H}(\overline{x}) + o(1) \right). \tag{4.22}$$

From (4.20) and (4.22) we finally get that

$$\|w\|_{L^2}^2 \mathcal{H}(\overline{x}) + o(1) \leq \|w\|_{L^2}^2 \mathcal{H}(x_0) + o(1)$$

and taking the limit as $m \to \infty$, we get

$$\mathcal{H}(\overline{x}) \leq \mathcal{H}(x_0) \tag{4.23}$$

completing the proof of the theorem. □

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César E. Torres Ledesma
Departamento de Matemáticas, Universidad Nacional de Trujillo, Av. Juan Pablo II s/n. Trujillo-Perú
E-mail address: `ctl_576@yahoo.es`