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Hecke Bound of Vector-valued Modular Forms and its Relationship with Cuspidality

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Abstract. In this paper, we prove that if the Fourier coefficients of a vector-valued modular form satisfy the Hecke bound, then it is cuspidal. Furthermore, we obtain an analogous result with regard to Jacobi forms by applying an isomorphism between vector-valued modular forms and Jacobi forms. As an application, we prove a result on the growth of the number of representations of m by a positive definite quadratic form Q.

1. Introduction

It is known that the Fourier coefficients of scalar-valued cusp forms satisfy the Hecke bound, i.e., if $f(z) = \sum_{n>0} a(n)e^{2\pi inz}$ is a cusp form of weight k, then it satisfies

$$a(n) = O(n^{k/2})$$

as $n \to \infty$. On the other hand, it is natural to ask a question whether the converse problem is true, i.e., whether one can conclude that a given general modular form is cuspidal if its Fourier coefficients satisfy the Hecke bound. This problem was initiated and investigated by Kohnen for scalar-valued modular forms in [8]. Later on, Schmoll [13] solved this problem for scalar-valued modular forms of even integral weight $k > 2$ on congruence subgroups $\Gamma_0(N)$. Similar results has been also obtained for other modular forms such as half-integral weight modular forms, Hilbert modular forms, Siegel modular forms, Jacobi forms, etc. (for more details, see [1, 4, 9–12].)

In this paper, we study the converse problem for vector-valued modular forms. A general method used before in proving the converse problems is dependent on the exact formula of the Fourier coefficients of the Eisenstein series and the Hecke theory which are complicated to describe for general modular forms on general groups. To overcome this difficulty, we use meromorphic continuations of vector-valued L-functions associated with

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given vector-valued modular forms. Moreover, we also obtain an analogous result with regard to Jacobi forms by applying an isomorphism between vector-valued modular forms and Jacobi forms. As an application, we prove a result on the growth of the number of representations of m by a positive definite quadratic form Q.

To describe in detail, we introduce the basic notation and terminology which will be used throughout the paper. Let $k \in \frac{1}{2}\mathbb{Z}$ and Γ be a finite index subgroup of $\mathrm{SL}(2,\mathbb{Z})$. We denote by $\{Q_0,\dots,Q_t,V_1,\dots,V_s\}$ a fixed set of generators of Γ, where $Q_0,\dots,Q_t$ are parabolic generators and $V_1,\dots,V_s$ are non-parabolic generators. Let p be a positive integer and $\rho\colon \Gamma \to \mathrm{GL}_p(\mathbb{C})$ be a p-dimensional complex representation such that $\rho(Q_i)$ is diagonal and of finite order for each i. Let χ be a multiplier system of weight k for the group $\mathrm{SL}(2,\mathbb{Z})$, i.e., $|\chi(\gamma)| = 1$, $\chi(-I) = e^{\pi i k}$, and χ satisfies the consistency condition

$$\chi(\gamma_3)(c_3\tau + d_3)^k = \chi(\gamma_1)\chi(\gamma_2)(c_1\gamma_2\tau + d_1)^k(c_2\tau + d_2)^k,$$

where $\gamma_3 = \gamma_1\gamma_2$ and $\gamma_i = \left(\begin{smallmatrix} * & * \\ c_i & d_i \end{smallmatrix}\right)$, $i = 1, 2,$ and 3. We denote the standard basis of $\mathbb{C}^p$ by $(\mathbf{e}_j)_{j=1}^p$.

A vector-valued modular form of weight k, multiplier system χ, and type ρ on Γ is the sum $f = \sum_{j=1}^p f_j\mathbf{e}_j$ of holomorphic functions f_j on the upper half-plane $\mathbb{H}$ such that $f|_{k,\chi}\gamma = \rho(\gamma)f$ for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, where the operator $|_{k,\chi}$ is defined by

$$(f|_{k,\chi}\gamma)(\tau) := \chi^{-1}(\gamma)(c\tau + d)^{-k} f(\gamma\tau).$$

Here, $\gamma\tau := (a\tau + b)/(c\tau + d)$ and we take $\arg z$ to lie in the range $-\pi \le \arg z < \pi$ for any complex number $z \neq 0$. For each $\gamma \in \mathrm{SL}(2,\mathbb{Z})$, the function $f|_{k,\chi}\gamma$ has a Fourier expansion of the form

$$\sum_{j=1}^p f_{j,\gamma}(\tau)\mathbf{e}_j = \sum_{j=1}^p \sum_{n+\kappa_{j,\gamma}\ge 0} a_{j,\gamma}(n) e^{2\pi i (n+\kappa_{j,\gamma})\tau/\lambda_\gamma}\mathbf{e}_j, \tag{1.1}$$

where $\kappa_{j,\gamma}$ is a real number with $0 \le \kappa_{j,\gamma} < 1$ and $\lambda_\gamma \in \mathbb{Z}$. For simplicity, let $a_j(n) = a_{j,I}(n)$, where I denotes the identity matrix. Furthermore, let $M_{k,\chi,\rho}(\Gamma)$ (resp. $S_{k,\chi,\rho}(\Gamma)$) denote the space of all vector-valued modular (resp. cusp) forms of of weight k, multiplier system χ, and type ρ on Γ (for additional details, see Section 2).

In [7], it is shown that for a given vector-valued cusp form $f(\tau)$ in $S_{k,\chi,\rho}(\mathrm{SL}(2,\mathbb{Z}))$ there is a constant α depending only on ρ such that $a_{j,\gamma}(n) = O(n^{k/2+\alpha})$ for every $1 \le j \le p$ and for every $\gamma \in \mathrm{SL}(2,\mathbb{Z})$, as $n \to \infty$ (see also [14]). In particular, if ρ is a unitary representation, then $\alpha = 0$. The following theorem shows that the converse is also true.

Theorem 1.1. *Suppose that $k > 2 + 2\alpha$ and $f \in M_{k,\chi,\rho}(\Gamma)$ with an irreducible representation ρ has a Fourier expansion as in* (1.1)*. If there exists at least one l such that the Fourier coefficients $a_{l,\gamma}(n)$ for each $\gamma \in \mathrm{SL}(2,\mathbb{Z})$ satisfy the growth condition $a_{l,\gamma}(n) = O(n^{k/2+\alpha})$ as $n \to \infty$, then f is cuspidal.*

Remark 1.2. (1) Theorem 1.1 proves the vector-valued version of the conjecture of Böcherer and Das [1] for $\mathrm{SL}(2,\mathbb{Z})$.

(2) We can remove the condition of irreducibility of ρ if we impose a stronger condition to get a similar result as follows. Suppose that $k > 2+2\alpha$ and $f \in M_{k,\chi,\rho}(\Gamma)$ has a Fourier expansion as in (1.1). If for every $1 \le l \le p$ and $\gamma \in \mathrm{SL}(2,\mathbb{Z})$, the Fourier coefficients $a_{l,\gamma}(n)$ satisfy the growth condition $a_{l,\gamma}(n) = O(n^{k/2+\alpha})$ as $n \to \infty$, then f is cuspidal. This result will be used later to prove a corresponding result on Jacobi forms.

Next, we consider Jacobi forms. A Jacobi form is a holomorphic function of two variables ($\tau \in \mathbb{H}$ and $z \in \mathbb{C}$) that satisfies modular and elliptic transformation properties with holomorphicity conditions at cusps. Let $J_{k,m,\chi}(\Gamma^J)$ (resp. $S_{k,m,\chi}(\Gamma^J)$) denote the space of Jacobi forms (resp. Jacobi cusp forms) of weight k, index m, and multiplier system χ on Γ^J, where Γ^J denotes a Jacobi group $\Gamma \ltimes \mathbb{Z}^2$. If $F \in J_{k,m,\chi}(\Gamma^J)$, then F has a Fourier expansion of the form

$$\sum_{\substack{n,r\in\mathbb{Z}\\ D_\gamma\ge 0}} c_\gamma(n,r) e^{2\pi i(n+\kappa_\gamma)\tau/\lambda_\gamma} e^{2\pi i r z} \tag{1.2}$$

for each $\gamma \in \mathrm{SL}(2,\mathbb{Z})$, where $D_\gamma = D_\gamma(n,r) := 4[(n+\kappa_\gamma)/\lambda_\gamma]m - r^2$ (for additional details, see Section 3). In the following theorem, we obtain a result about the cuspidality of a Jacobi form.

Theorem 1.3. *Let $k > 5/2$ and $m \in \mathbb{Z}$. Suppose that the Fourier coefficients of $F \in J_{k,m,\chi}(\Gamma^J)$ satisfy*

$$c_\gamma(n,r) = O(D_\gamma^{k/2-1/4}) \tag{1.3}$$

for every $\gamma \in \mathrm{SL}(2,\mathbb{Z})$, as $D_\gamma \to \infty$. Then, F is cuspidal.

To prove this, we use an isomorphism φ from the space of Jacobi forms of weight k to the space of vector-valued modular forms of weight $k-1/2$ with a unitary representation ρ'. Then the bound in (1.3) implies that Fourier coefficients of $\varphi(F)$ satisfies $O\big(n^{\frac{1}{2}\left(k-\frac{1}{2}\right)}\big)$. With this and Remark 1.2(2), one can prove Theorem 1.3. Note that since the representation ρ' is unitary, we take $\alpha = 0$ when we apply Remark 1.2(2).

As an example of a vector-valued modular form, we consider a theta series associated with a positive definite quadratic form. For a positive integer n, let $Q(x)$ be a positive definite quadratic form of rank n over $\mathbb{R}$, and $B(x,y)$ denote its associated bilinear form. Let L be a lattice of rank n such that $Q(x) \in \mathbb{Z}$ for all $x \in L$, and L^* denote its dual lattice defined by $L^* := \{x \in \mathbb{R}^n \mid B(x,y) \in \mathbb{Z} \text{ for all } y \in L\}$. We define a vector-valued theta series

$$\theta_{Q,L}(\tau) := \sum_{h \in L^*/L} \theta_{Q,L}(\tau, h)\mathbf{e}_h,$$

where $\theta_{Q,L}(\tau, h) = \sum_{x \in L+h} e^{2\pi i Q(x)\tau}$. Then $\theta_{Q,L}$ satisfies (for example, see [5, Corollary 1.7])

$$\theta_{Q,L}(\tau+1, h) = e^{2\pi i Q(h)}\theta_{Q,L}(\tau, h)$$

and

$$\theta_{Q,L}\left(-\frac{1}{\tau}, h\right) = \tau^{n/2}\frac{\sqrt{i}^{-n}}{\sqrt{|L^*/L|}}\sum_{k \in L^*/L} e^{-2\pi i B(k,h)}\theta_{Q,L}(\tau, k).$$

Therefore, $\theta_{Q,L}$ is a vector-valued modular form of weight $n/2$ on $\mathrm{SL}(2,\mathbb{Z})$ associated with the Weil representation $\rho_{Q,L}$ (for the definition of the Weil representation, see [2, Section 1]). The mth Fourier coefficient of $\theta_{Q,L}(\tau, h)$ is

$$R_{Q,L}(m, h) := |\{x \in L + h \mid Q(x) = m\}|.$$

As an application of Theorem 1.1, we obtain a result on the growth of $R_{Q,L}(m, h)$ as $m \to \infty$.

Theorem 1.4. *If $n \geq 5$, then for any $M > 0$, there are infinitely many $m \in \mathbb{N}$ such that*

$$R_{Q,L}(m, h) > Mm^{n/4}.$$

The remainder of this paper is organized as follows. In Section 2, we present basics about vector-valued modular forms and vector-valued L-functions. In Section 3, we review the theory of Jacobi forms focusing on the theta expansion. Finally, in Section 4, we prove the main results.

2. Vector-valued modular forms

In this section, we recall some basic facts about vector-valued modular forms and vector-valued L-functions. We start with the precise definition of a vector-valued modular form.

Definition 2.1. A vector-valued modular form of weight k, multiplier system χ, and type ρ on Γ is the sum $f = \sum_{j=1}^{p} f_j\mathbf{e}_j$ of holomorphic functions in $\mathbb{H}$ which satisfies

(1) $f|_{k,\chi}\gamma = \rho(\gamma)f$ for all $\gamma \in \Gamma$,

(2) for each $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$, the function $f|_{k,\chi}\gamma$ has a Fourier expansion as in (1.1).

Furthermore, if each $a_{j,\gamma}(n)$ is zero when $n+\kappa_{j,\gamma}$ is nonpositive, then f is called a vector-valued cusp form.

For each j, the constant term of $f_{j,\gamma}$ is defined by $C_{j,\gamma} := \sum_{n+\kappa_{j,\gamma}=0} a_{j,\gamma}(n)$. Further, we define the constant term of $f|_{k,\chi}\gamma$ by

$$C_{f,\gamma} := \sum_{j=1}^{p} C_{j,\gamma}\mathbf{e}_j = \sum_{j=1}^{p} \sum_{n+\kappa_{j,\gamma}=0} a_{j,\gamma}(n)\mathbf{e}_j.$$

Now, we introduce vector-valued L-functions attached to given vector-valued modular forms. Let $f = \sum_{j=1}^{p} f_j\mathbf{e}_j \in M_{k,\chi,\rho}(\Gamma)$ with a Fourier expansion as in (1.1). The associated L-function of f and γ is defined by

$$L_\gamma(f,s) := \sum_{j=1}^{p} L(f_{j,\gamma},s)\mathbf{e}_j,$$

where each component $L(f_{j,\gamma},s)$ is given by the usual Dirichlet series of $f_{j,\gamma}$, namely,

$$L(f_{j,\gamma},s) := \sum_{n+\kappa_{j,\gamma}>0} \frac{a_{j,\gamma}(n)}{((n+\kappa_{j,\gamma})/\lambda_\gamma)^s}.$$

To use later, we also define $\Lambda(f_{j,\gamma},s) := (2\pi)^{-s}\Gamma(s)L(f_{j,\gamma},s)$ as usual. We consider an analytic continuation and locations of poles of $L(f_{j,\gamma},s)$. For this reason, we need the following lemma.

Lemma 2.2. *If $f \in M_{k,\chi,\rho}(\Gamma) \setminus S_{k,\chi,\rho}(\Gamma)$ and ρ is irreducible, then for each $1 \le j \le p$, there exists at least one element $\gamma_j \in \Gamma$ such that f_{j,γ_j} has a non-zero constant term.*

Proof. Suppose that there is no such $\gamma_j \in \Gamma$. Since we fix a standard basis, we can consider $\rho(\gamma)$ as a $p \times p$ matrix. Let L_j be the space spanned by the jth rows of matrices $\rho(\gamma)$ for $\gamma \in \Gamma$. Note that the constant term of $f_{j,\gamma}$ is zero if and only if $(\rho(\gamma)C_{f,I})_j = 0$. Therefore, we see that $\dim_{\mathbb{C}} L_j < p$.

Since ρ is irreducible, its dual representation ρ^* is also irreducible. The space A generated by

$$\{\rho^*(\gamma)\mathbf{e}_j \mid \gamma \in \Gamma\}$$

is nontrivial and invariant by the representation ρ^*. Therefore, A is a non-zero subrepresentation of ρ^*. Since ρ^* is irreducible, A must be $\mathbb{C}^p$. Note that the dual representation ρ^* is defined by $\rho^*(\gamma) = \rho(\gamma^{-1})^t$, where γ^t denotes the transpose of γ. We see that A is the same as L_j. This is a contradiction since $\dim_{\mathbb{C}} L_j < p$. Therefore, there is at least one element $\gamma_j \in \Gamma$ such that f_{j,γ_j} has a non-zero constant term. □

Proposition 2.3. *Assume that $f \in M_{k,\chi,\rho}(\Gamma)\backslash S_{k,\chi,\rho}(\Gamma)$ and ρ is irreducible. Fix $1 \le l \le p$ and let $\gamma \in \Gamma$ such that $f_{l,\gamma}$ has a non-zero constant term $C_{l,\gamma}$. Then, $L(f_{l,\gamma S^{-1}}, s)$ has a meromorphic continuation on $\mathbb{C}$ with a simple pole at $s = k$, where $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.*

Proof. First, note that $f_{\gamma S^{-1}}|_{k,\chi} S = f_\gamma$. This implies that

$$\chi(S)^{-1}\tau^{-k} f_{\gamma S^{-1}}\left(-\frac{1}{\tau}\right) = f_\gamma(\tau).$$

Let

$$\psi_\gamma(v) := f_\gamma(iv) - C_{f,\gamma},$$

where $C_{f,\gamma}$ is the constant term of a vector-valued modular form f_γ. Then, we have

$$\psi_{\gamma S^{-1}}(v) = \chi(S) i^k v^{-k} \psi_\gamma\left(\frac{1}{v}\right) + \chi(S) i^k v^{-k} C_{f,\gamma} - C_{f,\gamma S^{-1}}.$$

If we denote by $\psi_{j,\gamma}$ the jth component function of ψ_γ, then one can see that

$$\psi_{j,\gamma}(v) = f_{j,\gamma}(iv) - C_{j,\gamma}$$

and

$$\psi_{l,\gamma S^{-1}}(v) = \chi(S) i^k v^{-k} \psi_{l,\gamma}\left(\frac{1}{v}\right) + \chi(S) i^k v^{-k} C_{l,\gamma} - C_{l,\gamma S^{-1}}. \tag{2.1}$$

Note that by assumption, $C_{l,\gamma} \neq 0$.

Now, we compute the Mellin transform of $\psi_{l,\gamma S^{-1}}$. If we insert the Fourier coefficients of $f_{l,\gamma S^{-1}}(\tau)$, then we see that

$$\begin{aligned} \int_0^\infty \psi_{l,\gamma S^{-1}}(v) v^s \frac{dv}{v} &= \sum_{n+\kappa_{l,\gamma S^{-1}}>0} a_{l,\gamma S^{-1}}(n) \int_0^\infty e^{-2\pi(n+\kappa_{l,\gamma S^{-1}})v/\lambda_{\gamma S^{-1}}} v^s \frac{dv}{v} \\ &= (2\pi)^{-s}\Gamma(s) L(f_{l,\gamma S^{-1}}, s) = \Lambda(f_{l,\gamma S^{-1}}, s). \end{aligned}$$

On the other hand, we can use the inversion formula in (2.1) to compute the Mellin transform of $\psi_{l,\gamma S^{-1}}$. First, we separate the integral into two parts

$$\int_0^\infty \psi_{l,\gamma S^{-1}}(v) v^s \frac{dv}{v} = \int_0^1 \psi_{l,\gamma S^{-1}}(v) v^s \frac{dv}{v} + \int_1^\infty \psi_{l,\gamma S^{-1}}(v) v^s \frac{dv}{v}. \tag{2.2}$$

Next, we compute the first integral in (2.2) using the inversion formula and change of variables

$$\begin{aligned} \int_0^1 \psi_{l,\gamma S^{-1}}(v) v^s \frac{dv}{v} &= \int_0^1 \left(\chi(S) i^k v^{-k} \psi_{l,\gamma}\left(\frac{1}{v}\right) + \chi(S) i^k v^{-k} C_{l,\gamma} - C_{l,\gamma S^{-1}}\right) v^s \frac{dv}{v} \\ &= \chi(S) i^k \int_1^\infty \psi_{l,\gamma}(v) v^{k-s} \frac{dv}{v} + \chi(S) i^k C_{l,\gamma} \int_0^1 v^{s-k-1}\, dv \\ &\quad - C_{l,\gamma S^{-1}} \int_0^1 v^{s-1}\, dv. \end{aligned}$$

Therefore, we have

$$\begin{aligned}\Lambda(f_{l,\gamma S^{-1}}, s) &= \int_0^\infty \psi_{l,\gamma S^{-1}}(v) v^s \frac{dv}{v} \\ &= \int_1^\infty (\psi_{l,\gamma S^{-1}}(v) v^s + \chi(S) i^k \psi_{l,\gamma}(v) v^{k-s}) \frac{dv}{v} + \chi(S) i^k C_{l,\gamma} \frac{1}{s-k} - C_{l,\gamma S^{-1}} \frac{1}{s}.\end{aligned}$$

Note that

$$L(f_{l,\gamma S^{-1}}, s) = \Gamma(s)^{-1} (2\pi)^s \Lambda(f_{l,\gamma S^{-1}}, s).$$

It is well known that $\Gamma(s)^{-1}$ is an entire function with simple zeros at $s = 0, -1, -2, \ldots$, and it does not vanish elsewhere (for instance, see [15]). Therefore, $L(f_{l,\gamma S^{-1}}, s)$ has a meromorphic continuation on $\mathbb{C}$ with a simple pole at $s = k$. □

3. Jacobi forms

In this section, we review the theory of Jacobi forms, following the approach of Eichler and Zagier [6]. First, we fix the notation. Let $\Gamma^J = \Gamma \ltimes \mathbb{Z}^2$ be a Jacobi group and F be a function on $\mathbb{H} \times \mathbb{C}$. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z}), X = (\lambda, \mu) \in \mathbb{Z}^2$, and $m \in \mathbb{Z}$, we define

$$(F|_{k,m,\chi}\gamma)(\tau, z) := \chi(\gamma)^{-1}(c\tau + d)^{-k} e^{-2\pi i m \frac{cz^2}{c\tau+d}} F(\gamma(\tau, z))$$

and

$$(F|_m X)(\tau, z) := e^{2\pi i m(\lambda^2\tau + 2\lambda z)} F(\tau, z + \lambda\tau + \mu),$$

where $\gamma(\tau, z) = ((a\tau + b)/(c\tau + d), z/(c\tau + d))$. Then, Γ^J acts on the space of functions on $\mathbb{H} \times \mathbb{C}$ by

$$(F|_{k,m,\chi}(\gamma, X))(\tau, z) := (F|_{k,m,\chi}\gamma|_m X)(\tau, z).$$

With this operator, we define a Jacobi form.

Definition 3.1. A Jacobi form of weight k, index m, and multiplier system χ on Γ^J is a holomorphic function F on $\mathbb{H} \times \mathbb{C}$ satisfying

(1) $F|_{k,m,\chi}(\gamma, X) = F$ for every $(\gamma, X) \in \Gamma^J$,

(2) for each $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$, the function $F|_{k,m,\chi}\gamma$ has a Fourier expansion as in (1.2).

Furthermore, if $c_\gamma(n, r) \neq 0$ only if $D_\gamma > 0$ for all $\gamma \in \mathrm{SL}(2, \mathbb{Z})$, then it is called a Jacobi cusp form.

Theorem 3.2. [6, Section 5] *Let F be a holomorphic function that satisfies $F|_m X = F$ for every $X \in \mathbb{Z}^2$. Then,*

$$F(\tau, z) = \sum_{\mu \in \mathcal{N}} f_\mu(\tau) \theta_{m,\mu}(\tau, z) \tag{3.1}$$

with uniquely determined holomorphic functions $f_\mu(\tau)$ on $\mathbb{H}$, where $\mathcal{N} = \mathbb{Z}/2m\mathbb{Z}$.

The expansion in Theorem 3.2 is called the theta expansion. The theta expansion gives an isomorphism between the space of Jacobi forms and the space of vector-valued modular forms. More precisely, let F be a Jacobi form in $J_{k,m,\chi}(\Gamma^J)$. By Theorem 3.2, a Jacobi form F has the theta expansion

$$F(\tau, z) = \sum_{\mu \in \mathcal{N}} f_\mu(\tau)\theta_{m,\mu}(\tau, z).$$

Then, one can check that a vector-valued function

$$\sum_{\mu \in \mathcal{N}} f_\mu \mathbf{e}_\mu$$

is a vector-valued modular form in $M_{k-1/2,\chi',\rho'}(\Gamma)$ for some χ' and ρ' (for example, see [3, Section 2]). Here, ρ' is essentially the Weil representation.

Theorem 3.3. [6, Section 5] *The theta expansion gives an isomorphism between* $J_{k,m,\chi}(\Gamma^J)$ *and* $M_{k-1/2,\chi',\rho'}(\Gamma)$. *Furthermore, it sends Jacobi cusp forms to vector-valued cusp forms.*

4. Proofs of the main theorems

For the proof of Theorem 1.1, we observe the following fact. Let f be a vector-valued modular form in $M_{k,\chi,\rho}(\Gamma)$. For each $\gamma \in \mathrm{SL}(2,\mathbb{Z})$, a vector-valued function $f|_{k,\chi}\gamma$ is a vector-valued modular form in $M_{k,\chi,\rho_\gamma}(\Gamma_\gamma)$, where $\Gamma_\gamma = \gamma^{-1}\Gamma\gamma$ and ρ_γ is a representation on Γ_γ defined by

$$\rho_\gamma(\gamma^{-1}A\gamma) := \rho(A)$$

for $A \in \Gamma$. For simplicity, let f_γ denote $f|_{k,\chi}\gamma$. Note that $(f_\gamma)_j = f_{j,\gamma}$ for every $1 \le j \le p$ and $\gamma \in \mathrm{SL}(2,\mathbb{Z})$.

Proof of Theorem 1.1. Let f be a vector-valued modular form contained in $M_{k,\chi,\rho}(\Gamma)$. For each $\gamma \in \Gamma$, the function $f|_{k,\chi}\gamma$ has a Fourier expansion of the form

$$\sum_{j=1}^{p} \sum_{n+\kappa_{j,\gamma} \ge 0} a_{j,\gamma}(n) e^{2\pi i(n+\kappa_{j,\gamma})\tau/\lambda_\gamma} \mathbf{e}_j.$$

Suppose that for such γ, the Fourier coefficients $a_{i,\gamma}(n)$ satisfy the growth condition

$$a_{i,\gamma}(n) = O(n^{k/2+\alpha}), \tag{4.1}$$

as $n \to \infty$. Here, α is a non-negative constant depending only on ρ.

Suppose that f is not cuspidal. By Lemma 2.2, we have an element $\gamma \in \Gamma$ such that $f_{i,\gamma}$ has the non-zero constant term $C_{i,\gamma}$. Then, $L(f_{i,\gamma S^{-1}}, s)$ has a meromorphic continuation on $\mathbb{C}$ with a simple pole at $s = k$ by Proposition 2.3. On the other hand, by the growth condition of the Fourier coefficients in (4.1), the series

$$L(f_{i,\gamma S^{-1}}, s) = \sum_{n+\kappa_{i,\gamma S^{-1}}>0} \frac{a_{i,\gamma S^{-1}}(n)}{\big((n+\kappa_{i,\gamma S^{-1}})/\lambda_{\gamma S^{-1}}\big)^s}$$

converges absolutely if $\mathrm{Re}(s) > k/2 + \alpha + 1$. By the assumption, we have $k > 2\alpha + 2$. This implies that $k > k/2 + \alpha + 1$. Therefore, $L(f_{i,\gamma S^{-1}}, k)$ converges absolutely, and hence, $L(f_{i,\gamma S^{-1}}, s)$ cannot have a pole at $s = k$. This is a contradiction. Hence, f is a vector-valued cusp form. □

Proof of Theorem 1.3. Let $F \in J_{k,m,\chi}(\Gamma^J)$ with the given growth condition on its Fourier coefficients, that is, the function $F|_{k,m,\chi}\gamma$ has a Fourier expansion

$$\sum_{\substack{n,r\in\mathbb{Z}\\ D_\gamma\geq 0}} c_\gamma(n,r) e^{2\pi i(n+\kappa_\gamma)\tau/\lambda_\gamma} e^{2\pi i r z}$$

with the growth condition $c_\gamma(n,r) = O(D_\gamma^{k/2-1/4})$ for every $\gamma \in \mathrm{SL}(2,\mathbb{Z})$, as $D_\gamma \to \infty$. Since $(F|_{k,m,\chi}|\gamma)|_m X = F|_{k,m,\chi}\gamma$ for any $X \in \mathbb{Z}^2$ and for any $\gamma \in \mathrm{SL}(2,\mathbb{Z})$, by (3.1) we have

$$(F|_{k,m,\chi}\gamma)(\tau, z) = \sum_{\mu\in\mathcal{N}} f_{\gamma,\mu}(\tau)\theta_{m,\mu}(\tau, z).$$

As in [6, Section 5], nth Fourier coefficient of $f_{\gamma,\mu}$ is the same with the $c_\gamma\left((n+\mu^2)/(4m), r\right)$.

Moreover, by Theorem 3.3, the vector-valued function

$$f = \sum_{\mu\in\mathcal{N}} f_{I,\mu}\mathbf{e}_\mu$$

is a vector-valued modular form in $M_{k-1/2,\chi',\rho'}(\Gamma)$. Let $\gamma \in \mathrm{SL}(2,\mathbb{Z})$. We write a Fourier expansion of $f|_{k-1/2,\chi'}\gamma$ as

$$\sum_{j=1}^{2m} f_j(\tau)\mathbf{e}_j = \sum_{j=1}^{2m} \sum_{n+\kappa_{j,\gamma}\geq 0} a_{j,\gamma}(n) e^{2\pi i(n+\kappa_{j,\gamma})\tau/\lambda_\gamma}\mathbf{e}_j.$$

Note that

$$\sum_{\mu\in\mathcal{N}} f_{\gamma,\mu}\mathbf{e}_\mu = \rho(\gamma)(f|_{k-1/2,\chi'}\gamma).$$

This gives a relationship between the Fourier coefficients $c_\gamma(n,r)$ and $a_{j,\gamma}(n)$, which determines the asymptotic behavior of $a_{j,\gamma}(n)$

$$a_{j,\gamma}(n) = O\big(n^{\frac{1}{2}\left(k-\frac{1}{2}\right)}\big)$$

as $n \to \infty$. Since ρ' is unitary, we use Remark 1.2(2) for $\alpha = 0$ to see that f is cuspidal. This implies that $F \in S_{k,m,\chi}(\Gamma^J)$ by Theorem 3.3. □

Proof of Theorem 1.4. Suppose that $n \geq 5$. Then, the weight of $\theta_{Q,L}$ is strictly larger than 2. Since $Q(0) = 0$, $\theta_{Q,L}(\tau, 0)$ has a constant term at its Fourier expansion. Therefore, $\theta_{Q,L}$ is not a vector-valued cusp form. By Theorem 1.1, for each $h \in L^*/L$, $R_{Q,L}(m,h)$ does not satisfy

$$R_{Q,L}(m,h) = O(m^{n/4}),$$

as $m \to \infty$. This implies that for any $M > 0$ there are infinitely many $m \in \mathbb{N}$ such that

$$R_{Q,L}(m,h) > Mm^{n/4}. \qquad \square$$

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