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Exact Controllability for Wave Equations with Switching Controls

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Abstract. In this paper, we analyze the exact controllability problem for wave equations endowed with switching controls. The goal is to control the dynamics of the system by switching among different actuators such that, in each instant of time, there are as few active actuators as possible. We prove that the system is exactly controllable under suitable geometric control conditions.

1. Introduction

Let (M, g) be an n-dimensional compact smooth Riemannian manifold with a boundary ∂M. Let $T > 0$. Denote by Δ the Laplace-Beltrami operator on M.

The main purpose of this paper is to study the exact controllability problems of the wave equation with switching controls. Let us consider the following two controlled wave equations

$$\begin{cases} y_{tt} - \Delta y = \sum_{i=1}^{m} \chi_{E_i}\chi_{\omega_i} f_i & \text{in } (0,T) \times M, \\ y = 0 & \text{on } (0,T) \times \partial M, \\ y(0) = y_0, \; y_t(0) = y_1 & \text{in } M \end{cases} \tag{1.1}$$

and

$$\begin{cases} z_{tt} - \Delta z = 0 & \text{in } (0,T) \times M, \\ z = \sum_{i=1}^{m} \chi_{F_i}\chi_{\Gamma_i} h_i & \text{on } (0,T) \times \partial M, \\ z(0) = z_0, \; z_t(0) = z_1 & \text{in } M. \end{cases} \tag{1.2}$$

In (1.1) (resp. (1.2)), $(y_0, y_1) \in H_0^1(M) \times L^2(M)$ (resp. $(z_0, z_1) \in L^2(M) \times H^{-1}(M)$), $m \in \mathbb{N}$, and for $i = 1, \ldots, m$, ω_i is an open subset of M (resp. Γ_i is an open subset of Γ), E_i (resp. F_i) is an open subset of $(0,T)$ such that $E_i \cap E_j = \emptyset$ (resp. $F_i \cap F_j = \emptyset$) for $i \neq j$, $f_i \in L^2(E_i \times \omega_i)$ (resp. $h_i \in L^2(F_i \times \Gamma_i)$).

The exact controllability of (1.1) and (1.2) are formulated respectively as follows.

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Definition 1.1. For any given initial data $(y_0, y_1) \in H_0^1(M) \times L^2(M)$ and $(y_2, y_3) \in H_0^1(M) \times L^2(M)$, one can find controls $\{f_i\}_{i=1}^m$ such that the corresponding solution to (1.1) satisfies that $(y(T), y_t(T)) = (y_2, y_3)$.

Definition 1.2. For any given initial data $(z_0, z_1) \in L^2(M) \times H^{-1}(M)$ and $(z_2, z_3) \in L^2(M) \times H^{-1}(M)$, one can find controls $\{h_i\}_{i=1}^m$ such that the corresponding solution to (1.1) satisfies that $(z(T), z_t(T)) = (z_2, z_3)$.

To guarantee the exact controllability of systems (1.1) and (1.2), we introduce the following two conditions, respectively.

Condition 1.3. *Every optics associated with the symbol of the wave operator issued at* $t = 0$ *intersects the set* $\bigcup_{i=1}^m (E_i \times \omega_i)$.

Condition 1.4. *Every optics associated with the symbol of the wave operator issued at* $t = 0$ *intersects the set* $\bigcup_{i=1}^m (F_i \times \Gamma_i)$ *at a non-diffractive point.*

Some detailed descriptions are given in Section 2 for readers who are not familiar with concepts such as "optics associated with the wave operator" and "non-diffractive point".

We have the following results for the exact controllability of (1.1) and (1.2).

Theorem 1.5. *System* (1.1) *is exactly controllable, provided that Condition* 1.3 *holds.*

Theorem 1.6. *System* (1.2) *is exactly controllable, provided that Condition* 1.4 *holds.*

By the standard Hilbert Uniqueness Method (see [9] for example), in order to prove Theorems 1.5 and 1.6, we only need to establish an internal observability estimate and a boundary observability estimate for the corresponding adjoint systems, respectively. Now, let us formulate the observability problems.

First, we consider a wave equation as follows:

$$\begin{cases} v_{tt} - \Delta v = 0 & \text{in } (0,T) \times M, \\ v = 0 & \text{on } (0,T) \times \partial M, \\ v(0) = v_0, \ v_t(0) = v_1 & \text{in } M, \end{cases} \tag{1.3}$$

where $(v_0, v_1) \in L^2(M) \times H^{-1}(M)$. The exact controllability of system (1.1) is implied by the following observability estimate

$$|v_0|^2_{L^2(M)} + |v_1|^2_{H^{-1}(M)} \leq C \sum_{i=1}^m \int_{E_i} \int_{\omega_i} |v|^2 \, dx dt, \tag{1.4}$$

where C is a constant which is independent of (v_0, v_1).

Next, we introduce the following wave equation:

$$\begin{cases} w_{tt} - \Delta w = 0 & \text{in } (0,T) \times M, \\ w = 0 & \text{on } (0,T) \times \partial M, \\ w(0) = w_0, \ w_t(0) = w_1 & \text{in } M, \end{cases} \tag{1.5}$$

where $(w_0, w_1) \in H_0^1(M) \times L^2(M)$. Once we prove that

$$|w_0|^2_{H_0^1(M)} + |w_1|^2_{L^2(M)} \leq C \sum_{i=1}^{m} \int_{F_i} \int_{\Gamma_i} \left| \frac{\partial w}{\partial \nu} \right|^2 d\Gamma dt, \tag{1.6}$$

where C is a constant independent of (z_0, z_1), we obtain the exact controllability of (1.2) immediately.

We have the following two results.

Theorem 1.7. *Inequality* (1.4) *is true, provided that Condition* 1.3 *holds.*

Theorem 1.8. *Inequality* (1.6) *is true, provided that Condition* 1.4 *holds.*

Remark 1.9. By the classical result of Gaussian beam (see [12]), we know that if Condition 1.3 (resp. Condition 1.4) is untrue, inequality (1.4) (resp. inequality (1.6)) cannot hold. Therefore, system (1.1) (resp. system (1.2)) is not exactly controllable.

Control systems in real applications are often endowed with several actuators. Switching controllers arise in many fields of applications (see [4, 10, 11, 13, 14, 17] for example). There are many reasons for using switching controls, such as to minimize the control cost, to optimize the control time, to decouple disturbances, etc. The main motivation to consider systems (1.1) and (1.2) is that in many control systems governed by wave equations, one actor cannot work for a long time. One should stop it if it works for some time. Otherwise it may be destroyed. Then, one should put at least two actors on the system. When one is stopped, the other works. Controllability problems for wave equations have been studied extensively in the literature (see [1–3, 5, 6, 8, 9, 15, 16] and the rich references therein). As far as we know, there is no result about the controllability problems for wave equations with switching controls. Although the main idea of proofs of Theorems 1.7 and 1.8 are the same as the one in [2,3], we believe that it deserves to provide complete proofs for them.

2. Some preliminaries

In this section, for the convenience of readers, we recall some useful results for the propagation of singularities of the solution to a wave equation involved in a manifold with a nonempty boundary. Although there is nothing new, we present it here for the sake of completeness and the readers' convenience. More details can be found in [7].

For a smooth manifold N, write $\dot{T}^*N$ for the set $T^*N \setminus (N \times \{0\})$. Write Q for the interior of the cylinder $(-\infty,+\infty) \times M$, ∂Q for the set $(-\infty,+\infty) \times \partial M$, and $\overline{Q}$ for the closure of Q. Let O be a neighborhood of Q such that $Q \subset\subset O$. Denote by $\dot{T}^*\overline{Q}$ the restriction of $\dot{T}^*O$ on $\overline{Q}$. Let $\dot{T}_b^*Q = \dot{T}^*Q \cup \dot{T}^*\partial Q$ and write $\dot{T}_{\partial Q}^*$ for the conormal bundle to ∂Q in O. Let π be the canonical projection

$$\pi\colon \dot{T}^*\overline{Q} \setminus \dot{T}_{\partial Q}^* \to \dot{T}_b^*Q.$$

Equip $\dot{T}_b^*Q$ with the topology induced by π. For any $\xi \in T^*M$, denoted by $|\xi|_g$ the norm of ξ with respect to the metric g. Let $p = \tau^2 - |\xi|_g^2$ and

$$\mathrm{Char}(p) = \big\{(t,x,\tau,\xi) : (t,x,\tau,\xi) \in \dot{T}^*\overline{Q}, \tau^2 - |\xi|_g^2 = 0\big\}, \quad \Sigma_b = \pi(\mathrm{Char}(p)).$$

The cotangent bundle to the boundary is the disjoint union of the elliptic set $\mathcal{E}$, the hyperbolic set $\mathcal{H}$ and the glancing set $\mathcal{G}$, which are consisted by points $\rho \in \dot{T}^*\partial Q$ such that p has, respectively, no zero in $\pi^{-1}(\rho)$, two simple zeroes in $\pi^{-1}(\rho)$ and a double zero in $\pi^{-1}(\rho)$.

Let $\rho_0 \in \mathcal{G}$ and $p_0 \in \mathrm{Char}(p)$ such that $\pi(\beta_0) = \rho_0$. Let $\gamma\colon s \to \dot{T}_b^*Q$ be the integral curve of

$$H_p \triangleq \left(\frac{\partial p}{\partial \tau}\frac{\partial}{\partial t}, \frac{\partial p}{\partial \xi^1}\frac{\partial}{\partial x^1}, \dots, \frac{\partial p}{\partial \xi^n}\frac{\partial}{\partial x^n}, -\frac{\partial p}{\partial t}\frac{\partial}{\partial \tau}, -\frac{\partial p}{\partial x^1}\frac{\partial}{\partial \xi^1}, \dots, -\frac{\partial p}{\partial x^n}\frac{\partial}{\partial \xi^n}\right)$$

such that $\gamma(0) = \beta_0$. Then, γ is tangent to ∂Q at β_0. Denote by $\mathcal{G}^k$ $(k \geq 2)$ the set such that the order of the contact of γ with ∂Q is exactly k. Let $\Sigma_b^{2,-}$ be the set such that $\beta(s) \in \dot{T}^*Q$ for $0 < |s| \leq \delta$ with δ small enough, and $\Sigma_b^{2,+}$ the set such that $\beta(s) \notin \dot{T}^*Q$ for $0 < |s| \leq \delta$, where δ is an arbitrary positive number.

Now we recall the definition of a ray associated with p.

Definition 2.1. A ray associated with p is a continuous curve $\gamma\colon I \to \Sigma_b$, where $I \subset \mathbb{R}$ is an open interval, such that the following conditions hold:

(1) If $\gamma(s_0) \in \Sigma_b \cap \dot{T}^*Q$, then γ is differentiable at s_0 and $\gamma'(s_0) = H_p(\gamma(s_0))$.

(2) If $\gamma(s_0) \in (\Sigma_b \cap \dot{T}^*Q) \cup \Sigma_b^{2,-}$, then there is a $\delta > 0$ such that $\gamma(s_0) \in \Sigma_b \cap \dot{T}^*Q$ for $0 < |s - s_0| < \delta$.

(3) If $\gamma(s_0) \in \Sigma_b^{2,+}$, then there is a $\delta > 0$ such that $\gamma(s) \in \Sigma_b^{2,+}$ for $|s - s_0| < \delta$. Further, γ is differentiable at s_0 (as a curve in $\Sigma_b^{2,+}$) and $\gamma'(s_0) = H_q(\gamma(s_0))$, where $q(t,x;\tau,\xi) = |\xi|_b^2 - \tau^2$, where $|\xi|_b$ is the length of $\xi \in T^*\Gamma$ for the metric induced by (M,g) on the boundary Γ.

(4) If $\gamma(s_0) \in \mathcal{G}^3$ and $\{\widetilde{\gamma}^+(s), \widetilde{\gamma}^-(s)\}$ are the (at most) two points in $\mathrm{Char}(p)$ such that $\pi(\widetilde{\gamma}^+(s)) = \pi(\widetilde{\gamma}^-(s)) = \gamma(s)$ and $\widetilde{\gamma}^+(s_0) = \widetilde{\gamma}^-(s_0)$, then

$$\lim_{s\to s_0} \frac{\widetilde{\gamma}^+(s) - \widetilde{\gamma}^+(s_0)}{s - s_0} = H_p(\widetilde{\gamma}^+(s_0)) \quad \text{and} \quad \lim_{s\to s_0} \frac{\widetilde{\gamma}^-(s) - \widetilde{\gamma}^-(s_0)}{s - s_0} = H_p(\widetilde{\gamma}^-(s_0)).$$

Definition 2.2. The projection of a ray γ to Q is called an optics associated with the symbol p.

Let u be an extendible distribution on Q. Let us give the definition of the wavefront set up to the boundary.

Definition 2.3. For any $s \in \mathbb{R}$, if $\rho \notin \mathrm{WF}_b^s(u)$ then

(1) $\rho \notin \mathrm{WF}^s(u)$ for $\rho \in \dot{T}^*Q$;

(2) there exists a tangential pseudodifferential operator A, which is elliptic at ρ, such that $Au \in H^s(\overline{Q})$ for $\rho \in \dot{T}^*\partial Q$.

At last, we recall the definition for a non-diffractive point.

Definition 2.4. A point $\rho \in \dot{T}^*\partial Q$ is non-diffractive if $\rho \in \mathcal{E} \cup \mathcal{H}$, or if $\rho \in \mathcal{G}$ and $\beta \in \mathrm{Char}(p)$ is the unique point such that $\pi(\beta) = \rho$, the ray γ through β with $\gamma(0) = \beta$ satisfies that for any $\varepsilon > 0$, there exists an $s \in (-\varepsilon, \varepsilon)$ such that $\gamma(s) \notin \dot{T}^*\overline{Q}$.

With the above notations, we give the following results. Proofs of them can be found in [2].

Lemma 2.5. *Let u be an extendible distribution in Q such that $u_{tt} - \Delta u = 0$ in Q and $u = 0$ on ∂Q. Let γ be a ray. If $\rho \in \gamma \subset \Sigma_b$ satisfies that $\rho \notin \mathrm{WF}^s(u)$ $(s = 0, 1)$, then $\gamma \cap \mathrm{WF}^s(u) = \emptyset$ $(s = 0, 1)$.*

Lemma 2.6. *Let u be an extendible distribution in Q such that $u_{tt} - \Delta u = 0$ and ρ a non-diffractive point. If $\rho \notin \mathrm{WF}^1(u|_{\partial Q}) \cup \mathrm{WF}^0(\frac{\partial u}{\partial \nu}|_{\partial Q})$, then $\rho \notin \mathrm{WF}_b^1(u)$.*

3. Proofs of Theorems 1.7 and 1.8

This section is devoted to the proofs of Theorems 1.7 and 1.8.

Proof of Theorem 1.7. Without loss of generality, we assume that $m = 2$. To begin with, let us define the following spaces:

$$\mathcal{X} \triangleq \big\{v : v \text{ solves equation (1.3) with some } (v_0, v_1) \in L^2(M) \times H^{-1}(M)\big\},$$

endowed with the norm

$$|v|_{\mathcal{X}} = \Big(|v_0|^2_{L^2(M)} + |v_1|^2_{H^{-1}(M)}\Big)^{1/2},$$

and

$$\mathcal{Y} \triangleq \big\{v \in H^{-1}((0,T)\times M) : v_{tt} - \Delta v = 0, v = 0 \text{ on } (0,T)\times\partial M,$$
$$v \in L^2((E_1\times\omega_1)\cup(E_2\times\omega_2))\big\},$$

endowed with the norm

$$|v|_{\mathcal{Y}} \triangleq \Big(\int_{E_1}\int_{\omega_1}|v|^2\,dxdt + \int_{E_2}\int_{\omega_2}|v|^2\,dxdt + |v|^2_{H^{-1}((0,T)\times M)}\Big)^{1/2}.$$

It is a simple matter to see that $\mathcal{X}$ is embedded into $\mathcal{Y}$ continuously. Further, by Lemma 2.5 and noting that $(E_1\times\omega_1)\cup(E_2\times\omega_2)$ satisfies Condition 1.3, we know that $\mathcal{Y}$ is also embedded into $\mathcal{X}$. Therefore, we find that $\mathcal{X} = \mathcal{Y}$.

Next, we show that $\mathcal{Y}$ is a Banach space. Indeed, let $\{v_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathcal{Y}$. From the definition of the norm of $\mathcal{Y}$, we see $\{v_n\}_{n=1}^{\infty}$ is also a Cauchy sequence in $H^{-1}((0,T)\times M)$. Then, there is a $v\in H^{-1}((0,T)\times M)$ such that

$$\lim_{n\to\infty} v_n = v \quad \text{in } H^{-1}((0,T)\times M). \tag{3.1}$$

This shows that $v_{tt} - \Delta v = 0$ in $D'((0,T)\times M)$.

Further, by the definition of the norm in $\mathcal{Y}$ again, we know that $\{\chi_{E_1\times\omega_1}v_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^2(E_1\times\omega_1)$. Hence, there is a $\widetilde{v}\in L^2(E_1\times\omega_1)$ such that

$$\lim_{n\to\infty}\chi_{E_1\times\omega_1}v_n = \widetilde{v} \quad \text{in } L^2(E_1\times\omega_1).$$

On the other hand, from (3.1), we know that for any $\varphi\in H_0^1(E_1\times\omega_1)\subset H_0^1((0,T)\times M)$, it holds that

$$\begin{aligned}
(\widetilde{v},\varphi)_{L^2(E_1\times\omega_1)} &= \lim_{n\to\infty}(\chi_{E_1\times\omega_1}v_n,\varphi)_{L^2(E_1\times\omega_1)}\\
&= \lim_{n\to\infty}(\chi_{E_1\times\omega_1}v_n,\varphi)_{H^{-1}(E_1\times\omega_1),H_0^1(E_1\times\omega_1)}\\
&= \lim_{n\to\infty}(v_n,\varphi)_{H^{-1}((0,T)\times M),H_0^1((0,T)\times M)}\\
&= (v,\varphi)_{H^{-1}((0,T)\times M),H_0^1((0,T)\times M)},
\end{aligned}$$

which gives

$$\widetilde{v} = v \text{in } H^{-1}(E_1\times\omega_1).$$

Thus, we see that

$$v|_{E_1\times\omega_1}\in L^2(E_1\times\omega_1).$$

Similarly, we get

$$v|_{E_2\times\omega_2}\in L^2(E_2\times\omega_2).$$

Therefore, we obtain that $v\in\mathcal{X}$, which completes the proof of our claim.

Now, by means of the closed graph theorem, we know the identity map between $\mathcal{X}$ and $\mathcal{Y}$ is bicontinuous. Hence, we know that there is a constant $C>0$ such that for every $(v_0,v_1)\in L^2(M)\times H^{-1}(M)$,

$$|v|_{\mathcal{X}}\le C|v|_{\mathcal{Y}},$$

which implies that

$$\begin{aligned}&|v_0|^2_{L^2(M)}+|v_1|^2_{H^{-1}(M)}\\ &\le C\left(\int_{E_1}\int_{\omega_1}|v|^2\,dxdt+\int_{E_2}\int_{\omega_2}|v|^2\,dxdt+|v|^2_{H^{-1}((0,T)\times M)}\right).\end{aligned}\tag{3.2}$$

It remains to get rid of the second terms on the right-hand side of (3.2). For this, we only need to show that

$$|v|^2_{H^{-1}((0,T)\times M)}\le C\left(\int_{E_1}\int_{\omega_1}|v|^2\,dxdt+\int_{E_2}\int_{\omega_2}|v|^2\,dxdt\right)$$

for a constant C which does not depend on $v\in\mathcal{X}$. We complete this task by a contradiction argument. Assume that there is a sequence $\big\{(v_0^{(n)},v_1^{(n)})\big\}_{n=1}^\infty\subset L^2(M)\times H^{-1}(M)$ such that

$$\big|(v_0^{(n)},v_1^{(n)})\big|_{L^2(M)\times H^{-1}(M)}=1\quad\text{for all } n\in\mathbb{N}$$

and that

$$\int_{E_1}\int_{\omega_1}\big|v^{(n)}\big|^2\,dxdt+\int_{E_2}\int_{\omega_2}\big|v^{(n)}\big|^2\,dxdt\le\frac{1}{n}\big|v^{(n)}\big|^2_{H^{-1}((0,T)\times M)}.$$

Since $|v^{(n)}|_{\mathcal{X}}=1$, we know that $\{v^{(n)}\}_{n=1}^\infty$ is bounded in $L^2((0,T)\times M)$. Then, there exist a $v\in L^2((0,T)\times M)$ and a subsequence $\{v^{(n_k)}\}_{k=1}^\infty\subset\{v^{(n)}\}_{n=1}^\infty$ such that

$$v^{(n_k)}\text{ converges to } v \text{ weakly in } L^2((0,T)\times M)\text{ as } k\to\infty.$$

It is clear that v satisfies that

$$\begin{cases}v_{tt}-\Delta v=0 & \text{in } D'((0,T)\times M),\\ v=0 & \text{on } (0,T)\times\Gamma,\\ v=0 & \text{in } (E_1\times\omega_1)\cup(E_2\times\omega_2).\end{cases}$$

Let us define the following space:

$$\mathcal{N}\overset{\triangle}{=}\big\{v\in\mathcal{X}: v_{tt}-\Delta v=0, v=0\text{ in }(E_1\times\omega_1)\cup(E_2\times\omega_2)\big\}.$$

The task now is to prove that $\mathcal{N} = \{0\}$. Since

$$v = 0 \quad \text{in } (E_1 \times \omega_1) \cup (E_2 \times \omega_2),$$

we have

$$v \in H^1((E_1 \times \omega_1) \cup (E_2 \times \omega_2)).$$

From Lemma 2.5 and the assumption that $(E_1 \times \omega_1) \cup (E_2 \times \omega_2)$ satisfies Condition 1.3, we see

$$\mathcal{N} \subset H^1((0,T) \times M).$$

Then, by the Sobolev embedding theorem, we know any bounded subset of $\mathcal{N}$ is compact in $L^2((0,T) \times M)$. This implies that $\mathcal{N}$ is a finite dimensional subspace of $L^2((0,T) \times M)$. Further, since $\overline{v} = v_t$ satisfies

$$\begin{cases} \overline{v}_{tt} - \Delta \overline{v} = 0 & \text{in } D'((0,T) \times M), \\ \overline{v} = 0 & \text{on } (0,T) \times \Gamma, \\ \overline{v} = 0 & \text{in } (E_1 \times \omega_1) \cup (E_2 \times \omega_2), \end{cases}$$

we find that $\partial_t v \in \mathcal{N}$. Therefore, if $\mathcal{N} \neq \{0\}$, we know the restriction of ∂_t on $\mathcal{N}$ must have an eigenvalue λ and an eigenfunction $\xi \neq 0$ in $(0,T) \times M$. Then, we get

$$\begin{cases} \partial_t \xi = \lambda \xi & \text{in } (0,T) \times M, \\ \xi(0) = \eta & \text{in } M, \end{cases}$$

where $\eta \in H^1(M)$. Hence, we see

$$\xi = e^{\lambda t} \eta \quad \text{in } (0,T) \times M.$$

Moreover, from $\xi = 0$ in $(E_1 \times \omega_1) \cup (E_2 \times \omega_2)$, we get $\eta = 0$ in $\omega_1 \cup \omega_2$. On the other hand, thanks to $\xi \in \mathcal{N}$, we know that η solves

$$\begin{cases} (-\Delta + \lambda^2)\eta = 0 & \text{in } M, \\ \eta = 0 & \text{on } \Gamma, \\ \eta = 0 & \text{in } \omega_1 \cup \omega_2. \end{cases}$$

From the classical unique continuation property for solutions of elliptic equations, we get that $\eta = 0$ in M. This contradicts our assumptions that $\mathcal{N} \neq \{0\}$.

By means of $\mathcal{N} = \{0\}$, we know

$$v^{(n_k)} \text{ converges to } 0 \text{ weakly in } L^2((0,T) \times M) \text{ as } k \to \infty.$$

Thus, we find

$$v^{(n_k)} \text{ converges to 0 strongly in } H^{-1}((0,T)\times M) \text{ as } k\to\infty.$$

Therefore, we obtain

$$\lim_{k\to\infty}\left(\int_{E_1}\int_{\omega_1}\left|v^{(n_k)}\right|^2 dxdt+\int_{E_2}\int_{\omega_2}\left|v^{(n_k)}\right|^2 dxdt+\left|v^{(n_k)}\right|^2_{H^{-1}((0,T)\times M)}\right)=0,$$

which contradicts the fact that $\left|(v_0^{(n_k)},v_0^{(n_k)})\right|=1$ and the inequality (3.2). This completes the proof of Theorem 1.7. □

Proof of Theorem 1.8. Let

$$\mathcal{Z}\triangleq\left\{w: w \text{ solves equation (1.5) with some } (w_0,w_1)\in H_0^1(M)\times L^2(M)\right\},$$

endowed with the norm

$$|w|_{\mathcal{Z}}=\left(|w_0|^2_{H_0^1(M)}+|w_1|^2_{L^2(M)}\right)^{1/2},$$

and

$$\mathcal{W}\triangleq\Big\{w\in L^2((0,T)\times M): w_{tt}-\Delta w=0, w|_{(0,T)\times\partial M}=0,$$
$$\frac{\partial w}{\partial\nu}\Big|_{(F_1\times\Gamma_1)\cup(F_2\times\Gamma_2)}\in L^2((F_1\times\Gamma_1)\cup(F_2\times\Gamma_2))\Big\}$$

endowed with the norm

$$|w|_{\mathcal{W}}\triangleq\left(\int_{E_1}\int_{\Gamma_1}\left|\frac{\partial w}{\partial\nu}\right|^2 d\Gamma dt+\int_{E_2}\int_{\Gamma_2}\left|\frac{\partial w}{\partial\nu}\right|^2 d\Gamma dt+|w|^2_{L^2((0,T)\times M)}\right)^{1/2}.$$

It is clear that $\mathcal{Z}$ is a Banach space. Further, by an argument similar to proving that $\mathcal{Y}$ is a Banach space, we can easily get that $\mathcal{W}$ is a Banach space.

It is clear that $\mathcal{Z}$ can be embedded into $\mathcal{W}$ continuously. On the other hand, for any $w\in\mathcal{W}$, we claim that $w\in\mathcal{Z}$. For showing this, we need to prove that for any $w\in\mathcal{W}$ and $\rho\in T_b^*(Q)\cap\{t=0\}$, it holds that $\rho\notin \mathrm{WF}^1(w)$. If $\rho\notin\Sigma_b$, noting that $w_{tt}-\Delta w=0$ and $w|_{\partial Q}=0$, one has $\rho\notin\mathrm{WF}^1(w)$. If $\rho\in\Sigma_b$, let $\gamma(s,\rho)$ be the ray through ρ. By Condition 1.4, there exists a non-diffractive point $\rho_0=\gamma(s_0,\rho)$ such that $\rho_0\in T^*\partial Q|_{(F_1\times\Gamma_1)\cup(F_2\times\Gamma_2)}$. Since $w|_{\partial Q}=0$ and

$$\frac{\partial w}{\partial\nu}\Big|_{(F_1\times\Gamma_1)\cup(F_2\times\Gamma_2)}\in L^2((F_1\times\Gamma_1)\cup(F_2\times\Gamma_2)),$$

we get

$$\rho_0\notin\mathrm{WF}_b^1(w|_{\partial Q})\cup\mathrm{WF}_b^0\left(\frac{\partial w}{\partial\nu}\Big|_{\partial Q}\right).$$

From Lemma 2.6, we obtain that $\rho_0 \notin \mathrm{WF}_b^1(w)$. Then, from Lemma 2.5, we get $\rho \notin \mathrm{WF}_b^1(w)$. Hence, the claim is proved. Then, by the closed graph theorem, we know there is a constant $C > 0$ such that for any $w \in \mathcal{Z}$,

$$|w|_{\mathcal{Z}} \leq C|w|_{\mathcal{W}}. \tag{3.3}$$

Now we only need to prove that there is a constant $C > 0$ such that for any $w \in \mathcal{Z}$,

$$|w|^2_{L^2((0,T)\times M)} \leq C\left(\int_{F_1}\int_{\Gamma_1}\left|\frac{\partial w}{\partial \nu}\right|^2 d\Gamma dt + \int_{F_2}\int_{\Gamma_2}\left|\frac{\partial w}{\partial \nu}\right|^2 d\Gamma dt\right). \tag{3.4}$$

We achieve this goal by a contradiction argument. If (3.4) is untrue, then we can find a sequence $\left\{(w_0^{(n)}, w_1^{(n)})\right\}_{n=1}^{\infty} \subset H_0^1(M) \times L^2(M)$ such that the corresponding solutions $\{w^{(n)}\}_{n=1}^{\infty} \subset \mathcal{Z}$ satisfying

$$\left|w^{(n)}\right|_{L^2((0,T)\times M)} = 1 \quad \text{for } n = 1, 2, \ldots \tag{3.5}$$

and

$$\int_{F_1}\int_{\Gamma_1}\left|\frac{\partial w^{(n)}}{\partial \nu}\right|^2 d\Gamma dt + \int_{F_2}\int_{\Gamma_2}\left|\frac{\partial w^{(n)}}{\partial \nu}\right|^2 d\Gamma dt \leq \frac{1}{n}. \tag{3.6}$$

From (3.3), (3.5) and (3.6), we have that $\{w^{(n)}\}_{n=1}^{\infty}$ is a bounded subset of $H^1((0,T)\times M)$. Therefore, there is a subsequence $\left\{w^{(n_k)}\right\}_{k=1}^{\infty} \subset \left\{w^{(n)}\right\}_{n=1}^{\infty}$ and a $w \in H^1((0,T) \times M)$ such that

$$w^{(n_k)} \to w \quad \text{weakly in } H^1((0,T) \times M) \text{ as } k \to \infty$$

and

$$\frac{\partial w}{\partial \nu}\Big|_{(F_1\times\Gamma_1)\cup(F_2\times\Gamma_2)} = 0.$$

Thus, we have

$$w^{(n_k)} \to w \quad \text{strongly in } L^2((0,T) \times M) \text{ as } k \to \infty.$$

This, together with (3.5), implies that

$$|w|_{L^2((0,T)\times M)} = 1.$$

Let

$$\mathcal{O} \triangleq \left\{w \in \mathcal{W} : \frac{\partial w}{\partial \nu}\Big|_{(F_1\times\Gamma_1)\cup(F_2\times\Gamma_2)} = 0\right\}.$$

Now we only need to prove that $\mathcal{O} = \{0\}$. For any $w \in \mathcal{O}$, by (3.3), we know $w \in H^1((0,T) \times M)$. Hence, we find

$$\begin{cases} v = w_t \in L^2((0,T) \times M), \\ v_{tt} - \Delta v = 0 & \text{in } (0,T) \times M, \\ \frac{\partial v}{\partial \nu}\Big|_{(F_1\times\Gamma_1)\cup(F_2\times\Gamma_2)} = 0, \\ v|_{(0,T)\times\partial M} = 0. \end{cases}$$

Therefore, we get $v \in \mathcal{O}$, which means that $\partial_t \mathcal{O} \subset \mathcal{O}$. Utilizing (3.3) again, we find the two norms $|\cdot|_{H^1((0,T)\times M)}$ and $|\cdot|_{L^2((0,T)\times M)}$ are equivalent in $\mathcal{O}$. Thus, $\mathcal{O}$ is finite dimensional. Let λ be an eigenvalue of ∂_t on $\mathcal{O}$ and ζ the corresponding eigenfunction. Then we see

$$\begin{cases} \partial_t \zeta = \lambda \zeta & \text{in } (0,T) \times M, \\ \zeta(0) = \varsigma & \text{in } (0,T) \times M, \end{cases}$$

where $\varsigma \in H_0^1(M)$. Since ζ is a solution to (1.3), we know that ς solves

$$\begin{cases} (\lambda^2 - \Delta)\varsigma = 0 & \text{in } M, \\ \varsigma = 0 & \text{on } \Gamma. \end{cases}$$

Since $\frac{\partial \zeta}{\partial \nu}\big|_{(F_1\times\Gamma_1)\cup(F_2\times\Gamma_2)} = 0$, we have that $\frac{\partial \varsigma}{\partial \nu} = 0$ on $\Gamma_1 \times \Gamma_2$. Then, from the unique continuation property for elliptic equations, we conclude that $\varsigma = 0$, which implies that $\zeta = 0$. Hence, we prove that $\mathcal{O} = \{0\}$. $\square$

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