

An Existence Result for Discrete Anisotropic Equations

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Abstract. A critical point result is exploited in order to prove that a class of discrete anisotropic boundary value problems possesses at least one solution under an asymptotical behaviour of the potential of the nonlinear term at zero. Some recent results are extended and improved. Some examples are presented to demonstrate the applications of our main results.

1. Introduction

In this note we consider an anisotropic difference equation with Dirichlet type boundary condition of the form

$$(1.1) \quad \begin{cases} -\Delta(\alpha(k)|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)) = f(k, u(k)) & k \in [1, T], \\ u(0) = u(T+1) = 0, \end{cases}$$

where $T \geq 2$ is a fixed positive integer number, $[1, T]$ is the discrete interval $\{1, \dots, T\} \subset \mathbb{N}$, $u(k) \in \mathbb{R}$ for all $k \in [1, T]$, $\Delta u(k-1) = u(k) - u(k-1)$ is the forward difference operator, $\alpha: [1, T+1] \rightarrow (0, +\infty)$, and $p: [0, T] \rightarrow (1, +\infty)$ are some fixed functions; $f: [1, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous function. Let

$$p^- = \min_{k \in [0, T]} p(k), \quad p^+ = \max_{k \in [0, T]} p(k)$$

and

$$\alpha^- = \min_{k \in [1, T+1]} \alpha(k), \quad \alpha^+ = \max_{k \in [1, T+1]} \alpha(k).$$

In recent years, a great deal of work has been done in the study of the existence of solutions for discrete boundary value problems (BVPs), by which a number of physical, computer science, mechanical engineering, control systems, artificial or biological neural networks, phenomena are described. Recently, there is a trend to study difference equations by using fixed point theory, lower and upper solutions method, variational methods and

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critical point theory, Morse theory and the mountain-pass theorem, and many interesting results have been obtained, see for instance, [1, 3–8, 10, 11, 13–20, 22, 23, 25–27] and the references therein. For example, Henderson and Thompson in [14] gave conditions on the nonlinear term involving pairs of discrete lower and discrete upper solutions which led to the existence of at least three solutions of discrete two-point boundary value problems, and in a special case of the nonlinear term, they gave growth conditions on the function and applied their general result to show the existence of three positive solutions. Concerning the applications of critical point theory and variational methods, Tian et al. in [26], investigated the existence of solutions of second-order discrete Sturm-Liouville boundary value problem (BVPs) with a p -Laplacian. Mihăilescu et al. in [19], obtained the existence of a continuous spectrum for a family of discrete boundary value problems. Bonanno and Candito [3], investigated the multiplicity of solutions for nonlinear difference equations involving the p -Laplacian. Galewski and Głąb in [7], studied a parametric version of the problem (1.1), in the case $\alpha \equiv 1$. First, they applied the direct method of the calculus of variations and the mountain pass theorem in order to reach the existence of at least one nontrivial solution. Secondly, they derived some version of a discrete three critical point theorem which they applied in order to get the existence of at least two nontrivial solutions. Galewski and Wieteska in [10], studied the existence of solutions of a system of anisotropic discrete boundary value problems using critical point theory, while in [?], they derived the intervals of the numerical parameter for which the parametric version of the problem (1.1) has at least 1, exactly 1, or at least 2 positive solutions. They also derived some useful discrete inequalities. In [20], the existence of infinitely many solutions for perturbed nonlinear difference equations with discrete Dirichlet boundary conditions, was discussed. Steglański in [25], obtained the existence of infinitely many solutions for a parametric version the problem (1.1). In [8], a parametric version of the problem (1.1) was studied and a new multiplicity results have been established combining, first, the Bonanno local minimum theorem with the mountain pass theorem, next, a two local minimum theorem of Bonanno again with the well known three critical point theorem of Pucci and Serrin, see also [4], for a complete overview on these topics. In [13] the existence of at least three distinct solutions for a perturbed anisotropic discrete Dirichlet problem was studied.

Motivated by this large interest on the subject in the current literature, in the present paper we are looking for the existence of at least one solution for the problem (1.1) and its parametric version by employing Ricceri's variational principle [24, Theorem 2.5]. Precisely, in Theorem 3.1 we obtain the existence of at least one solution for the problem (1.1) requiring an algebraic condition on the nonlinear term f . We present Example 3.2 in which the hypotheses of Theorem 3.1 are fulfilled. Also in Theorem 3.3 a parametric version of

the result of Theorem 3.1 is successively discussed in which, for small values of the parameter and requiring an additional asymptotical behaviour of the potential at zero if $f(k, 0) = 0$ for all $k \in [1, T]$, the existence of one non-trivial solution is established; see Remark 3.7. Moreover, we deduce the existence of solutions for small positive values of the parameter λ such that the corresponding solutions have smaller and smaller energies as the parameter goes to zero; see Remark 3.8. Finally, some consequences of the main results and examples are shown.

The paper is organized as follows. In Section 2, we recall some basic definitions and our main tool, while Section 3 is devoted to main results.

2. Preliminaries

In the present paper E denotes a finite dimensional real Banach space and $I_\lambda : E \rightarrow \mathbb{R}$ is a functional satisfying the following structure hypothesis:

$$I_\lambda(u) := \Phi(u) - \lambda\Psi(u) \quad \text{for all } u \in E,$$

where λ is a positive real parameter and $\Phi, \Psi : E \rightarrow \mathbb{R}$ are two functions of class C^1 on E with Φ coercive, i.e., $\lim_{\|u\| \rightarrow +\infty} \Phi(u) = +\infty$.

In this framework a finite dimensional variant of celebrated Ricceri’s variational principle [24, Theorem 2.1] is the following, see also [4, Theorems 3.3 and 3.4], where a new proof is given in the finite dimensional setting, whenever $\inf \Phi = \Phi(0) = \Psi(0) = 0$.

Theorem 2.1. *For every $r > \inf_E \Phi$, let φ be the function defined as*

$$\varphi(r) := \inf_{u \in \Phi^{-1}(-\infty, r)} \frac{\sup_{v \in \Phi^{-1}(-\infty, r)} \Psi(v) - \Psi(u)}{r - \Phi(u)}.$$

Then, for every $r > \inf_E \Phi$ and every $\lambda \in (0, 1/\varphi(r))$, the restriction of the functional I_λ to $\Phi^{-1}(-\infty, r)$ admits a global minimum, which is a critical point (precisely a local minimum) of I_λ in E .

We refer the interested reader to the papers [9, 12, 21] in which Theorem 2.1 has been successfully get the existence of at least one non-trivial solution for boundary value problems.

Here $\|\cdot\|$ stays for the norm

$$\|u\| := \left(\sum_{k=1}^{T+1} |\Delta u(k-1)|^2 \right)^{1/2}$$

of the T -dimensional Banach space

$$E := \{u : [0, T + 1] \rightarrow \mathbb{R} : u(0) = u(T + 1) = 0\}.$$

Put

$$F(k, t) := \int_0^t f(k, \xi) d\xi \quad \text{for all } (k, t) \in [1, T] \times \mathbb{R}.$$

Lemma 2.2. (a) For $p^- \geq 2$ and $\|u\| \leq 1$, there exists positive constant C_{p^-} such that

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \leq C_{p^-} \|u\|^{p^-}$$

for $u \in E$.

(b) For $p^+ \geq 2$ and $\|u\| \geq 1$, there exists positive constant C_{p^+} such that

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \leq C_{p^+} \|u\|^{p^+}$$

for $u \in E$.

Proof. By considering [10, Lemma 5(a)(b)(c)], we have

$$\begin{aligned} \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} &\leq \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p^-} = \sum_{k=1}^{T+1} |u(k) - u(k-1)|^{p^-} \\ &\leq \sum_{k=1}^{T+1} 2^{p^- - 1} \left(|u(k)|^{p^-} + |u(k-1)|^{p^-} \right) \\ &= C_1 \sum_{k=1}^T |u(k)|^{p^-} \leq C_1(T+1)C_2 \|u\|^{p^-}, \end{aligned}$$

where C_1 and C_2 are two positive constants. Choosing $C_{p^-} = C_1(T+1)C_2$, (a) follows.

Now using the same lemma, we obtain

$$\begin{aligned} \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} &\leq \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p^+} = \sum_{k=1}^{T+1} |u(k) - u(k-1)|^{p^+} \\ &\leq 2^{p^+ - 1} \left(\sum_{k=1}^{T+1} |u(k)|^{p^+} + |u(k-1)|^{p^+} \right) \\ &= C_3 \sum_{k=1}^T |u(k)|^{p^+} \leq C_3(T+1)C_4 \|u\|^{p^+}, \end{aligned}$$

where C_3 and C_4 are two positive constants, and by choosing $C_{p^+} = C_3(T+1)C_4$, (b) follows. \square

3. Main results

We formulate our main result as follows:

Theorem 3.1. *Assume that*

$$(3.1) \quad \sup_{\gamma > 0} \frac{\gamma^{p^-}}{\sum_{k=1}^T \max_{|t| \leq \gamma} F(k, t)} > \frac{T^{p^-} p^+}{\alpha^-}.$$

Then, the problem (1.1) admits at least one solution (maybe trivial) in E .

Proof. Our goal is to apply Theorem 2.1 to the problem (1.1). We consider the functionals Φ, Ψ for every $u \in E$, defined by

$$(3.2) \quad \Phi(u) = \sum_{k=1}^{T+1} \frac{\alpha(k)}{p(k-1)} |\Delta u(k-1)|^{p(k-1)}$$

and

$$(3.3) \quad \Psi(u) = \sum_{k=1}^T F(k, u(k)),$$

and we denote $I(u) = \Phi(u) - \Psi(u)$ for every $u \in E$. We observe that the functionals Φ and Ψ satisfy the required conditions in Theorem 2.1. It is well known that Ψ is a differentiable functional whose differential at the point $u \in E$ is

$$\Psi'(u)(v) = \sum_{k=1}^T f(k, u(k))v(k)$$

for every $v \in E$, as well as is sequentially weakly upper semicontinuous. Also Φ is coercive (see [23, Lemma 2.1] and [25, Lemma 2.2]). Moreover, Φ is continuously differentiable whose differential at the point $u \in E$ is

$$\Phi'(u)(v) = \sum_{k=1}^{T+1} \alpha(k) |\Delta u(k-1)|^{p(k-1)-2} \Delta u(k-1) \Delta v(k-1)$$

for every $v \in E$. Furthermore, Φ is sequentially weakly lower semicontinuous. Therefore, we observe that the regularity assumptions on Φ and Ψ , as requested in Theorem 2.1 are verified. Taking into account that

$$\sum_{k=1}^{T+1} \Delta u(k-1) \Delta v(k-1) = - \sum_{k=1}^T \Delta(\Delta u(k-1))v(k) \quad \text{for all } u, v \in E,$$

we verify that a vector $u \in E$ is a solution of the problem (1.1) if and only if u is a critical point of the function I . We now look on the existence of a critical point of the functional I in E . From the condition (3.1), there exists $\bar{\gamma} > 0$ such that

$$(3.4) \quad \frac{\bar{\gamma}^{p^-}}{\sum_{k=1}^T \max_{|t| \leq \bar{\gamma}} F(k, t)} > \frac{T^{p^-} p^+}{\alpha^-}.$$

Choose

$$r = \frac{\alpha^-}{T^{p^-} p^+} \bar{\gamma}^{p^-}.$$

Moreover, for every $u \in E$, from the estimate $\Phi(u) < r$ implies that

$$\sum_{k=1}^{T+1} \frac{\alpha(k)}{p(k-1)} |\Delta u(k-1)|^{p(k-1)} < r$$

for all $k \in [1, T]$. Then

$$|\Delta u(k-1)| < \left(\frac{p(k-1)r}{\alpha(k)} \right)^{1/[p(k-1)]} \leq \left(\frac{p^+ r}{\alpha^-} \right)^{1/p^-}$$

for all $k \in [1, T]$. From this and since $u \in E$ we deduce by easy induction

$$\begin{aligned} |u(k)| &\leq |\Delta u(k-1)| + |u(k-1)| < \left(\frac{p^+ r}{\alpha^-} \right)^{1/p^-} + |u(k-1)| \\ &\leq k \left(\frac{p^+ r}{\alpha^-} \right)^{1/p^-} \leq T \left(\frac{p^+ r}{\alpha^-} \right)^{1/p^-} = \bar{\gamma} \end{aligned}$$

for all $k \in [1, T]$. From the definition of r , it follows that

$$\Phi^{-1}(-\infty, r) = \{u \in E : \Phi(u) < r\} \subseteq \{u \in E : |u| \leq \bar{\gamma}\}.$$

Therefore, we have that

$$\Psi(u) = \sum_{k=1}^T F(k, u(k)) \leq \sum_{k=1}^T \max_{|t| \leq \bar{\gamma}} F(k, t)$$

for every $u \in E$ such that $\Phi(u) < r$. Then

$$\sup_{\Phi(u) < r} \Psi(u) \leq \sum_{k=1}^T \max_{|t| \leq \bar{\gamma}} F(k, t).$$

By a simple computation and from the definition of $\varphi(r)$, since $0 \in \Phi^{-1}(-\infty, r)$ and $\Phi(0) = \Psi(0) = 0$, one has

$$\begin{aligned} \varphi(r) &= \inf_{u \in \Phi^{-1}(-\infty, r)} \frac{\left(\sup_{v \in \Phi^{-1}(-\infty, r)} \Psi(v) \right) - \Psi(u)}{r - \Phi(u)} \leq \frac{\sup_{v \in \Phi^{-1}(-\infty, r)} \Psi(v)}{r} \\ &\leq \frac{T^{p^-} p^+}{\alpha^-} \frac{\sum_{k=1}^T \max_{|t| \leq \bar{\gamma}} F(k, t)}{\bar{\gamma}^{p^-}}. \end{aligned}$$

At this point, observe that

$$(3.5) \quad \varphi(r) \leq \frac{T^{p^-} p^+ \sum_{k=1}^T \max_{|t| \leq \gamma} F(k, t)}{\alpha^- \gamma^{p^-}}.$$

Consequently, by (3.4) and (3.5) one has $\varphi(r) < 1$. Hence, since $1 \in (0, 1/\varphi(r))$, applying Theorem 2.1 the functional I admits at least one critical point (local minima) $\tilde{u} \in \Phi^{-1}(-\infty, r)$. The proof is complete. \square

In the following, we present an example in which the hypotheses of Theorem 3.1 are satisfied.

Example 3.2. Consider the problem

$$(3.6) \quad \begin{cases} -\Delta(\alpha(k)|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)) = f(u) & k \in [1, 3], \\ u(0) = u(4) = 0, \end{cases}$$

where $\alpha(k) = 2 + \cos(\pi(k-1))$ for $k = 1, 2, 3, 4$, $p(k) = 5 + \ln(k+1)$ for $k = 0, 1, 2, 3$ and

$$f(t) = \frac{1}{10^6}(5t^4 + 3e^t)$$

for every $t \in \mathbb{R}$. By the expression of f we have

$$F(t) = \frac{1}{10^6}(t^5 + 3e^t - 3)$$

for every $t \in \mathbb{R}$. By direct calculations, we obtain $p^- = 5$, $p^+ = 5 + \ln(4)$ and $\alpha^- = 1$. Since

$$\sup_{\gamma > 0} \frac{\gamma^5}{\sum_{k=1}^3 \max_{|t| \leq \gamma} F(t)} > 3^5(5 + \ln(4)) = \frac{T^{p^-} p^+}{\alpha^-},$$

we observe that all assumptions of Theorem 3.1 are fulfilled. Hence, Theorem 3.1 implies that the problem (3.6), admits at least one solution in $\{u: [0, 4] \rightarrow \mathbb{R} : u(0) = u(4) = 0\}$.

We note that Theorem 3.1 can be exploited establishing the existence of at least one solution for the following parametric version of the problem (1.1),

$$(3.7) \quad \begin{cases} -\Delta(\alpha(k)|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)) = \lambda f(k, u(k)) & k \in [1, T], \\ u(0) = u(T+1) = 0, \end{cases}$$

where λ is a positive parameter. More precisely, we have the following existence result.

Theorem 3.3. *For every λ small enough, i.e.,*

$$\lambda \in \left(0, \frac{\alpha^-}{T^{p^-} p^+} \sup_{\gamma > 0} \frac{\gamma^{p^-}}{\sum_{k=1}^T \max_{|t| \leq \gamma} F(k, t)} \right),$$

the problem (3.7) admits at least one solution $u_\lambda \in E$ (maybe trivial).

Proof. Fix λ as in the conclusion. Choose Φ and Ψ as given in the proof of Theorem 3.1, and put $I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$ for every $u \in E$. Let us pick

$$0 < \lambda < \frac{\alpha^-}{T^{p^-} p^+} \sup_{\gamma > 0} \frac{\gamma^{p^-}}{\sum_{k=1}^T \max_{|t| \leq \gamma} F(k, t)}.$$

Hence, there exists $\bar{\gamma} > 0$ such that

$$\lambda \frac{T^{p^-} p^+}{\alpha^-} < \frac{\bar{\gamma}^{p^-}}{\sum_{k=1}^T \max_{|t| \leq \bar{\gamma}} F(k, t)}.$$

Choose $r = \frac{\alpha^-}{T^{p^-} p^+} \bar{\gamma}^{p^-}$. By the same notations as in the proof of Theorem 3.1, one has

$$\varphi(r) \leq \frac{\sup_{v \in \Phi^{-1}(-\infty, r)} \Psi(v)}{r} \leq \frac{T^{p^-} p^+ \sum_{k=1}^T \max_{|t| \leq \bar{\gamma}} F(k, t)}{\alpha^- \bar{\gamma}^{p^-}} < \frac{1}{\lambda}.$$

In other words, $\lambda \in (0, 1/\varphi(r))$. Thanks to Theorem 2.1, there exists $u_\lambda \in \Phi^{-1}(-\infty, r)$ such that $I'_\lambda(u_\lambda) = 0$ and since the critical points of the functional I_λ are the solutions of the problem (3.7) we have the conclusion. \square

Now, we give some remarks of our results.

Remark 3.4. Let $p^+ \geq 2$. In Theorem 3.3 we looked for the critical points of the functional I_λ naturally associated with the problem (3.7). We note that, in general, I_λ can be unbounded from below in E .

Indeed, let us take $f(\xi) = |\xi|^{\gamma-p^+} \xi^{p^+-1}$ for $\xi \in \mathbb{R}$ with $\gamma > p^+$. Then, from Lemma 2.2(b), we have

$$\begin{aligned} I_\lambda(u) &= \Phi(u) - \lambda \sum_{k=1}^T F(u(k)) \leq \frac{\alpha^+ C_{p^+}}{p^-} \|u\|^{p^+} - \lambda \frac{2^{-\gamma}}{\gamma} \sum_{k=1}^{T+1} |\Delta u(k-1)|^\gamma \\ &\leq \frac{\alpha^+ C_{p^+}}{p^-} \|u\|^{p^+} - \lambda \frac{C_0^\gamma}{\gamma} \|u\|^\gamma \rightarrow -\infty, \end{aligned}$$

where $C_0 = 2^{-\gamma}(T+1)^{(2-\gamma)/2}$ (see [10, Lemma 5(c)]), as $\|u\| \rightarrow +\infty$. Hence, we can not use direct minimization to find critical points of the functional I_λ . However in this example direct maximization provides additional solution as the argument of a maximum as the functional is anti-coercive.

Remark 3.5. For fixed $\bar{\gamma} > 0$ let $\frac{\bar{\gamma}^{p^-}}{\sum_{k=1}^T \max_{|t| \leq \bar{\gamma}} F(k, t)} > \frac{T^{p^-} p^+}{\alpha^-}$. Then the result of Theorem 3.3 holds with $\|u_\lambda\|_\infty \leq \bar{\gamma}$ is the ensured solution in E .

Remark 3.6. If in Theorem 3.1 the function $f(k, \xi) \geq 0$ for every $k \in [1, T]$ and $\xi \in \mathbb{R}$, the condition (3.1) takes the following more simple form

$$(3.8) \quad \sup_{\gamma > 0} \frac{\gamma^{p^-}}{\sum_{k=1}^T F(k, \gamma)} > \frac{T^{p^-} p^+}{\alpha^-}.$$

Moreover, if the following assumption holds

$$\limsup_{\gamma \rightarrow +\infty} \frac{\gamma^{p^-}}{\sum_{k=1}^T F(k, \gamma)} > \frac{T^{p^-} p^+}{\alpha^-},$$

then the condition (3.8) is automatically verified.

Remark 3.7. Let $p^- \geq 2$. By the similar arguments as given in the proof of [2, Theorem 3.5] if in Theorem 3.3, $f(k, 0) \neq 0$ for all $k \in [1, T]$, then the ensured solution is obviously non-trivial. On the other hand, the non-triviality of the solution can be achieved also in the case $f(k, 0) = 0$ for all $k \in [1, T]$ requiring the extra condition at zero, that is there are discrete intervals $[1, T_1] \subseteq [1, T]$ and $[1, T_2] \subset [1, T_1]$ where $T_1, T_2 \geq 2$, such that

$$\limsup_{\xi \rightarrow 0^+} \frac{\inf_{k \in [1, T_2]} F(k, \xi)}{|\xi|^{p^-}} = +\infty$$

and

$$\liminf_{\xi \rightarrow 0^+} \frac{\inf_{k \in [1, T_1]} F(k, \xi)}{|\xi|^{p^-}} > -\infty.$$

Indeed, let $0 < \bar{\lambda} < \lambda^*$ where

$$\lambda^* = \frac{\alpha^-}{T^{p^-} p^+} \sup_{\gamma > 0} \frac{\gamma^{p^-}}{\sum_{k=1}^T \max_{|t| \leq \gamma} F(k, t)}.$$

Then, there exists $\bar{\gamma} > 0$ such that

$$\bar{\lambda} \frac{T^{p^-} p^+}{\alpha^-} < \frac{\bar{\gamma}^{p^-}}{\sum_{k=1}^T \max_{|t| \leq \bar{\gamma}} F(k, t)}.$$

Let Φ and Ψ be as given in (3.2) and (3.3), respectively. Due to Theorem 2.1, for every $\lambda \in (0, \bar{\lambda})$ there exists a critical point of $I_\lambda = \Phi - \lambda\Psi$ such that $u_\lambda \in \Phi^{-1}(-\infty, r_\lambda)$ where $r_\lambda = \frac{\alpha^-}{T^{p^-} p^+} \bar{\gamma}^{p^-}$. In particular, u_λ is a global minimum of the restriction of I_λ to $\Phi^{-1}(-\infty, r_\lambda)$. We will prove that u_λ cannot be trivial. By the same arguments as given in [8, Remark 3.4], we observe that

$$\limsup_{\|u\| \rightarrow 0^+} \frac{\Psi(u)}{\Phi(u)} = +\infty.$$

Hence, there exists a sequence $\{w_n\} \subset E$ strongly converging to zero such that, for n large enough, $w_n \in \Phi^{-1}(-\infty, r)$ and

$$I_\lambda(w_n) = \Phi(w_n) - \lambda\Psi(w_n) < 0.$$

Since u_λ is a global minimum of the restriction of I_λ to $\Phi^{-1}(-\infty, r)$, we conclude that

$$(3.9) \quad I_\lambda(u_\lambda) < 0,$$

so that u_λ is not trivial.

Remark 3.8. From (3.9) we easily observe that the map

$$(3.10) \quad (0, \lambda^*) \ni \lambda \mapsto I_\lambda(u_\lambda)$$

is negative. Also, one has

$$\lim_{\lambda \rightarrow 0^+} \|u_\lambda\| = 0.$$

Indeed, bearing in mind that Φ is coercive and for every $\lambda \in (0, \lambda^*)$ the solution $u_\lambda \in \Phi^{-1}(-\infty, r)$, one has that there exists a positive constant L such that $\|u_\lambda\| \leq L$ for every $\lambda \in (0, \lambda^*)$. Then, there exists a positive constant N such that

$$(3.11) \quad \left| \sum_{k=1}^T f(k, u_\lambda(k))u_\lambda(k) \right| \leq N\|u_\lambda\| \leq NL$$

for every $\lambda \in (0, \lambda^*)$. Since u_λ is a critical point of I_λ , we have $I'_\lambda(u_\lambda)(v) = 0$ for every $v \in E$ and every $\lambda \in (0, \lambda^*)$. In particular $I'_\lambda(u_\lambda)(u_\lambda) = 0$, that is,

$$\Phi'(u_\lambda)(u_\lambda) = \lambda \sum_{k=1}^T f(k, u_\lambda(k))u_\lambda(k)$$

for every $\lambda \in (0, \lambda^*)$. Then, it follows

$$0 \leq \alpha^- \sum_{k=1}^{T+1} |\Delta u_\lambda(k-1)|^{p(k-1)} \leq \Phi'(u_\lambda)(u_\lambda) = \lambda \sum_{k=1}^T f(k, u_\lambda(k))u_\lambda(k)$$

for any $\lambda \in (0, \lambda^*)$. Letting $\lambda \rightarrow 0^+$, by (3.11), we get

$$\lim_{\lambda \rightarrow 0^+} \|u_\lambda\| = 0.$$

Then, we have obviously the desired conclusion. Finally, we have to show that the map

$$\lambda \mapsto I_\lambda(u_\lambda)$$

is strictly decreasing in $(0, \lambda^*)$. We see that for any $u \in E$, one has

$$(3.12) \quad I_\lambda(u) = \lambda \left(\frac{\Phi(u)}{\lambda} - \Psi(u) \right).$$

Now, let us fix $0 < \lambda_1 < \lambda_2 < \lambda^*$ and let u_{λ_i} be the global minimum of the functional I_{λ_i} restricted to $\Phi(-\infty, r)$ for $i = 1, 2$. Also, set

$$m_{\lambda_i} = \left(\frac{\Phi(u_{\lambda_i})}{\lambda_i} - \Psi(u_{\lambda_i}) \right) = \inf_{v \in \Phi^{-1}(-\infty, r)} \left(\frac{\Phi(v)}{\lambda_i} - \Psi(v) \right)$$

for every $i = 1, 2$. Clearly, (3.10) together with (3.12) and the positivity of λ imply that

$$(3.13) \quad m_{\lambda_i} < 0 \quad \text{for } i = 1, 2.$$

Moreover

$$(3.14) \quad m_{\lambda_2} \leq m_{\lambda_1},$$

due to the fact that $0 < \lambda_1 < \lambda_2$. Then, by (3.12)–(3.14) and again by the fact that $0 < \lambda_1 < \lambda_2$, we get

$$I_{\lambda_2}(u_{\lambda_2}) = \lambda_2 m_{\lambda_2} \leq \lambda_2 m_{\lambda_1} < \lambda_1 m_{\lambda_1} = I_{\lambda_1}(u_{\lambda_1}),$$

so that the map $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $\lambda \in (0, \lambda^*)$. The arbitrariness of $\lambda < \lambda^*$ shows that $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $(0, \lambda^*)$.

Here we give a direct application of Theorem 3.3, Remarks 3.7 and 3.8.

Example 3.9. Consider the following problem

$$(3.15) \quad \begin{cases} -\Delta(\alpha(k)|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)) \\ \quad = \lambda(\varepsilon(k)|u(k)|^{r-2}u(k) + \tau(k)|u(k)|^{s-2}u(k)) \quad k \in [1, T], \\ \Delta u(0) = \Delta u(T+1) = 0 \end{cases}$$

where $r \in (1, p)$, $s \in (p, +\infty)$ and $\varepsilon, \tau: [1, T] \rightarrow \mathbb{R}$ are two continuous positive functions. Due to Theorem 3.3 taking Remarks 3.7 and 3.8 into account, the problem (3.15) possesses at least one non-trivial solution $u_\lambda \in E$ such that

$$\lim_{\lambda \rightarrow 0^+} \|u_\lambda\| = 0$$

and the real function

$$\lambda \rightarrow \sum_{k=1}^{T+1} \frac{\alpha(k)}{p(k-1)} |\Delta u(k-1)|^{p(k-1)} - \lambda \sum_{k=1}^T F(k, u(k))$$

is negative and strictly decreasing in

$$\left(0, \frac{\alpha^-}{T^{p^-} p^+} \sup_{\gamma > 0} \frac{\gamma^{p^-}}{\sum_{k=1}^T \max_{|t| \leq \gamma} F(k, t)} \right).$$

Remark 3.10. If f is non-negative then the solution ensured in Theorem 3.3 is non-negative. This follows directly from [11].

Remark 3.11. We observe that Theorem 3.3 is a bifurcation result in the sense that the pair $(0, 0)$ belongs to the closure of the set

$$\{(u_\lambda, \lambda) \in E \times (0, +\infty) : u_\lambda \text{ is a non-trivial solution of (3.7)}\}$$

in $E \times \mathbb{R}$. Indeed, by Theorem 3.3 we have that

$$\|u_\lambda\| \rightarrow 0 \quad \text{as } \lambda \rightarrow 0.$$

Hence, there exist two sequences $\{u_j\}$ in E and $\{\lambda_j\}$ in \mathbb{R}^+ (here $u_j = u_{\lambda_j}$) such that

$$\lambda_j \rightarrow 0^+ \quad \text{and} \quad \|u_j\| \rightarrow 0,$$

as $j \rightarrow +\infty$. Moreover, we emphasize that due to the fact that the map

$$(0, \lambda^*) \ni \lambda \mapsto I_\lambda(u_\lambda)$$

is strictly decreasing, for every $\lambda_1, \lambda_2 \in (0, \lambda^*)$, with $\lambda_1 \neq \lambda_2$, the solutions u_{λ_1} and u_{λ_2} ensured by Theorem 3.3 are different.

Now, we deduce the following straightforward consequence of Theorem 3.3. Precisely, we consider the following autonomous problem

$$(3.16) \quad \begin{cases} -\Delta(\alpha(k)|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)) = \lambda f(u(k)) & k \in [1, T], \\ u(0) = u(T+1) = 0, \end{cases}$$

where $f: \mathbb{R} \rightarrow \mathbb{R}$ is a non-negative continuous function. Put

$$F(\xi) = \int_0^\xi f(t) dt \quad \text{for all } \xi \in \mathbb{R}.$$

Theorem 3.12. *Assume that*

$$\lim_{\xi \rightarrow 0^+} \frac{F(\xi)}{\xi^{p^-}} = +\infty.$$

Then, for each

$$\lambda \in \Lambda = \left(0, \frac{\alpha^-}{T^{p^-+1}p^+} \sup_{\gamma>0} \frac{\gamma^{p^-}}{F(\gamma)} \right),$$

the problem (3.16) admits at least one non-trivial solution $u_\lambda \in E$ such that

$$\lim_{\lambda \rightarrow 0^+} \|u_\lambda\| = 0$$

and the real function

$$\lambda \rightarrow \sum_{k=1}^{T+1} \frac{\alpha(k)}{p(k-1)} |\Delta u_\lambda(k-1)|^{p(k-1)} - \lambda \sum_{k=1}^T F(u_\lambda(k))$$

is negative and strictly decreasing in Λ .

Now, we present the following example to illustrate Theorem 3.12.

Example 3.13. Let $T = 4$. We consider the problem

$$\begin{cases} \Delta(\alpha(k)|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)) = \lambda f(u) & k \in [1, 4], \\ u(0) = u(5) = 0, \end{cases}$$

where $\alpha(k) = 1 + \sin^2(\pi k/2)$ for all $k = 1, 2, 3, 4, 5$, $p(k) = 10^{k+1}$ for $k = 0, 1, 2, 3, 4$,

$$f(t) = \frac{1}{10^5}(10t^9 + 4e^{4t})$$

for every $t \in \mathbb{R}$. By simple computations, we have

$$F(t) = \frac{1}{10^5}(t^{10} + e^{4t} - 1)$$

for every $t \in \mathbb{R}$. Taking into account that $p^- = 10$, $p^+ = 10^5$ and $\alpha^- = 1$, we observe that all the assumptions of Theorem 3.12 are satisfied, and it implies that the problem (3.6) for each $\lambda \in (0, 1/4^{11})$, admits at least one non-trivial solution u_λ in $\{u: [0, 5] \rightarrow \mathbb{R} : \Delta u(0) = \Delta u(5) = 0\}$ such that

$$\lim_{\lambda \rightarrow 0^+} \|u_\lambda\| = 0$$

and the real function

$$\lambda \rightarrow \sum_{k=1}^5 \frac{\alpha(k)}{p(k-1)} |\Delta u_\lambda(k-1)|^{p(k-1)} - \frac{\lambda}{10^5} \sum_{k=1}^4 \left(u_\lambda^{10}(k) + e^{4u_\lambda(k)} - 1 \right)$$

is negative and strictly decreasing in $(0, 1/4^{11})$.

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