ON THE MINIMIZATION OF SOME NONCONVEX DOUBLE OBSTACLE PROBLEMS

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We consider a nonconvex variational problem for which the set of admissible functions consists of all Lipschitz functions located between two fixed obstacles. It turns out that the value of the minimization problem at hand is equal to zero when the obstacles do not touch each other; otherwise, it might be positive. The results are obtained through the construction of suitable minimizing sequences. Interpolating these minimizing sequences in some discrete space, a numerical analysis is then carried out.

1. Introduction

Let Ω denote a bounded and convex domain in $\mathbb{R}^n$ ($n \geq 2$) of boundary $\partial\Omega$ and of closure $\overline{\Omega}$. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that

$$\varphi(w_i) = 0 \quad \forall i = 1,\dots,p, \tag{1.1}$$

$$\varphi(w) > 0 \quad \forall w \neq w_i, \ i = 1,\dots,p, \tag{1.2}$$

where w_i's are p elements of $\mathbb{R}^n$ ($p \geq 2$). For instance, in solid-solid phase transformations, the function φ could be some elastic energy that vanishes at wells w_i's. These wells stand for the stress-free states of a body represented by Ω. For further details about the physical background and also the mathematical modelling, we refer the reader to [2, 3] and the references quoted therein. We assume that, for some physical reasons (e.g., some loads applied to the body Ω), the deformations are constrained not to exceed some fixed obstacles. To make precise the formulation of our

problem, let α, β, and G be three Lipschitz continuous functions, that is,

$$\alpha, \beta, G \in W^{1,\infty}(\Omega) \tag{1.3}$$

such that

$$\alpha(x) \leq G(x) \leq \beta(x) \quad \text{in } \Omega. \tag{1.4}$$

Here $W^{1,\infty}(\Omega)$ is the usual Sobolev space of all weakly differentiable functions $u : \Omega \to \mathbb{R}$ such that u and $|\nabla u|$ are essentially bounded. For a further discussion of Sobolev spaces, we refer the reader to [1]. We denote by $\mathcal{K}$ the following set:

$$\mathcal{K}(\Omega) := \mathcal{K} = \{v \in W^{1,\infty}(\Omega) : \alpha(x) \leq v(x) \leq \beta(x) \text{ in } \Omega, \, v(x) = G(x) \text{ on } \partial\Omega\}. \tag{1.5}$$

Then we consider the following problem:

$$I := \inf_{v \in \mathcal{K}} \int_{\Omega} \varphi(\nabla v(x)) dx. \tag{1.6}$$

More precisely, we consider the case when

$$\nabla G(x) \in \text{Co}(w_i) \quad \text{a.e. in } \Omega, \tag{1.7}$$

where $\text{Co}(w_i)$ denotes the convex hull of the wells w_i's.

   For the numerical purpose and in order to simplify, we will assume that Ω is polyhedral. Let $(\mathcal{T}_h)$ be a regular family of triangulations of Ω (see [11]). This means that

$$\forall h > 0 \quad \begin{cases} \forall K \in \mathcal{T}_h, \, K \text{ is an } N\text{-simplex}, \\ \max_{K \in \mathcal{T}_h}(h_K) = h, \\ \exists \nu > 0 \text{ such that } \forall K \in \mathcal{T}_h, \, \dfrac{h_K}{\rho_K} \leq \nu, \end{cases} \tag{1.8}$$

where h_K is the diameter of the N-simplex K and ρ_K its roundness (i.e., the largest diameter of the balls that could fit in K). If $P_1(K)$ is the space of polynomials of degree 1 on K, we set

$$V_h = \{v : \Omega \longrightarrow \mathbb{R} \text{ continuous}, \, v/_K \in P_1(K), \, \forall K \in \mathcal{T}_h\},$$
$$\mathcal{K}_h := \{v \in V_h : \alpha(p) \leq v(p) \leq \beta(p) \, \forall p \in \Sigma_h^0, \, v(p) = G(p) \, \forall p \in \Sigma_h \cap \partial\Omega\}, \tag{1.9}$$

where

$$\Sigma_h = \{p \in \overline{\Omega} \mid p \text{ is a vertex of } K \in \mathcal{T}_h\},$$
$$\Sigma_h^0 = \{p \in \Sigma_h \mid p \notin \partial\Omega\}. \tag{1.10}$$

Then we denote by I_h the following approximate version of (1.6):

$$I_h := \inf_{\mathcal{K}_h} \int_\Omega \varphi(\nabla v(x)) \, dx. \tag{1.11}$$

The (double) obstacle problems were intensively studied by many authors in the case of convex variational problems, that is, the function φ in problem (1.6) is assumed to be convex. In this paper, we would like to investigate such problems in the presence of nonconvexity. Recall that the question of existence or nonexistence of minimizers for scalar nonconvex variational problems with homogeneous boundary conditions is closed (see [10]). One can also see [9] for a contribution in this direction. The case of nonhomogeneous boundary conditions was first studied by Chipot (see [5]) and then other results were obtained in [6, 7]. The main concern of our paper is to compute the value of I and obtain estimates for $|I_h - I|$ in terms of h. We denote by Ω' the subset of Ω where the obstacles do not touch each other, that is,

$$\Omega' := \{x \in \Omega : \alpha(x) < \beta(x)\}. \tag{1.12}$$

Roughly speaking, our main results are summarized in what follows. Under some assumptions, one has

$$I := \inf_{v \in \mathcal{K}} \int_\Omega \varphi(\nabla v(x)) \, dx = \int_{\Omega \setminus \Omega'} \varphi(\nabla G(x)) \, dx. \tag{1.13}$$

Moreover, if $|\Omega \setminus \Omega'| = 0$ ($|\Omega \setminus \Omega'|$ is the Lebesgue measure of $\Omega \setminus \Omega'$), one has, for h small enough,

$$0 \le I_h \le C h^{1/2}, \tag{1.14}$$

where C is a constant independent of h.

The plan of our paper will be as follows. In Section 2, we present all the intermediate results that we will need to prove our main results which are exposed in Section 3. We will try as far as possible to have our paper self-contained.

2. Preliminary results

In this section, we collect all the ingredients which will be useful to prove our results. First we have the following lemma.

LEMMA 2.1. *If G is a Lipschitz continuous function satisfying (1.7), then there exists a Lipschitz function $\tilde{G}$ defined in $\mathbb{R}^n$ such that*

$$\tilde{G} = G \quad in\ \Omega. \tag{2.1}$$

Moreover,

$$\bigwedge_{i=1}^{p} w_i \cdot (x - y) \le \tilde{G}(x) - \tilde{G}(y) \le \bigvee_{i=1}^{p} w_i \cdot (x - y) \quad \forall x, y \in \mathbb{R}^n, \tag{2.2}$$

where $\bigwedge$ and $\bigvee$ denote, respectively, the infimum and supremum of functions and $a \cdot b$ is the scalar product of $a, b \in \mathbb{R}^n$.

Proof. Applying the mean value theorem after regularization (see [8] for details), one has

$$G(x) - G(y) \ge \bigwedge_{i=1}^{p} w_i \cdot (x - y) \quad for\ x, y \in \Omega. \tag{2.3}$$

Then let

$$\tilde{G}(x) = \inf_{y \in \Omega} \left\{ G(y) - \bigwedge_{i=1}^{p} w_i \cdot (y - x) \right\}, \quad x \in \mathbb{R}^n. \tag{2.4}$$

It is clear that $\tilde{G}(x) \le G(x)$ in Ω. Moreover, using (2.3), one has

$$\tilde{G}(x) \ge G(x) \quad in\ \Omega, \tag{2.5}$$

so that

$$\tilde{G}(x) = G(x) \quad in\ \Omega. \tag{2.6}$$

We now prove that

$$\bigwedge_{i=1}^{p} w_i \cdot (x - y) \le \tilde{G}(x) - \tilde{G}(y) \le \bigvee_{i=1}^{p} w_i \cdot (x - y) \quad for\ x, y \in \mathbb{R}^n. \tag{2.7}$$

For every $x, y \in \mathbb{R}^n$, one has

$$\tilde{G}(x) = \inf_{z \in \Omega} \left\{ G(z) - \bigwedge_{i=1}^{p} w_i \cdot (z - y) + \bigwedge_{i=1}^{p} w_i \cdot (z - y) - \bigwedge_{i=1}^{p} w_i \cdot (z - x) \right\}.$$

(2.8)

Since

$$\bigwedge_{i=1}^{p} w_i \cdot (z - y) - \bigwedge_{i=1}^{p} w_i \cdot (z - x) \geq \bigwedge_{i=1}^{p} w_i \cdot (x - y),$$

(2.9)

one gets

$$\tilde{G}(x) \geq \inf_{z \in \Omega} \left\{ G(z) - \bigwedge_{i=1}^{p} w_i \cdot (z - y) \right\} + \bigwedge_{i=1}^{p} w_i \cdot (x - y),$$

(2.10)

that is,

$$\tilde{G}(x) - \tilde{G}(y) \geq \bigwedge_{i=1}^{p} w_i \cdot (x - y).$$

(2.11)

Moreover, for every $x, y \in \mathbb{R}^n$, one has

$$\bigvee_{i=1}^{p} w_i \cdot (x - y) = - \bigwedge_{i=1}^{p} w_i \cdot (y - x).$$

(2.12)

Then the second inequality in (2.7) follows easily using the first inequality proved above. This completes the proof of the lemma. □

Now let $(v_1, \ldots, v_q)$ be a basis of the space W spanned by the wells w_i's and denote by x_z the points of the lattice of size $h^{1/2}$ spanned by the v_i's, that is, for any $z = (z_1, \ldots, z_q) \in \mathbb{Z}^q$, set

$$x_z = \sum_{i=1}^{q} z_i h^{1/2} v_i.$$

(2.13)

Then we define the functions Λ_h and V_h by

$$\Lambda_h(x) = \bigvee_{z \in \mathbb{Z}^q} \left(\bigwedge_{i=1}^{p} w_i \cdot (x - x_z) + \tilde{G}(x_z) \right),$$

$$V_h(x) = \bigwedge_{z \in \mathbb{Z}^q} \left(\bigvee_{i=1}^{p} w_i \cdot (x - x_z) + \tilde{G}(x_z) \right).$$

(2.14)

By a unit cell of the lattice spanned by the $h^{1/2}v_i$'s we mean a set of the type

$$C_z = x_z + \left\{ \sum_{i=1}^{q} \beta_i h^{1/2} v_i | \beta_i \in [0,1] \right\}. \tag{2.15}$$

Then one has the following lemma.

LEMMA 2.2. *Denote by C_{z_0} a unit cell spanned by $h^{1/2}v_i$'s and by E the set*

$$E = \left\{ z \in \mathbb{Z}^q | z_i = 0 \text{ or } 1 \ \forall i = 1, \ldots, q \right\}. \tag{2.16}$$

Assume that G verifies ([1.7](#)); if $\dim(W) > 1$, assume in addition that the family $(w_p - w_1, \ldots, w_p - w_{p-1})$ is a basis of W and $(v_1, \ldots, v_{p-1})$ is its dual, where $\dim(W)$ denotes the dimension of W. Then,

$$\Lambda_h(x) = \bigvee_{z' \in z_0 + E} \left(\bigwedge_{i=1}^{p} w_i \cdot (x - x_{z'}) + \tilde{G}(x_{z'}) \right),$$
$$V_h(x) = \bigwedge_{z' \in z_0 + E} \left(\bigvee_{i=1}^{p} w_i \cdot (x - x_{z'}) + \tilde{G}(x_{z'}) \right). \tag{2.17}$$

Proof. We give here an astute proof for the case when W is a one-dimensional space. This case was not treated in [6] where a technical proof for higher dimensions is given. Let $z_0, z \in \mathbb{Z}$ and $x \in C_{z_0} := [x_{z_0}, x_{z_0+1}]$. One has either $x_{z_0} \in [x_z, x]$ or $x_{z_0+1} \in [x_z, x]$. We assume that $x_{z_0} \in [x_z, x]$; the other case can be handled similarly. There exists $\lambda \in [0,1]$ such that

$$x_{z_0} = \lambda x_z + (1 - \lambda)x. \tag{2.18}$$

Therefore

$$x - x_{z_0} = \lambda(x - x_z), \qquad x_{z_0} - x_z = (1 - \lambda)(x - x_z). \tag{2.19}$$

One has

$$\bigwedge_{i=1}^{p} w_i \cdot (x - x_{z_0}) + \tilde{G}(x_{z_0})$$
$$= \bigwedge_{i=1}^{p} w_i \cdot (x - x_{z_0}) + \tilde{G}(x_{z_0}) - \tilde{G}(x_z) + \tilde{G}(x_z) \tag{2.20}$$

so that

$$\bigwedge_{i=1}^{p} w_i \cdot (x - x_{z_0}) + \tilde{G}(x_{z_0}) \geq \bigwedge_{i=1}^{p} w_i \cdot (x - x_{z_0}) + \bigwedge_{i=1}^{p} w_i \cdot (x_{z_0} - x_z) + \tilde{G}(x_z),$$

(2.21)

but

$$\bigwedge_{i=1}^{p} w_i \cdot (x - x_{z_0}) + \bigwedge_{i=1}^{p} w_i \cdot (x_{z_0} - x_z)$$

$$= \lambda \bigwedge_{i=1}^{p} w_i \cdot (x - x_z) + (1 - \lambda) \bigwedge_{i=1}^{p} w_i \cdot (x - x_z)$$

(2.22)

so that

$$\bigwedge_{i=1}^{p} w_i \cdot (x - x_{z_0}) + \bigwedge_{i=1}^{p} w_i \cdot (x_{z_0} - x_z) = \bigwedge_{i=1}^{p} w_i \cdot (x - x_z).$$

(2.23)

Hence

$$\bigwedge_{i=1}^{p} w_i \cdot (x - x_{z_0}) + \tilde{G}(x_{z_0}) \geq \bigwedge_{i=1}^{p} w_i \cdot (x - x_z) + \tilde{G}(x_z).$$

(2.24)

Replacing the above infima by suprema and using the second inequality in Lemma 2.1, we obtain the result for the function V_h. □

We will also need the following lemma.

LEMMA 2.3. *If G satisfies (1.7),*

$$G(x) - C^* h^{1/2} \leq \Lambda_h(x) \leq G(x) \leq V_h(x) \leq G(x) + C^* h^{1/2} \quad \forall x \in \Omega, \quad (2.25)$$

where $C^* = 2q \max_i \|w_i\| \max_i \|v_i\|$ ($\|\cdot\|$ denotes the Euclidian norm).

Proof. Let $z \in \mathbb{Z}^q$ and $x \in \Omega$. Using Lemma 2.1, one has

$$\bigwedge_{i=1}^{p} w_i \cdot (x - x_z) + \tilde{G}(x_z) \leq G(x) \quad \forall x_z, \quad (2.26)$$

so that

$$\Lambda_h(x) \leq G(x). \quad (2.27)$$

Now we denote by x' the component of x on $P_W(\Omega)$, the orthogonal projection of Ω onto W. There exists z_0 such that $x' \in C_{z_0}$, then x' can be written as follows:

$$x' = x_{z_0} + \sum_{i=1}^{q} \beta_i h^{1/2} v_i, \quad \beta_i \in [0,1]. \tag{2.28}$$

Hence

$$\|x' - x_{z_0}\| \le q \max_i \|v_i\| h^{1/2} \tag{2.29}$$

so that

$$\left| \bigwedge_{i=1}^{p} w_i \cdot (x - x_{z_0}) \right| = \left| \bigwedge_{i=1}^{p} w_i \cdot (x' - x_{z_0}) \right| \\ \le q \max_i \|w_i\| \max_i \|v_i\| h^{1/2}. \tag{2.30}$$

Since

$$\Lambda_h(x) \ge \bigwedge_{i=1}^{p} w_i \cdot (x - x_{z_0}) + \tilde{G}(x_{z_0}), \tag{2.31}$$

one gets

$$\Lambda_h(x) \ge -q \max_i \|w_i\| \max_i \|v_i\| h^{1/2} + \tilde{G}(x_{z_0}). \tag{2.32}$$

Since

$$\tilde{G}(x_{z_0}) = \tilde{G}(x_{z_0}) - G(x) + G(x) \ge \bigwedge_{i=1}^{p} w_i \cdot (x_{z_0} - x) + G(x), \tag{2.33}$$

one obtains by (2.30)

$$\Lambda_h(x) \ge -2q \max_i \|w_i\| \max_i \|v_i\| h^{1/2} + G(x). \tag{2.34}$$

Replacing again the above infima by suprema, one easily obtains the remaining inequalities. This completes the proof of the lemma.   □

Remark 2.4. We have seen that condition (1.7) implies (2.3). The two conditions are actually equivalent. Indeed, if (2.3) is verified, then, due to (2.25), the sequence (Λ_h) converges uniformly to G. Since $\nabla \Lambda_h(x) = w_i$

almost everywhere in Ω, one has at least for a subsequence that $\nabla \Lambda_h \rightharpoonup \nabla G$ in $L^\infty(\Omega)$ weak $*$. Let B be any ball included in Ω. Since

$$\frac{1}{|B|}\int_B \nabla \Lambda_h(x)dx \in \mathrm{Co}\,(w_i) \tag{2.35}$$

and $\mathrm{Co}(w_i)$ is a closed set, one has

$$\frac{1}{|B|}\int_B \nabla G(x)dx \in \mathrm{Co}\,(w_i). \tag{2.36}$$

Using the Lebesgue differentiation theorem, one obtains (1.7).

3. Statement and proof of the main results

First we assume that the obstacles do not touch each other, that is,

$$\alpha < \beta \quad \text{in } \overline{\Omega}. \tag{3.1}$$

Thus the following constants are positive:

$$M_1 := \inf_{x \in \Omega_1}\,(G(x) - \alpha(x)), \qquad M_2 := \inf_{x \in \Omega_2}\,(\beta(x) - G(x)), \tag{3.2}$$

where Ω_1 and Ω_2 denote the following sets:

$$\Omega_1 := \left\{ x \in \Omega : G(x) > \frac{\alpha(x) + \beta(x)}{2} \right\}, \qquad \Omega_2 := \Omega \backslash \Omega_1. \tag{3.3}$$

Then one has the following theorem.

THEOREM 3.1. *Assume that φ is a continuous function satisfying (1.1) and (1.2). If (1.7) holds, then*

$$I := \inf_{u \in \mathcal{K}}\int_\Omega \varphi(\nabla u(x))dx = 0. \tag{3.4}$$

Moreover, if Ω is polyhedral, then, under the assumptions of Lemma 2.2, there exists a constant C independent of h, $0 < h < \inf((M_1/C^*)^2, (M_2/C^*)^2, 1)$ such that

$$I := \inf_{u \in \mathcal{K}_h}\int_\Omega \varphi(\nabla u(x))dx \le Ch^{1/2}, \tag{3.5}$$

where $C^* = 2q\max_i\|w_i\|\max_i\|v_i\|.$

Proof. We consider the following Lipschitz functions:

$$u_1^h(x) = \Lambda_h \vee \left(G(x) - \text{dist}\,(x, \partial\Omega_1)\right) \quad \text{if } x \in \Omega_1,$$
$$u_2^h(x) = V_h(x) \wedge \left(G(x) + \text{dist}\,(x, \partial\Omega_2)\right) \quad \text{if } x \in \Omega_2, \tag{3.6}$$

where $\text{dist}(x, \partial\Omega_i)$ denotes the distance from x to the boundary $\partial\Omega_i$ of Ω_i, $i = 1, 2$. The functions Λ_h and V_h are defined by (2.14). Then one has

$$\alpha(x) \le u_1^h(x) \le G(x) \quad \text{in } \Omega_1, \tag{3.7}$$
$$G(x) \le u_2^h(x) \le \beta(x) \quad \text{in } \Omega_2. \tag{3.8}$$

We prove (3.7); (3.8) can be proved the same way. According to Lemma 2.3, one has

$$u_1^h(x) \le G(x) \quad \forall x \in \Omega_1. \tag{3.9}$$

On the other hand, one has

$$u_1^h(x) \ge \left(G(x) - C^* h^{1/2}\right) \vee \left(G(x) - \text{dist}\,(x, \partial\Omega_1)\right)$$
$$\ge G(x) - C^* h^{1/2}. \tag{3.10}$$

Thus

$$u_1^h(x) \ge G(x) - \alpha(x) - C^* h^{1/2} + \alpha(x) \tag{3.11}$$

so that

$$u_1^h(x) \ge M_1 - C^* h^{1/2} + \alpha(x), \tag{3.12}$$

where M_1 is defined by (3.2). Since $0 < h < (M_1/C^*)^2$, one gets

$$u_1^h(x) \ge \alpha(x). \tag{3.13}$$

It is clear by (2.25) that the functions u_i^h's coincide with G at the boundaries of Ω_i's, respectively. Since $\partial\Omega \subset \partial\Omega_1 \cup \partial\Omega_2$, the function

$$u_h(x) = \begin{cases} u_1^h(x), & \text{if } x \in \Omega_1, \\ u_2^h(x), & \text{if } x \in \Omega_2, \end{cases} \tag{3.14}$$

coincides with G at the boundary of Ω. Moreover, u_h is a Lipschitz function. Indeed, let $x \in \Omega_1$ and $y \in \Omega_2$. There exists $z \in \partial\Omega_1 \cap \partial\Omega_2$ such that

$$z \in [x, y]. \tag{3.15}$$

Thus

$$
\begin{aligned}
\left|u_h(x) - u_h(y)\right| &= \left|u_1^h(x) - u_2^h(y)\right| \\
&\leq \left|u_1^h(x) - u_1^h(z)\right| + \left|u_2^h(z) - u_2^h(y)\right|
\end{aligned}
\tag{3.16}
$$

since $u_1^h(z) = u_2^h(z) = G(z)$. Hence

$$
\left|u_h(x) - u_h(y)\right| \leq C\{|x - z| + |z - y|\} = C|x - y|,
\tag{3.17}
$$

where C is a constant independent of x, y, and h (in the sequel, we will denote by C every constant which does not depend on h). Then the function u_h belongs to $\mathcal{K}$. Now we can prove (3.4). Due to (2.25), one has

$$
\begin{aligned}
u_h &= \Lambda_h \quad \text{on } \{x \in \Omega_1 : \text{dist}\,(x, \partial\Omega_1) \geq C^* h^{1/2}\}, \\
u_h &= V_h \quad \text{on } \{x \in \Omega_2 : \text{dist}\,(x, \partial\Omega_2) \geq C^* h^{1/2}\}.
\end{aligned}
\tag{3.18}
$$

Therefore, $\nabla u_h = w_i$ except in a neighborhood N_h of the boundaries $\partial\Omega_i$ of measure less than $Ch^{1/2}$. Using (1.1) and the fact that ∇u_h is uniformly bounded, one obtains

$$
0 \leq I = \int_{N_h} \varphi(\nabla u_h(x))\,dx \leq Ch^{1/2}
\tag{3.19}
$$

for every $0 < h < \inf((M_1/C^*)^2, (M_2/C^*)^2)$, which obviously implies (3.4). Due to (2.25) again, one has

$$
\left|G(x) - \Lambda_h(x)\right| \leq C^* h^{1/2}, \quad \left|G(x) - V_h(x)\right| \leq C^* h^{1/2}, \quad \forall x \in \overline{\Omega}, \tag{3.20}
$$

so that

$$
\left|G(x) - u_h(x)\right| \leq Ch^{1/2} \quad \forall x \in \overline{\Omega}.
\tag{3.21}
$$

Let $\hat{u}_h$ denote the interpolate of u_h. Clearly, $\hat{u}_h \in \mathcal{K}_h$ and one has

$$
\left|u_h(x) - \hat{u}_h(x)\right| \leq Ch.
\tag{3.22}
$$

We prove (3.22). Let $x \in \Omega$. There exists an N-simplex $K \in \mathcal{T}_h$ such that $x \in K$. Let y be any vertex of the N-simplex K. One has

$$
u_h(y) = \hat{u}_h(y).
\tag{3.23}
$$

Therefore, one has

$$
\left|u_h(x) - \hat{u}_h(x)\right| \leq \left|u_h(x) - u_h(y)\right| + \left|\hat{u}_h(x) - \hat{u}_h(y)\right|.
\tag{3.24}
$$

Using the mean value theorem, one obtains

$$\left| u_h(x) - \hat{u}_h(x) \right| \leq \left\{ \sup_{z \in K} \| \nabla u_h(z) \| + \sup_{z \in K} \| \nabla \hat{u}_h(z) \| \right\} \| x - y \|. \qquad (3.25)$$

Since ∇u_h is uniformly bounded, $\nabla \hat{u}_h$ is also uniformly bounded (see [4]). Moreover, $\| x - y \| \leq h_K \leq h$. Thus there exists a constant C so that (3.22) holds. Therefore, using the triangle inequality (recall that $h < 1$), one gets

$$\left| \hat{u}_h(x) - G(x) \right| \leq C h^{1/2}. \qquad (3.26)$$

Notice that

$$\nabla \hat{u}_h = w_i \qquad (3.27)$$

except maybe on the set S composed of simplices where interpolation occurred. Since on this set $\nabla \hat{u}_h$ remains bounded, one has

$$\int_\Omega \varphi(\nabla \hat{u}_h(x)) dx = \int_S \varphi(\nabla \hat{u}_h(x)) dx \leq C|S|, \qquad (3.28)$$

where $|S|$ is the Lebesgue measure of S. When

$$\operatorname{dist}\left(x, \partial\Omega_1 \cup \partial\Omega_2\right) \geq C^* h^{1/2}, \qquad (3.29)$$

one has

$$u_h(x) = \Lambda_h(x) \text{ or } V_h. \qquad (3.30)$$

So, in the set where $\operatorname{dist}(x, \partial\Omega_1 \cup \partial\Omega_2) \geq C^* h^{1/2} + h$, one has that $\hat{u}_h$ equals the interpolate of Λ_h or V_h. We denote by S_1 the set

$$S_1 = \left\{ x \in \Omega : \operatorname{dist}\left(x, \partial\Omega_1 \cup \partial\Omega_2\right) \geq C^* h^{1/2} + h \right\}. \qquad (3.31)$$

Hence

$$|S| \leq \left| S \cap S_1 \right| + \left| \Omega \backslash S_1 \right|. \qquad (3.32)$$

First we have

$$\left| \Omega \backslash S_1 \right| \leq C h^{1/2}. \qquad (3.33)$$

To estimate $|S \cap S_1|$, one sees that interpolation occurs on an h-neighborhood of the set where Λ_h and V_h have discontinuity in their gradients.

Clearly, Λ_h and V_h have a jump in their gradients on a unit cell of the lattice spanned by $h^{1/2}v_i$'s when one of the functions

$$w_i \cdot (x - x_z) + \tilde{G}(x_z) \tag{3.34}$$

is equal to another. These two functions are equal on a set of $(q-1)$-dimensional measure bounded by $Ch^{(q-1)/2}$. Since in Lemma 2.2 the supremum and infimum are taken on a finite number of functions, it is clear that

$$\left|S \cap S_1\right| \le Ch^{(q-1)/2} \cdot h \cdot N(h), \tag{3.35}$$

where $N(h)$ is the number of cells of size $h^{1/2}$ included in $P_W(\Omega)$. Clearly, $N(h)h^{q/2}$ is less than or equal to the q-dimensional measure of $P_W(\Omega)$. Therefore,

$$\left|S \cap S_1\right| \le Ch^{1/2}. \tag{3.36}$$

Combining (3.28), (3.33), and (3.36), one obtains

$$\int_\Omega \varphi(\nabla \hat{u}_h)\,dx \le Ch^{1/2}. \tag{3.37}$$

This completes the proof of the theorem.   □

Remark 3.2. We chose the cells C_z's of size $h^{1/2}$ among those of size h^α, $\alpha \in (0,1)$, since they provide the best estimate (see [6] for details).

Remark 3.3. Notice that (3.4) is still valid if Ω is the union of a finite number of convex domains. Moreover, a polyhedral domain can be divided into a finite number of disjoint convex polyhedral domains. Using the same construction as above in every such subdomain, one can see that the estimate obtained in Theorem 3.1 is obviously still valid.

Remark 3.4. The continuous problem (1.6) does not admit in general a minimizer. Indeed, we assume that $q < n$ and

$$\nabla G \ne w_i \quad \text{in a set of positive measure.} \tag{3.38}$$

There exists $v \in \mathbb{R}^n$ such that

$$w_i \cdot v = 0 \quad \forall i = 1,2,\ldots,p. \tag{3.39}$$

If problem (1.6) admits a minimizer u, by (3.4), (1.1), and (1.2), one has

$$\nabla u = w_i \quad \text{a.e. in } \Omega. \tag{3.40}$$

Using the variant of Poincaré's inequality

$$\int_\Omega |u(x) - G(x)| dx \le C \int_\Omega |(\nabla u(x) - \nabla G(x)) \cdot v| dx, \qquad (3.41)$$

(1.7), (3.39), and (3.40), one deduces that

$$u = G. \qquad (3.42)$$

But the assertions (3.38) and (3.40) are incompatible. Therefore, problem (1.6) cannot admit a minimizer. On the other hand and by a compactness argument, the discrete problem I_h admits minimizers.

We end this paper by considering the case where the two obstacles touch each other. We denote by Ω' the following open set:

$$\Omega' := \{x \in \Omega : \alpha(x) < \beta(x)\}. \qquad (3.43)$$

One has, for every $u \in \mathcal{K}$,

$$u(x) = G(x) = \alpha(x) = \beta(x) \quad \text{in } \Omega \backslash \Omega', \qquad (3.44)$$

$$\int_\Omega \varphi(\nabla u(x)) dx = \int_{\Omega'} \varphi(\nabla u(x)) dx + \int_{\Omega \backslash \Omega'} \varphi(\nabla G(x)) dx. \qquad (3.45)$$

Since

$$u = G \quad \text{on } \partial\Omega', \qquad (3.46)$$

one has

$$\int_\Omega \varphi(\nabla u(x)) dx \ge \inf_{v \in \mathcal{K}(\Omega')} \int_{\Omega'} \varphi(\nabla v(x)) dx + \int_{\Omega \backslash \Omega'} \varphi(\nabla G(x)) dx. \quad (3.47)$$

Therefore,

$$\inf_{v \in \mathcal{K}(\Omega)} \int_\Omega \varphi(\nabla v(x)) dx \ge \inf_{v \in \mathcal{K}(\Omega')} \int_{\Omega'} \varphi(\nabla v(x)) dx + \int_{\Omega \backslash \Omega'} \varphi(\nabla G(x)) dx. \qquad (3.48)$$

Now let $u \in \mathcal{K}(\Omega')$. We define $\tilde{u} \in \mathcal{K}(\Omega)$ as follows:

$$\tilde{u} = u \quad \text{in } \Omega', \qquad \tilde{u} = G \quad \text{in } \Omega \backslash \Omega'. \qquad (3.49)$$

Then

$$\inf_{v\in\mathcal{K}(\Omega)}\int_\Omega\varphi(\nabla v(x))\,dx \le \int_\Omega\varphi(\nabla\tilde{u}(x))\,dx$$
$$= \int_{\Omega'}\varphi(\nabla u(x))\,dx + \int_{\Omega\setminus\Omega'}\varphi(\nabla G(x))\,dx. \tag{3.50}$$

Therefore,

$$\inf_{v\in\mathcal{K}(\Omega)}\int_\Omega\varphi(\nabla v(x))\,dx = \inf_{v\in\mathcal{K}(\Omega')}\int_{\Omega'}\varphi(\nabla v(x))\,dx + \int_{\Omega\setminus\Omega'}\varphi(\nabla G(x))\,dx. \tag{3.51}$$

Then one has the following theorem.

THEOREM 3.5. *Assume that φ is a continuous function satisfying (1.1) and (1.2). If (1.7) holds,*

$$\inf_{u\in\mathcal{K}(\Omega)}\int_\Omega\varphi(\nabla u(x))\,dx = \int_{\Omega\setminus\Omega'}\varphi(\nabla G(x))\,dx. \tag{3.52}$$

Proof. Due to (3.51), it suffices to prove that

$$\inf_{u\in\mathcal{K}(\Omega')}\int_{\Omega'}\varphi(\nabla u(x))\,dx = 0. \tag{3.53}$$

We denote by Ω'_ε the following set:

$$\Omega'_\varepsilon = \{x\in\Omega' : \operatorname{dist}(x,\partial\Omega') > \varepsilon\}. \tag{3.54}$$

Let $u\in\mathcal{K}(\Omega'_\varepsilon)$. We extend u to Ω' by setting

$$u = G \quad \text{in } \Omega'\setminus\Omega'_\varepsilon. \tag{3.55}$$

The extension of u to Ω' belongs to $\mathcal{K}(\Omega')$ and

$$\int_{\Omega'}\varphi(\nabla u(x))\,dx = \int_{\Omega'_\varepsilon}\varphi(\nabla u(x))\,dx + \int_{\Omega'\setminus\Omega'_\varepsilon}\varphi(\nabla G(x))\,dx. \tag{3.56}$$

Thus

$$\int_{\Omega'_\varepsilon}\varphi(\nabla u(x))\,dx + \int_{\Omega'\setminus\Omega'_\varepsilon}\varphi(\nabla G(x))\,dx \ge \inf_{v\in\mathcal{K}(\Omega')}\int_{\Omega'}\varphi(\nabla v(x))\,dx, \tag{3.57}$$

which implies that

$$\inf_{v \in \mathcal{K}(\Omega')} \int_{\Omega'} \varphi(\nabla v(x)) dx \leq \inf_{v \in \mathcal{K}(\Omega'_\varepsilon)} \int_{\Omega'_\varepsilon} \varphi(\nabla v(x)) dx + \int_{\Omega' \backslash \Omega'_\varepsilon} \varphi(\nabla G(x)) dx.$$

(3.58)

But we know that

$$\inf_{v \in \mathcal{K}(\Omega'_\varepsilon)} \int_{\Omega'_\varepsilon} \varphi(\nabla v(x)) dx = 0$$

(3.59)

since in the set Ω_ε the obstacles do not touch each other. Thus

$$0 \leq \inf_{v \in \mathcal{K}(\Omega')} \int_{\Omega'} \varphi(\nabla v(x)) dx \leq C' |\Omega' \backslash \Omega'_\varepsilon| \leq C'' \varepsilon,$$

(3.60)

where C' and C'' are constants independent of ε. Hence,

$$\inf_{v \in \mathcal{K}(\Omega')} \int_{\Omega'} \varphi(\nabla v(x)) dx = 0,$$

(3.61)

and this completes the proof of the theorem. $\square$

Remark 3.6. Using (3.45), one should notice that if u is a minimizer of

$$\inf_{v \in \mathcal{K}(\Omega)} \int_{\Omega} \varphi(\nabla v(x)) dx,$$

(3.62)

then the restriction of u to Ω' is also a minimizer of

$$\inf_{v \in \mathcal{K}(\Omega')} \int_{\Omega'} \varphi(\nabla v(x)) dx.$$

(3.63)

Therefore, the continuous problem (1.6) cannot admit a minimizer in general. Moreover, if $\nabla G(x) \neq w_i$ in a subset of $\Omega \backslash \Omega'$ of positive measure, one has

$$I = \inf_{v \in \mathcal{K}(\Omega)} \int_{\Omega} \varphi(\nabla v(x)) dx = \int_{\Omega \backslash \Omega'} \varphi(\nabla G(x)) dx > 0.$$

(3.64)

Acknowledgments

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