MATRIX VARIATE KUMMER-DIRICHLET DISTRIBUTIONS

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The multivariate Kummer-Beta and multivariate Kummer-Gamma families of distributions have been proposed and studied recently by Ng and Kotz. These distributions are extensions of Kummer-Beta and Kummer-Gamma distributions. In this article we propose and study matrix variate generalizations of multivariate Kummer-Beta and multivariate Kummer-Gamma families of distributions.

1. Introduction

The Kummer-Beta and Kummer-Gamma families of distributions are defined by the density functions

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\left\{{}_1F_1(\alpha;\alpha+\beta;-\lambda)\right\}^{-1}\exp(-\lambda u)u^{\alpha-1}(1-u)^{\beta-1}, \quad 0<u<1, \tag{1.1}$$

$$\left\{\Gamma(\alpha)\Psi(\alpha,\alpha-\gamma+1;\xi)\right\}^{-1}\exp(-\xi v)v^{\alpha-1}(1+v)^{-\gamma}, \quad v>0, \tag{1.2}$$

respectively, where $\alpha > 0$, $\beta > 0$, $\xi > 0$, $-\infty < \gamma,\lambda < \infty$, ${}_1F_1$, and Ψ are confluent hypergeometric functions. These distributions are extensions of Gamma and Beta distributions, and for $\alpha < 1$ (and certain values of λ and γ) yield bimodal distributions on finite and infinite ranges, respectively. These distributions are used (i) in the Bayesian analysis of queueing system where posterior distribution of certain basic parameters in $M/M/\infty$ queueing system is Kummer-Gamma and (ii) in common value auctions where the posterior distribution of "value of a single good" is Kummer-Beta. For properties and applications of these distributions the reader is referred to Ng and Kotz [7], Armero and Bayarri [1], and Gordy [2].

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As the corresponding multivariate generalization of these distributions, we have the following n-dimensional densities:

$$\frac{\Gamma\left(\sum_{i=1}^{n}\alpha_i+\beta\right)}{\prod_{i=1}^{n}\Gamma(\alpha_i)\Gamma(\beta)}\left\{{}_1F_1\left(\sum_{i=1}^{n}\alpha_i;\sum_{i=1}^{n}\alpha_i+\beta;-\lambda\right)\right\}^{-1}\exp\left(-\lambda\sum_{i=1}^{n}u_i\right)$$
$$\times\prod_{i=1}^{n}u_i^{\alpha_i-1}\left(1-\sum_{i=1}^{n}u_i\right)^{\beta-1},\quad 0<u_i<1,\ \sum_{i=1}^{n}u_i<1, \tag{1.3}$$

where $\alpha_i>0$, $i=1,\ldots,n$, $\beta>0$, $-\infty<\lambda<\infty$, and

$$\left\{\Gamma\left(\sum_{i=1}^{n}\alpha_i\right)\Psi\left(\sum_{i=1}^{n}\alpha_i,\sum_{i=1}^{n}\alpha_i-\gamma+1;\xi\right)\right\}^{-1}\exp\left(-\xi\sum_{i=1}^{n}v_i\right)$$
$$\times\prod_{i=1}^{n}v_i^{\alpha_i-1}\left(1+\sum_{i=1}^{n}v_i\right)^{-\gamma},\quad v_i>0, \tag{1.4}$$

where $\alpha_i>0$, $i=1,\ldots,n$, $\xi>0$, $-\infty<\gamma<\infty$, respectively. These distributions have been considered by Ng and Kotz [7] who refer to (1.3) and (1.4) as multivariate Kummer-Beta and multivariate Kummer-Gamma distributions, respectively. For $\lambda=0$, (1.1) and (1.3) reduce to Beta and Dirichlet distributions with probability density functions

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}u^{\alpha-1}(1-u)^{\beta-1},\quad 0<u<1,$$

$$\frac{\Gamma\left(\sum_{i=1}^{n}\alpha_i+\beta\right)}{\prod_{i=1}^{n}\Gamma(\alpha_i)\Gamma(\beta)}\prod_{i=1}^{n}u_i^{\alpha_i-1}\left(1-\sum_{i=1}^{n}u_i\right)^{\beta-1},\quad 0<u_i<1,\ \sum_{i=1}^{n}u_i<1, \tag{1.5}$$

respectively. Since (1.3) is an extension of Dirichlet distribution and a multivariate generalization of Kummer-Beta distribution, an appropriate nomenclature for this distribution would be *Kummer-Dirichlet distribution*. In the same vein, we may call (1.4) a Kummer-Dirichlet distribution. Further, in order to distinguish between these two distributions ((1.3) and (1.4)), we call them Kummer-Dirichlet type I and Kummer-Dirichlet type II distributions.

In this article we propose and study matrix variate generalizations of (1.3) and (1.4), respectively.

2. Matrix variate Kummer-Dirichlet distributions

We begin with a brief review of some definitions and notations. We adhere to standard notations (cf. Gupta and Nagar [3]). Let $A=(a_{ij})$ be a $p\times p$ matrix.

Then, A' denotes the transpose of A; $\text{tr}(A) = a_{11} + \cdots + a_{pp}$; $\text{etr}(A) = \exp(\text{tr}(A))$; $\det(A) =$ determinant of A; $A > 0$ means that A is symmetric positive definite and $A^{1/2}$ denotes the unique symmetric positive definite square root of $A > 0$. The multivariate gamma function $\Gamma_p(m)$ is defined as

$$\Gamma_p(m) = \pi^{p(p-1)/4} \prod_{j=1}^{p} \Gamma\left(m - \frac{j-1}{2}\right), \quad \text{Re}(m) > \frac{p-1}{2}, \tag{2.1}$$

where $\text{Re}(\cdot)$ denotes the real part of $(\cdot)$. It is straightforward to show that

$$\Gamma_p(m) = \int_{R>0} \det(R)^{m-(p+1)/2} \text{etr}(-R)\, dR, \quad \text{Re}(m) > \frac{p-1}{2}, \tag{2.2}$$

where the integral has been evaluated over the space of the $p \times p$ symmetric positive definite matrices. The integral representation of the confluent hypergeometric function ${}_1F_1$ is given by

$$\begin{aligned} {}_1F_1(a;b;X) = {} & \frac{\Gamma_p(b)}{\Gamma_p(a)\Gamma_p(b-a)} \\ & \times \int_{0<R<I_p} \det(R)^{a-(p+1)/2} \det\left(I_p - R\right)^{b-a-(p+1)/2} \text{etr}(XR)\, dR, \end{aligned} \tag{2.3}$$

where $\text{Re}(a) > (p-1)/2$ and $\text{Re}(b-a) > (p-1)/2$. The confluent hypergeometric function Ψ of a $p \times p$ symmetric matrix X is defined by

$$\begin{aligned} \Psi(a,c;X) = {} & \frac{1}{\Gamma_p(a)} \\ & \times \int_{R>0} \text{etr}(-XR) \det(R)^{a-(p+1)/2} \det\left(I_p + R\right)^{c-a-(p+1)/2} dR, \end{aligned} \tag{2.4}$$

where $\text{Re}(X) > 0$ and $\text{Re}(a) > (p-1)/2$.

Now we define the corresponding matrix variate generalizations of (1.3) and (1.4) as follows.

Definition 2.1. The $p \times p$ symmetric positive definite random matrices $U_1, \ldots, U_n$ are said to have the matrix variate Kummer-Dirichlet type I distribution with parameters $\alpha_1, \ldots, \alpha_n$, β and Λ, denoted by $(U_1, \ldots, U_n) \sim KD_p^I(\alpha_1, \ldots, \alpha_n, \beta, \Lambda)$, if their joint probability density function (pdf) is given by

$$K_1(\alpha_1,\ldots,\alpha_n,\beta,\Lambda)\operatorname{etr}\left(-\Lambda\sum_{i=1}^n U_i\right)$$
$$\times\prod_{i=1}^n \det(U_i)^{\alpha_i-(p+1)/2}\det\left(I_p-\sum_{i=1}^n U_i\right)^{\beta-(p+1)/2}, \tag{2.5}$$
$$0<U_i<I_p,\ 0<\sum_{i=1}^n U_i<I_p,$$

where $\alpha_i>(p-1)/2$, $i=1,\ldots,n$, $\beta>(p-1)/2$, $\Lambda(p\times p)$ is symmetric and $K_1(\alpha_1,\ldots,\alpha_n,\beta,\Lambda)$ is the normalizing constant.

Definition 2.2. The $p\times p$ symmetric positive definite random matrices $V_1,\ldots,V_n$ are said to have the matrix variate Kummer-Dirichlet type II distribution with parameters $\alpha_1,\ldots,\alpha_n,\gamma$ and Ξ, denoted by $(V_1,\ldots,V_n)\sim KD_p^{II}(\alpha_1,\ldots,\alpha_n,\gamma,\Xi)$, if their joint pdf is given by

$$K_2(\alpha_1,\ldots,\alpha_n,\gamma,\Xi)\operatorname{etr}\left(-\Xi\sum_{i=1}^n V_i\right)$$
$$\times\prod_{i=1}^n \det(V_i)^{\alpha_i-(p+1)/2}\det\left(I_p+\sum_{i=1}^n V_i\right)^{-\gamma},\quad V_i>0, \tag{2.6}$$

where $\alpha_i>(p-1)/2$, $i=1,\ldots,n$, $-\infty<\gamma<\infty$, $\Xi(p\times p)>0$, and $K_2(\alpha_1,\ldots,\alpha_n,\gamma,\Xi)$ is the normalizing constant.

The normalizing constants in (2.5) and (2.6) are given as

$$\{K_1(\alpha_1,\ldots,\alpha_n,\beta,\Lambda)\}^{-1}$$
$$=\int\cdots\int_{\substack{0<\sum_{i=1}^n U_i<I_p\\ U_i>0}}\operatorname{etr}\left(-\Lambda\sum_{i=1}^n U_i\right)$$
$$\times\prod_{i=1}^n \det(U_i)^{\alpha_i-(p+1)/2}\det\left(I_p-\sum_{i=1}^n U_i\right)^{\beta-(p+1)/2}\prod_{i=1}^n dU_i$$
$$=\frac{\prod_{i=1}^n\Gamma_p(\alpha_i)}{\Gamma_p\left(\sum_{i=1}^n\alpha_i\right)}\int_{0<U<I_p}\operatorname{etr}(-\Lambda U)\det(U)^{\sum_{i=1}^n\alpha_i-(p+1)/2}$$
$$\times\det(I_p-U)^{\beta-(p+1)/2}dU$$

$$= \frac{\prod_{i=1}^{n} \Gamma_p(\alpha_i)\Gamma_p(\beta)}{\Gamma_p\left(\sum_{i=1}^{n} \alpha_i + \beta\right)} {}_1F_1\left(\sum_{i=1}^{n} \alpha_i; \sum_{i=1}^{n} \alpha_i + \beta; -\Lambda\right), \tag{2.7}$$

$$\begin{aligned}\{K_2(\alpha_1,\ldots,\alpha_n,\gamma,\Xi)\}^{-1} \\ &= \int_{V_1>0} \cdots \int_{V_n>0} \operatorname{etr}\left(-\Xi\sum_{i=1}^{n} V_i\right) \\ &\quad \times \prod_{i=1}^{n} \det(V_i)^{\alpha_i-(p+1)/2} \det\left(I_p + \sum_{i=1}^{n} V_i\right)^{-\gamma} \prod_{i=1}^{n} dV_i \\ &= \frac{\prod_{i=1}^{n} \Gamma_p(\alpha_i)}{\Gamma_p\left(\sum_{i=1}^{n} \alpha_i\right)} \int_{V>0} \operatorname{etr}(-\Xi V) \det(V)^{\sum_{i=1}^{n} \alpha_i-(p+1)/2} \det\left(I_p + V\right)^{-\gamma} dV \\ &= \prod_{i=1}^{n} \Gamma_p(\alpha_i) \Psi\left(\sum_{i=1}^{n} \alpha_i, \sum_{i=1}^{n} \alpha_i - \gamma + \frac{p+1}{2}; \Xi\right),\end{aligned} \tag{2.8}$$

respectively, where ${}_1F_1$ and Ψ are confluent hypergeometric functions of matrix argument.

For $\Lambda = 0$, the matrix variate Kummer-Dirichlet type I distribution collapses to an ordinary matrix variate Dirichlet type I distribution with pdf

$$\frac{\Gamma_p\left(\sum_{i=1}^{n} \alpha_i + \beta\right)}{\prod_{i=1}^{n} \Gamma_p(\alpha_i)\Gamma_p(\beta)} \prod_{i=1}^{n} \det(U_i)^{\alpha_i-(p+1)/2} \det\left(I_p - \sum_{i=1}^{n} U_i\right)^{\beta-(p+1)/2}, \quad 0 < U_i < I_p,\ 0 < \sum_{i=1}^{n} U_i < I_p, \tag{2.9}$$

where $\alpha_i > (p-1)/2$, $i = 1,\ldots,n$, and $\beta > (p-1)/2$. A common notation to designate that $(U_1,\ldots,U_n)$ has this density is $(U_1,\ldots,U_n) \sim D_p^I(\alpha_1,\ldots,\alpha_n;\beta)$. For $\gamma = 0$, the matrix variate Kummer-Dirichlet type II density simplifies to the product of n matrix variate Gamma densities.

For $p = 1$, the densities in (2.5) and (2.6) simplify to Kummer-Dirichlet type I (multivariate Kummer-Beta) and Kummer-Dirichlet type II (multivariate Kummer-Gamma) densities, respectively. For $n = 1$, the matrix variate Kummer-Dirichlet type I and matrix variate Kummer-Dirichlet type II distributions reduce to the matrix variate Kummer-Beta and matrix variate Kummer-Gamma distributions, respectively. These two distributions have been studied by Nagar and Gupta [6] and Nagar and Cardeño [5]. Substituting

$n=1$ in (2.5) and (2.6), the matrix variate Kummer-Beta and matrix variate Kummer-Gamma densities are obtained as

$$K_1(\alpha,\beta,\Lambda)\operatorname{etr}(-\Lambda U)\det(U)^{\alpha-(p+1)/2} \times \det\left(I_p-U\right)^{\beta-(p+1)/2}, \quad 0<U<I_p, \tag{2.10}$$

$$K_2(\alpha,\gamma,\Xi)\operatorname{etr}(-\Xi V)\det(V)^{\alpha-(p+1)/2}\det\left(I_p+V\right)^{-\gamma}, \quad V>0,$$

respectively, where $\alpha>(p-1)/2$, $\beta>(p-1)/2$, $-\infty<\gamma<\infty$, $\Lambda=\Lambda'$, and $\Xi(p\times p)>0$. These two distributions are designated by $U\sim KB_p(\alpha,\beta,\Lambda)$ and $V\sim KG_p(\alpha,\gamma,\Xi)$. It may be noted that the matrix variate Kummer-Dirichlet distributions are special cases of the matrix variate Liouville distribution.

Using certain transformations, generalized matrix variate Kummer-Dirichlet distributions are generated as given in the next two theorems.

Theorem 2.3. *Let* $(U_1,\ldots,U_n)\sim KD_p^I(\alpha_1,\ldots,\alpha_n,\beta,\Lambda)$ *and* $\Psi_1,\ldots,\Psi_n,\Omega$ *be symmetric matrices such that* $\Omega>0$ *and* $\Omega-\sum_{i=1}^n\Psi_i>0$. *Define*

$$Z_i=\left(\Omega-\sum_{i=1}^n\Psi_i\right)^{1/2}U_i\left(\Omega-\sum_{i=1}^n\Psi_i\right)^{1/2}+\Psi_i, \quad i=1,\ldots,n. \tag{2.11}$$

Then $(Z_1,\ldots,Z_n)$ *have the generalized matrix variate Kummer-Dirichlet type I distribution with pdf*

$$\frac{K_1(\alpha_1,\ldots,\alpha_n,\beta,\Lambda)}{\det\left(\Omega-\sum_{i=1}^n\Psi_i\right)^{\sum_{i=1}^r\alpha_i+\beta-(p+1)/2}} \times\frac{\prod_{i=1}^n\det\left(Z_i-\Psi_i\right)^{\alpha_i-(p+1)/2}\det\left(\Omega-\sum_{i=1}^n Z_i\right)^{\beta-(p+1)/2}}{\operatorname{etr}\left\{\left(\Omega-\sum_{i=1}^n\Psi_i\right)^{-1/2}\Lambda\left(\Omega-\sum_{i=1}^n\Psi_i\right)^{-1/2}\sum_{i=1}^n\left(Z_i-\Psi_i\right)\right\}},$$

$$\Psi_i<Z_i<\Omega,\ i=1,\ldots,n,\ \sum_{i=1}^n Z_i<\Omega. \tag{2.12}$$

Proof. Making the transformation $U_i=(\Omega-\sum_{i=1}^n\Psi_i)^{-1/2}(Z_i-\Psi_i)(\Omega-\sum_{i=1}^r\Psi_i)^{-1/2}$, $i=1,\ldots,n$, with Jacobian $J(U_1,\ldots,U_n\to Z_1,\ldots,Z_n)=\det(\Omega-\sum_{i=1}^n\Psi_i)^{-n(p+1)/2}$ in (2.5), we get (2.12). □

If $(Z_1,\ldots,Z_n)$ has the pdf (2.12), then we write $(Z_1,\ldots,Z_n)\sim GKD_p^I(\alpha_1,\ldots,\alpha_n,\beta,\Lambda;\Omega;\Psi_1,\ldots,\Psi_n)$. Note that $GKD_p^I(\alpha_1,\ldots,\alpha_n,\beta,\Lambda;I_p;0,\ldots,0)\equiv KD_p^I(\alpha_1,\ldots,\alpha_n,\beta,\Lambda)$.

Theorem 2.4. *Let* $(V_1,\ldots,V_n)\sim KD_p^{II}(\alpha_1,\ldots,\alpha_n,\gamma,\Xi)$ *and* $\Psi_1,\ldots,\Psi_n,\Omega$ *be symmetric matrices such that* $\Omega>0$ *and* $\Omega+\sum_{i=1}^n\Psi_i>0$. *Define*

$$Y_i=\left(\Omega+\sum_{i=1}^n\Psi_i\right)^{1/2}V_i\left(\Omega+\sum_{i=1}^n\Psi_i\right)^{1/2}+\Psi_i,\quad i=1,\ldots,n. \tag{2.13}$$

Then, $(Y_1,\ldots,Y_n)$ *have the generalized matrix variate Kummer-Dirichlet type II distribution with pdf*

$$\frac{K_2(\alpha_1,\ldots,\alpha_n,\gamma,\Xi)}{\det\left(\Omega+\sum_{i=1}^n\Psi_i\right)^{\sum_{i=1}^n\alpha_i-\gamma}}$$

$$\times\frac{\prod_{i=1}^n\det\left(Y_i-\Psi_i\right)^{\alpha_i-(p+1)/2}\det\left(\Omega+\sum_{i=1}^nY_i\right)^{-\gamma}}{\operatorname{etr}\left\{\left(\Omega+\sum_{i=1}^n\Psi_i\right)^{-1/2}\Xi\left(\Omega+\sum_{i=1}^n\Psi_i\right)^{-1/2}\sum_{i=1}^n\left(Y_i-\Psi_i\right)\right\}},$$

$$Y_i>\Psi_i,\ i=1,\ldots,n. \tag{2.14}$$

Proof. Making the transformation $V_i=(\Omega+\sum_{i=1}^n\Psi_i)^{-1/2}(Y_i-\Psi_i)(\Omega+\sum_{i=1}^n\Psi_i)^{-1/2}$, $i=1,\ldots,n$, with the Jacobian $J(V_1,\ldots,V_n\to Y_1,\ldots,Y_n)=\det(\Omega+\sum_{i=1}^n\Psi_i)^{-n(p+1)/2}$ in (2.6), we get (2.14). □

If $(Y_1,\ldots,Y_n)$ has pdf (2.14), then we write $(Y_1,\ldots,Y_n)\sim GKD_p^{II}(\alpha_1,\ldots,\alpha_n,\gamma,\Xi;\Omega;\Psi_1,\ldots,\Psi_n)$. In this case $GKD_p^{II}(\alpha_1,\ldots,\alpha_n,\gamma;I_p;0,\ldots,0)\equiv KD_p^{II}(\alpha_1,\ldots,\alpha_n;\gamma,\Xi)$.

3. Properties

In this section, we study certain properties of matrix variate Kummer-Dirichlet type I and II distributions. It may be noted that for $\Lambda=\lambda I_p$, $\Xi=\xi I_p$ densities (2.5) and (2.6) are orthogonally invariant. That is, for any fixed orthogonal matrix $\Gamma(p\times p)$, the distribution of $(\Gamma U_1\Gamma',\ldots,\Gamma U_n\Gamma')$ is the same as the distribution of $(U_1,\ldots,U_n)$, and similarly the distribution of $(\Gamma V_1\Gamma',\ldots,\Gamma V_n\Gamma')$ is the same as that of $(V_1,\ldots,V_n)$. Our next two results give marginal and conditional distributions.

Theorem 3.1. *If* $(U_1,\ldots,U_n)\sim KD_p^{I}(\alpha_1,\ldots,\alpha_n,\beta,\Lambda)$, *then the joint marginal pdf of* $U_1,\ldots,U_m$, $m\le n$, *is given by*

$$K_1\left(\alpha_1,\ldots,\alpha_m,\sum_{i=m+1}^{n}\alpha_i+\beta,\Lambda\right)\operatorname{etr}\left(-\Lambda\sum_{i=1}^{m}U_i\right)$$
$$\times\prod_{i=1}^{m}\det(U_i)^{\alpha_i-(p+1)/2}\det\left(I_p-\sum_{i=1}^{m}U_i\right)^{\sum_{i=m+1}^{n}\alpha_i+\beta-(p+1)/2}$$
$$\times{}_1F_1\left(\sum_{i=m+1}^{n}\alpha_i;\sum_{i=m+1}^{n}\alpha_i+\beta;-\Lambda\left(I_p-\sum_{i=1}^{m}U_i\right)\right),$$
$$0<U_i<I_p,\ 0<\sum_{i=1}^{m}U_i<I_p, \tag{3.1}$$

and the conditional density of $(U_{m+1},\ldots,U_n)\,|\,(U_1,\ldots,U_m)$ *is given by*

$$\frac{K_1(\alpha_1,\ldots,\alpha_n,\beta,\Lambda)}{K_1\left(\alpha_1,\ldots,\alpha_m,\sum_{i=m+1}^{n}\alpha_i+\beta,\Lambda\right)}$$
$$\times\frac{\operatorname{etr}\left(-\Lambda\sum_{i=m+1}^{n}U_i\right)}{\det\left(I_p-\sum_{i=1}^{m}U_i\right)^{\sum_{i=m+1}^{n}\alpha_i+\beta-(p+1)/2}}$$
$$\times\frac{\prod_{i=m+1}^{n}\det(U_i)^{\alpha_i-(p+1)/2}\det\left(I_p-\sum_{i=1}^{m}U_i-\sum_{i=m+1}^{n}U_i\right)^{\beta-(p+1)/2}}{{}_1F_1\left(\sum_{i=m+1}^{n}\alpha_i;\sum_{i=m+1}^{n}\alpha_i+\beta;-\Lambda\left(I_p-\sum_{i=1}^{m}U_i\right)\right)},$$
$$0<U_i<I_p-\sum_{i=1}^{m}U_i,\ i=m+1,\ldots,n,\ \sum_{i=m+1}^{n}U_i<I_p-\sum_{i=1}^{m}U_i. \tag{3.2}$$

Proof. First we find the marginal density of $U_1,\ldots,U_{n-1}$ by integrating out U_n from the joint density of $U_1,\ldots,U_n$ as

$$K_1(\alpha_1,\ldots,\alpha_n,\beta,\Lambda)\int_{0<U_n<I_p-\sum_{i=1}^{n-1}U_i}\operatorname{etr}\left(-\Lambda\sum_{i=1}^{n}U_i\right)$$
$$\times\prod_{i=1}^{n}\det(U_i)^{\alpha_i-(p+1)/2}\det\left(I_p-\sum_{i=1}^{n}U_i\right)^{\beta-(p+1)/2}dU_n. \tag{3.3}$$

Now, substituting $Z_n = (I_p - \sum_{i=1}^{n-1} U_i)^{-1/2} U_n (I_p - \sum_{i=1}^{n-1} U_i)^{-1/2}$ with Jacobian $J(U_n \to Z_n) = \det(I_p - \sum_{i=1}^{n-1} U_i)^{(p+1)/2}$ in (3.2), we get

$$
\begin{aligned}
&K_1(\alpha_1,\ldots,\alpha_n,\beta,\Lambda)\,\mathrm{etr}\left(-\Lambda\sum_{i=1}^{n-1}U_i\right)\\
&\quad\times\prod_{i=1}^{n-1}\det(U_i)^{\alpha_i-(p+1)/2}\det\left(I_p-\sum_{i=1}^{n-1}U_i\right)^{\alpha_n+\beta-(p+1)/2}\\
&\quad\times\int_{0<Z_n<I_p}\mathrm{etr}\left[-\left(I_p-\sum_{i=1}^{n-1}U_i\right)^{1/2}\Lambda\left(I_p-\sum_{i=1}^{n-1}U_i\right)^{1/2}Z_n\right]\\
&\qquad\qquad\times\det(Z_n)^{\alpha_n-(p+1)/2}\det(I_p-Z_n)^{\beta-(p+1)/2}dZ_n.
\end{aligned}
\tag{3.4}
$$

But

$$
\begin{aligned}
&K_1(\alpha_1,\ldots,\alpha_n,\beta,\Lambda)\\
&\quad\times\int_{0<Z_r<I_p}\mathrm{etr}\left[-\left(I_p-\sum_{i=1}^{n-1}U_i\right)^{1/2}\Lambda\left(I_p-\sum_{i=1}^{n-1}U_i\right)^{1/2}Z_n\right]\\
&\qquad\qquad\times\det(Z_n)^{\alpha_n-(p+1)/2}\det(I_p-Z_n)^{\beta-(p+1)/2}dZ_n\\
&=K_1(\alpha_1,\ldots,\alpha_n,\beta,\Lambda)\frac{\Gamma_p(\alpha_n)\Gamma_p(\beta)}{\Gamma_p(\alpha_n+\beta)}\,{}_1F_1\left(\alpha_n;\alpha_n+\beta;-\Lambda\left(I_p-\sum_{i=1}^{n-1}U_i\right)\right)\\
&=K_1(\alpha_1,\ldots,\alpha_{n-1},\alpha_n+\beta,\Lambda)\,{}_1F_1\left(\alpha_n;\alpha_n+\beta;-\Lambda\left(I_p-\sum_{i=1}^{n-1}U_i\right)\right).
\end{aligned}
\tag{3.5}
$$

Hence, we get the joint density of $(U_1,\ldots,U_{n-1})$ as

$$
\begin{aligned}
&K_1(\alpha_1,\ldots,\alpha_{n-1},\alpha_n+\beta,\Lambda)\,\mathrm{etr}\left(-\Lambda\sum_{i=1}^{n-1}U_i\right)\prod_{i=1}^{n-1}\det(U_i)^{\alpha_i-(p+1)/2}\\
&\quad\times\det\left(I_p-\sum_{i=1}^{n-1}U_i\right)^{\alpha_n+\beta-(p+1)/2}{}_1F_1\left(\alpha_n;\alpha_n+\beta;-\Lambda\left(I_p-\sum_{i=1}^{n-1}U_i\right)\right).
\end{aligned}
\tag{3.6}
$$

Repeating this procedure $n-m$ times gives the marginal density of $(U_1,\ldots,$ $U_m)$ as

$$K_1\left(\alpha_1,\ldots,\alpha_m,\sum_{i=m+1}^{n}\alpha_i+\beta,\Lambda\right)\operatorname{etr}\left(-\Lambda\sum_{i=1}^{m}U_i\right)$$
$$\times\prod_{i=1}^{m}\det\left(U_i\right)^{\alpha_i-(p+1)/2}\det\left(I_p-\sum_{i=1}^{m}U_i\right)^{\sum_{i=m+1}^{n}\alpha_i+\beta-(p+1)/2}$$
$$\times{}_1F_1\left(\sum_{i=m+1}^{n}\alpha_i;\sum_{i=m+1}^{n}\alpha_i+\beta;-\Lambda\left(I_p-\sum_{i=1}^{m}U_i\right)\right). \tag{3.7}$$

Now, the second part of the theorem follows immediately. □

Corollary 3.2. *If* $(U_1,\ldots,U_n)\sim KD_p^{I}(\alpha_1,\ldots,\alpha_n,\beta,\Lambda)$, *then the marginal pdf of* U_i, $i=1,\ldots,n$ *is given by*

$$K_1\left(\alpha_i,\sum_{j=1(\neq i)}^{n}\alpha_j+\beta,\Lambda\right)\operatorname{etr}\left(-\Lambda U_i\right)\det\left(U_i\right)^{\alpha_i-(p+1)/2}$$
$$\times\det\left(I_p-U_i\right)^{\sum_{j=1(\neq i)}^{n}\alpha_j+\beta-(p+1)/2}$$
$$\times{}_1F_1\left(\sum_{j=1(\neq i)}^{n}\alpha_j;\sum_{j=1(\neq i)}^{n}\alpha_j+\beta;-\Lambda\left(I_p-U_i\right)\right),\quad 0<U_i<I_p. \tag{3.8}$$

It is interesting to note that the marginal density of U_i does not belong to the Kummer-Beta family and differs by an additional factor containing confluent hypergeometric function ${}_1F_1$.

In Theorem 3.3 we give results on marginal and conditional distributions for Kummer-Dirichlet type II distribution. Before doing so, we need to give an integral that will be used in the derivation of marginal distribution. From (2.6) and (2.8), we have

$$\int_{X>0}\int_{Y>0}\operatorname{etr}\left[-\Xi(X+Y)\right]\det(Y)^{a_1-(p+1)/2}$$
$$\times\det(X)^{a_2-(p+1)/2}\det\left(I_p+X+Y\right)^{-b}dX\,dY \tag{3.9}$$
$$=\Gamma_p\left(a_1\right)\Gamma_p\left(a_2\right)\Psi\left(a_1+a_2,a_1+a_2-b+\frac{p+1}{2};\Xi\right),$$

where $\operatorname{Re}(a_1)>(p-1)/2$, $\operatorname{Re}(a_2)>(p-1)/2$ and $\operatorname{Re}(\Xi)>0$. Substituting

$W = (I_p + X)^{-1/2}\, Y(I_p + X)^{-1/2}$ with the Jacobian $J(Y \to W) = \det(I_p + X)^{(p+1)/2}$ in (3.9) and integrating W, we obtain

$$\begin{aligned} &\int_{X>0} \operatorname{etr}(-\Xi X)\det(X)^{a_2-(p+1)/2}\det\left(I_p + X\right)^{a_1-b} \\ &\qquad\times \Psi\left(a_1, a_1 - b + \frac{p+1}{2}; \Xi(I_p + X)\right) dX \\ &\quad= \Gamma_p(a_2)\Psi\left(a_1 + a_2, a_1 + a_2 - b + \frac{p+1}{2}; \Xi\right). \end{aligned} \tag{3.10}$$

Now we turn to our problem of finding the marginal and conditional distributions.

Theorem 3.3. *If* $(V_1,\ldots,V_n) \sim KD_p^{II}(\alpha_1,\ldots,\alpha_n,\gamma,\Xi)$, *then the joint marginal pdf of* $V_1,\ldots,V_m$, $m \le n$, *is given by*

$$\begin{aligned} &\Gamma_p\left(\sum_{i=m+1}^{n}\alpha_i\right) K_2\left(\alpha_1,\ldots,\alpha_m,\sum_{i=m+1}^{n}\alpha_i,\gamma,\Xi\right)\operatorname{etr}\left(-\Xi\sum_{i=1}^{m}V_i\right) \\ &\qquad\times \prod_{i=1}^{m}\det\left(V_i\right)^{\alpha_i-(p+1)/2}\det\left(I_p + \sum_{i=1}^{m}V_i\right)^{-\gamma+\sum_{i=m+1}^{n}\alpha_i} \\ &\qquad\times \Psi\left(\sum_{i=m+1}^{n}\alpha_i, \sum_{i=m+1}^{n}\alpha_i - \gamma + \frac{p+1}{2}; \Xi\left(I_p + \sum_{j=1}^{m}V_j\right)\right), \\ &\qquad\qquad V_j > 0,\ j = 1,\ldots,m, \end{aligned} \tag{3.11}$$

and the conditional density of $(V_{m+1},\ldots,V_n) \mid (V_1,\ldots,V_m)$ *is given by*

$$\begin{aligned} &\frac{K_2\left(\alpha_1,\ldots,\alpha_n,\gamma,\Xi\right)}{\Gamma_p\left(\sum_{i=m+1}^{n}\alpha_i\right)K_2\left(\alpha_1,\ldots,\alpha_m,\sum_{i=m+1}^{n}\alpha_i,\gamma,\Xi\right)} \\ &\times \frac{\operatorname{etr}\left(-\Xi\sum_{i=m+1}^{n}V_i\right)}{\det\left(I_p + \sum_{i=1}^{m}V_i\right)^{-\gamma+\sum_{i=m+1}^{n}\alpha_i}} \\ &\times \frac{\prod_{i=m+1}^{n}\det\left(V_i\right)^{\alpha_i-(p+1)/2}\det\left(I_p + \sum_{i=1}^{m}V_i + \sum_{i=m+1}^{n}V_i\right)^{-\gamma}}{\Psi\left(\sum_{i=m+1}^{n}\alpha_i, \sum_{i=m+1}^{n}\alpha_i - \gamma + (p+1)/2; \Xi\left(I_p + \sum_{j=1}^{m}V_j\right)\right)}, \\ &\qquad\qquad V_i > 0,\ i = m+1,\ldots,n. \end{aligned} \tag{3.12}$$

Proof. In this case, to obtain the marginal density of $V_1,\ldots,V_{n-1}$, we substitute $W_n = (I_p + \sum_{i=1}^{n-1}V_i)^{-1/2}V_n(I_p + \sum_{i=1}^{n-1}V_i)^{-1/2}$ with the

Jacobian $J(V_n \to W_n) = \det(I_p + \sum_{i=1}^{n-1} V_i)^{(p+1)/2}$. Thus, the joint density of $V_1, \ldots, V_{n-1}$ is obtained as

$$\begin{aligned}
&K_2(\alpha_1, \ldots, \alpha_n, \gamma, \Xi) \operatorname{etr}\left(-\Xi \sum_{i=1}^{n-1} V_i\right) \\
&\quad \times \prod_{i=1}^{n-1} \det(V_i)^{\alpha_i-(p+1)/2} \det\left(I_p + \sum_{i=1}^{n-1} V_i\right)^{-\gamma+\alpha_n} \\
&\quad \times \int_{W_n>0} \operatorname{etr}\left[-\left(I_p + \sum_{i=1}^{n-1} V_i\right)^{1/2} \Xi \left(I_p + \sum_{i=1}^{n-1} V_i\right)^{1/2} W_n\right] \\
&\qquad\qquad \times \det(W_n)^{\alpha_n-(p+1)/2} \det(I_p + W_n)^{-\gamma}\, dW_n \qquad (3.13) \\
&\quad = \Gamma_p(\alpha_n) K_2(\alpha_1, \ldots, \alpha_n, \gamma, \Xi) \operatorname{etr}\left(-\Xi \sum_{i=1}^{n-1} V_i\right) \\
&\quad \times \prod_{i=1}^{n-1} \det(V_i)^{\alpha_i-(p+1)/2} \det\left(I_p + \sum_{i=1}^{n-1} V_i\right)^{-\gamma+\alpha_n} \\
&\quad \times \Psi\left(\alpha_n, \alpha_n - \gamma + \frac{p+1}{2}; \Xi\left(I_p + \sum_{i=1}^{n-1} V_i\right)\right).
\end{aligned}$$

Further, substituting $W_{n-1} = (I_p + \sum_{i=1}^{n-2} V_i)^{-1/2} V_{n-1} (I_p + \sum_{i=1}^{n-2} V_i)^{-1/2}$ with the Jacobian $J(V_{n-1} \to W_{n-1}) = \det(I_p + \sum_{i=1}^{n-2} V_i)^{(p+1)/2}$ in (3.13) and integrating W_{n-1} using (3.10), we get the joint marginal density of $V_1, \ldots, V_{n-2}$ as

$$\begin{aligned}
&\Gamma_p(\alpha_n) K_2(\alpha_1, \ldots, \alpha_n, \gamma, \Xi) \operatorname{etr}\left(-\Xi \sum_{i=1}^{n-2} V_i\right) \\
&\quad \times \prod_{i=1}^{n-2} \det(V_i)^{\alpha_i-(p+1)/2} \det\left(I_p + \sum_{i=1}^{n-2} V_i\right)^{-\gamma+\alpha_n+\alpha_{n-1}} \\
&\quad \times \int_{W_{n-1}>0} \operatorname{etr}\left[-\left(I_p + \sum_{i=1}^{n-2} V_i\right)^{1/2} \Xi \left(I_p + \sum_{i=1}^{n-2} V_i\right)^{1/2} W_{n-1}\right] \\
&\qquad\qquad \times \det(W_{n-1})^{\alpha_{n-1}-(p+1)/2} \det(I_p + W_{n-1})^{-\gamma+\alpha_n}
\end{aligned}$$

$$\times \Psi\left(\alpha_n, \alpha_n - \gamma + \frac{p+1}{2}; \left(I_p + \sum_{i=1}^{n-2} V_i\right)^{1/2} \times \Xi \left(I_p + \sum_{i=1}^{n-2} V_i\right)^{1/2} W_{n-1}\right) dW_{n-1}$$

$$= \Gamma_p(\alpha_n)\Gamma_p(\alpha_{n-1}) K_2(\alpha_1, \ldots, \alpha_n, \gamma, \Xi) \operatorname{etr}\left(-\Xi \sum_{i=1}^{n-2} V_i\right) \times \prod_{i=1}^{n-2} \det(V_i)^{\alpha_i - (p+1)/2} \det\left(I_p + \sum_{i=1}^{n-2} V_i\right)^{-\gamma+\alpha_n+\alpha_{n-1}}$$
$$\times \Psi\left(\alpha_n + \alpha_{n-1}, \alpha_n + \alpha_{n-1} - \gamma + \frac{p+1}{2}; \Xi\left(I_p + \sum_{i=1}^{n-2} V_i\right)\right). \tag{3.14}$$

Integrating out $V_{n-2}, \ldots, V_{m+1}$ similarly, we get the marginal density of $V_1, \ldots, V_m$ as

$$\prod_{i=m+1}^{n} \Gamma_p(\alpha_i) K_2(\alpha_1, \ldots, \alpha_n, \gamma, \Xi) \operatorname{etr}\left(-\Xi \sum_{i=1}^{m} V_i\right) \times \prod_{i=1}^{m} \det(V_i)^{\alpha_i - (p+1)/2} \det\left(I_p + \sum_{i=1}^{m} V_i\right)^{-\gamma + \sum_{i=m+1}^{n} \alpha_i}$$
$$\times \Psi\left(\sum_{i=m+1}^{n} \alpha_i, \sum_{i=m+1}^{n} \alpha_i - \gamma + \frac{p+1}{2}; \Xi\left(I_p + \sum_{i=1}^{m} V_i\right)\right). \tag{3.15}$$

The final expression of the marginal density of $V_1, \ldots, V_m$ is obtained by noting that

$$\prod_{i=m+1}^{n} \Gamma_p(\alpha_i) K_2(\alpha_1, \ldots, \alpha_n, \gamma, \Xi) = \Gamma_p\left(\sum_{i=m+1}^{n} \alpha_i\right) K_2\left(\alpha_1, \ldots, \alpha_m, \sum_{i=m+1}^{n} \alpha_i, \gamma, \Xi\right). \tag{3.16}$$

The derivation of the conditional density is now straightforward. □

Corollary 3.4. *If* $(V_1,\ldots,V_n) \sim KD_p^{II}(\alpha_1,\ldots,\alpha_n,\gamma,\Xi)$, *then the density of* V_i, $i=1,\ldots,n$ *is given by*

$$\Gamma_p\left(\sum_{j=1(\neq i)}^{n}\alpha_j\right) K_2\left(\alpha_i,\sum_{j=1(\neq i)}^{n}\alpha_j,\gamma,\Xi\right)\operatorname{etr}\left(-\Xi V_i\right)$$
$$\times\det\left(V_i\right)^{\alpha_i-(p+1)/2}\det\left(I_p+V_i\right)^{-\gamma+\sum_{j=1(\neq i)}^{n}\alpha_j}$$
$$\times\Psi\left(\sum_{j=1(\neq i)}^{n}\alpha_j,\sum_{j=1(\neq i)}^{n}\alpha_j-\gamma+\frac{p+1}{2};\Xi(I_p+V_i)\right),\quad V_i>0. \tag{3.17}$$

Note that the marginal density of V_i differs from the Kummer-Gamma density. It is a pdf with an additional factor containing confluent hypergeometric function Ψ.

Theorem 3.5. *Let* $(U_1,\ldots,U_n)\sim KD_p^{I}(\alpha_1,\ldots,\alpha_n,\beta,I_p)$ *and define*

$$W_i=\left(I_p-\sum_{i=1}^{m}U_i\right)^{-1/2}U_i\left(I_p-\sum_{i=1}^{m}U_i\right)^{-1/2},\quad i=m+1,\ldots,n. \tag{3.18}$$

Then the joint density of $(W_{m+1},\ldots,W_n)$ *is given by*

$$\frac{\Gamma_p\left(\sum_{j=m+1}^{n}\alpha_j+\beta\right)}{\prod_{i=m+1}^{n}\Gamma_p(\alpha_i)\Gamma_p(\beta)}\left\{{}_1F_1\left(\sum_{i=1}^{n}\alpha_i;\sum_{i=1}^{n}\alpha_i+\beta;-I_p\right)\right\}^{-1}$$
$$\times\operatorname{etr}\left(-\sum_{i=m+1}^{n}W_i\right)$$
$$\times\prod_{i=m+1}^{n}\det\left(W_i\right)^{\alpha_i-(p+1)/2}\det\left(I_p-\sum_{i=m+1}^{n}W_i\right)^{\beta-(p+1)/2}$$
$$\times{}_1F_1\left(\sum_{i=1}^{m}\alpha_i;\sum_{j=1}^{n}\alpha_j+\beta;-\left(I_p-\sum_{i=m+1}^{n}W_i\right)\right),$$
$$0<W_i<I_p,\ \sum_{i=1}^{m}W_i<I_p. \tag{3.19}$$

Proof. Transforming $W_i=(I_p-\sum_{i=1}^{m}U_i)^{-1/2}U_i(I_p-\sum_{i=1}^{m}U_i)^{-1/2}$, $i=m+1,\ldots,n$ with Jacobian $J(U_{m+1},\ldots,U_n\to W_{m+1},\ldots,W_n)=\det(I_p-$

$\sum_{i=1}^{m} U_i)^{(n-m)(p+1)/2}$, in the joint density of $(U_1,\dots,U_n)$, we get

$$K_1(\alpha_1,\dots,\alpha_n,\beta,I_p)\,\mathrm{etr}\left[-\sum_{i=m+1}^{n} W_i-\left(\sum_{i=1}^{m} U_i\right)\left(I_p-\sum_{i=m+1}^{n} W_i\right)\right]$$
$$\times\prod_{i=1}^{m}\det(U_i)^{\alpha_i-(p+1)/2}\det\left(I_p-\sum_{i=1}^{m} U_i\right)^{\sum_{j=m+1}^{n}\alpha_j+\beta-(p+1)/2}$$
$$\times\prod_{i=m+1}^{n}\det(W_i)^{\alpha_i-(p+1)/2}\det\left(I_p-\sum_{i=m+1}^{n} W_i\right)^{\beta-(p+1)/2},$$
$$0<U_i<I_p,\ i=m+1,\dots,n,\ \sum_{i=m+1}^{n} U_i<I_p,$$
$$0<W_i<I_p,\ i=1,\dots,m,\ \sum_{i=1}^{m} W_i<I_p. \tag{3.20}$$

Now, integrating $U_1,\dots,U_m$,

$$\underset{\substack{0<\sum_{i=1}^{m}U_i<I_p\\0<U_i<I_p}}{\int\cdots\int}\mathrm{etr}\left[-\left(\sum_{i=1}^{m} U_i\right)\left(I_p-\sum_{i=m+1}^{n} W_i\right)\right]$$
$$\times\prod_{i=1}^{m}\det(U_i)^{\alpha_i-(p+1)/2}$$
$$\times\det\left(I_p-\sum_{i=1}^{m} U_i\right)^{\sum_{i=m+1}^{n}\alpha_i+\beta-(p+1)/2}\prod_{i=1}^{m} dU_i$$
$$=\frac{\prod_{i=1}^{m}\Gamma_p(\alpha_i)}{\Gamma_p\left(\sum_{i=1}^{m}\alpha_i\right)}\int_{0<U<I_p}\mathrm{etr}\left[-\left(I_p-\sum_{i=m+1}^{n} W_i\right)U\right]$$
$$\times\det(U)^{\sum_{i=1}^{m}\alpha_i-(p+1)/2}$$
$$\times\det(I_p-U)^{\sum_{i=m+1}^{n}\alpha_i+\beta-(p+1)/2}dU$$
$$=\frac{\prod_{i=1}^{m}\Gamma_p(\alpha_i)\Gamma_p\left(\sum_{i=m+1}^{n}\alpha_i+\beta\right)}{\Gamma_p\left(\sum_{i=1}^{n}\alpha_i+\beta\right)}$$
$$\times{}_1F_1\left(\sum_{i=1}^{m}\alpha_i;\sum_{i=1}^{n}\alpha_i+\beta;-\left(I_p-\sum_{i=m+1}^{n} W_i\right)\right), \tag{3.21}$$

and using

$$K_1(\alpha_1,\ldots,\alpha_n,\beta,I_p)\frac{\prod_{i=1}^m\Gamma_p(\alpha_i)\Gamma_p\left(\sum_{i=m+1}^n\alpha_i+\beta\right)}{\Gamma_p\left(\sum_{i=1}^n\alpha_i+\beta\right)}$$

$$=\frac{\Gamma_p\left(\sum_{i=m+1}^n\alpha_i+\beta\right)}{\prod_{i=m+1}^n\Gamma_p(\alpha_i)\Gamma_p(\beta)}\left\{{}_1F_1\left(\sum_{i=1}^n\alpha_i;\sum_{i=1}^n\alpha_i+\beta;-I_p\right)\right\}^{-1}, \tag{3.22}$$

we get the desired result. □

Theorem 3.6. *Let* $(V_1,\ldots,V_n)\sim KD_p^{II}(\alpha_1,\ldots,\alpha_n,\gamma,I_p)$ *and define*

$$Z_i=\left(I_p+\sum_{i=1}^m V_i\right)^{-1/2}V_i\left(I_p+\sum_{i=1}^m V_i\right)^{-1/2},\quad i=m+1,\ldots,n. \tag{3.23}$$

Then the pdf of $(Z_{m+1},\ldots,Z_n)$ *is given by*

$$\left\{\prod_{i=m+1}^n\Gamma_p(\alpha_i)\Psi\left(\sum_{i=1}^n\alpha_i,\sum_{i=1}^n\alpha_i-\gamma+\frac{p+1}{2};I_p\right)\right\}^{-1}$$

$$\times\operatorname{etr}\left(-\sum_{i=m+1}^n Z_i\right)\prod_{i=m+1}^n\det(Z_i)^{\alpha_i-(p+1)/2}\det\left(I_p+\sum_{i=m+1}^n Z_i\right)^{-\gamma}$$

$$\times\Psi\left(\sum_{i=1}^m\alpha_i,\sum_{i=1}^n\alpha_i-\gamma+\frac{p+1}{2};\left(I_p+\sum_{i=m+1}^n Z_i\right)\right),\quad Z_i>0. \tag{3.24}$$

Proof. The proof is similar to the proof of Theorem 3.5. □

Theorem 3.7. *Let* $(U_1,\ldots,U_n)\sim KD_p^{I}(\alpha_1,\ldots,\alpha_n,\beta,\Lambda)$ *and define* $U=\sum_{i=1}^n U_i$ *and* $X_i=U^{-1/2}U_iU^{-1/2}$, $i=1,\ldots,n-1$. *Then*

(i) $(X_1,\ldots,X_{n-1})$ *and* U *are independent*,
(ii) $(X_1,\ldots,X_{n-1})\sim D_p^{I}(\alpha_1,\ldots,\alpha_{n-1};\alpha_n)$, *and*
(iii) $U\sim KB_p(\sum_{i=1}^n\alpha_i,\beta,\Lambda)$.

Proof. Substituting $U_i=U^{1/2}X_iU^{1/2}$, $i=1,\ldots,n-1$ and $U_n=U^{1/2}(I_p-\sum_{i=1}^{n-1}X_i)U^{1/2}$ with the Jacobian $J(U_1,\ldots,U_{n-1},U_n\to X_1,\ldots,X_{n-1},U)=$

$\det(U)^{(n-1)(p+1)/2}$ in the joint density of $(U_1,\dots,U_n)$, we get the desired result. □

Theorem 3.8. *Let* $(V_1,\dots,V_n) \sim \mathrm{KD}_p^{\mathrm{II}}(\alpha_1,\dots,\alpha_n,\gamma,\Xi)$ *and define* $V = \sum_{i=1}^n V_i$ *and* $Y_i = V^{-1/2}V_iV^{-1/2}$, $i=1,\dots,n-1$. *Then*

(i) $(Y_1,\dots,Y_{n-1})$ *and* V *are independent*,
(ii) $(Y_1,\dots,Y_{n-1}) \sim D_p^{\mathrm{I}}(\alpha_1,\dots,\alpha_{n-1};\alpha_n)$, *and*
(iii) $V \sim \mathrm{KG}_p(\sum_{i=1}^n \alpha_i,\gamma,\Xi)$.

Proof. The proof is similar to the proof of Theorem 3.7. □

In Theorems 3.9 and 3.10, we derive the joint pdfs of partial sums of random matrices distributed as matrix variate Kummer-Dirichlet type I or II.

Theorem 3.9. *Let* $(U_1,\dots,U_n) \sim \mathrm{KD}_p^{\mathrm{I}}(\alpha_1,\dots,\alpha_n,\beta,\Lambda)$ *and define*

$$U_{(i)} = \sum_{j=n_{i-1}^*+1}^{n_i^*} U_j, \quad \alpha_{(i)} = \sum_{j=n_{i-1}^*+1}^{n_i^*} \alpha_j, \quad n_0^* = 0,\ n_i^* = \sum_{j=1}^{i} n_j,\ i=1,\dots,\ell. \tag{3.25}$$

Then $(U_{(1)},\dots,U_{(\ell)}) \sim \mathrm{KD}_p^{\mathrm{I}}(\alpha_{(1)},\dots,\alpha_{(\ell)},\beta,\Lambda)$.

Proof. Make the transformation

$$U_{(i)} = \sum_{j=n_{i-1}^*+1}^{n_i^*} U_j, \qquad W_j = U_{(i)}^{-1/2}U_jU_{(i)}^{-1/2}, \tag{3.26}$$

$j = n_{i-1}^*+1,\dots,n_i^*-1$, $i=1,\dots,\ell$. The Jacobian of this transformation is given by

$$\begin{aligned} &J\big(U_1,\dots,U_n \longrightarrow W_1,\dots,W_{n_1-1},U_{(1)},\dots,W_{n_{\ell-1}^*+1},\dots,W_{n-1},U_{(\ell)}\big) \\ &\quad= \prod_{i=1}^{\ell} J\big(U_{n_{i-1}^*+1},\dots,U_{n_i^*} \longrightarrow W_{n_{i-1}^*+1},\dots,W_{n_i^*-1},U_{(i)}\big) \\ &\quad= \prod_{i=1}^{\ell} \det\big(U_{(i)}\big)^{(n_i-1)(p+1)/2}. \end{aligned} \tag{3.27}$$

Now, substituting from (3.26) and (3.27) in the joint density of $(U_1,\dots,U_n)$ given by (2.5), we get the joint density of $W_{n_{i-1}^*+1},\dots,W_{n_i^*-1},U_{(i)}$, where $i = 1, \ldots, \ell$, as

$$
\begin{aligned}
& K_1(\alpha_1, \ldots, \alpha_n, \beta, \Lambda) \operatorname{etr}\left(-\Lambda \sum_{i=1}^{\ell} U_{(i)}\right) \\
& \quad \times \prod_{i=1}^{\ell} \det\left(U_{(i)}\right)^{\alpha_{(i)}-(p+1)/2} \det\left(I_p - \sum_{i=1}^{\ell} U_{(i)}\right)^{\beta-(p+1)/2} \\
& \quad \times \prod_{i=1}^{\ell} \left\{ \prod_{j=n_{i-1}^*+1}^{n_i^*-1} \det\left(W_j\right)^{\alpha_j-(p+1)/2} \right. \\
& \qquad\qquad \left. \times \det\left(I_p - \sum_{j=n_{i-1}^*+1}^{n_i^*-1} W_j\right)^{\alpha_{n_i^*}-(p+1)/2} \right\},
\end{aligned}
\tag{3.28}
$$

where $0 < U_{(i)} < I_p$, $\sum_{i=1}^{\ell} U_{(i)} < I_p$, $0 < W_j < I_p$, $j = n_{i-1}^* + 1, \ldots, n_i^* - 1$, $\sum_{j=n_{i-1}^*+1}^{n_i^*-1} W_j < I_p$, $i = 1, \ldots, \ell$. From (3.28), it is easy to see that $(U_{(1)}, \ldots, U_{(\ell)})$ and $(W_{n_{i-1}^*+1}, \ldots, W_{n_i^*-1})$, $i = 1, \ldots, \ell$, are independently distributed. Further, $(U_{(1)}, \ldots, U_{(\ell)}) \sim \mathrm{KD}_p^{\mathrm{I}}(\alpha_{(1)}, \ldots, \alpha_{(\ell)}, \beta, \Lambda)$ and $(W_{n_{i-1}^*+1}, \ldots, W_{n_i^*-1}) \sim \mathrm{D}_p^{\mathrm{I}}(\alpha_{n_{i-1}^*+1}, \ldots, \alpha_{n_i^*-1}; \alpha_{n_i^*})$, where $i = 1, \ldots, \ell$. □

When $\ell = 1$, $\sum_{i=1}^{n} U_i \sim \mathrm{KB}_p(\sum_{i=1}^{n} \alpha_i, \beta, \Lambda)$.

Theorem 3.10. *Let* $(V_1, \ldots, V_n) \sim \mathrm{KD}_p^{\mathrm{II}}(\alpha_1, \ldots, \alpha_n, \gamma, \Xi)$ *and define*

$$
V_{(i)} = \sum_{j=n_{i-1}^*+1}^{n_i^*} V_j, \quad \alpha_{(i)} = \sum_{j=n_{i-1}^*+1}^{n_i^*} \alpha_j, \quad n_0^* = 0,\ n_i^* = \sum_{j=1}^{i} n_j,\ i = 1, \ldots, \ell.
\tag{3.29}
$$

Then $(V_{(1)}, \ldots, V_{(\ell)}) \sim \mathrm{KD}_p^{\mathrm{II}}(\alpha_{(1)}, \ldots, \alpha_{(\ell)}, \gamma, \Xi)$.

Proof. Make the transformation

$$
V_{(i)} = \sum_{j=n_{i-1}^*+1}^{n_i^*} V_j, \qquad Z_j = V_{(i)}^{-1/2} V_j V_{(i)}^{-1/2},
\tag{3.30}
$$

where $j = n_{i-1}^* + 1, \ldots, n_i^* - 1$, $i = 1, \ldots, \ell$. The Jacobian of this transformation is given by

$$\begin{aligned} J\big(V_1,\ldots,V_n &\longrightarrow Z_1,\ldots,Z_{n_1-1},V_{(1)},\ldots,Z_{n^*_{\ell-1}+1},\ldots,Z_{n-1},V_{(\ell)}\big) \\ &= \prod_{i=1}^{\ell} J\big(V_{n^*_{i-1}+1},\ldots,V_{n^*_i} \longrightarrow Z_{n^*_{i-1}+1},\ldots,Z_{n^*_i-1},V_{(i)}\big) \\ &= \prod_{i=1}^{\ell} \det\left(V_{(i)}\right)^{(n_i-1)(p+1)/2}. \end{aligned} \tag{3.31}$$

Now, substituting from (3.30) and (3.31) in the joint density of $(V_1,\ldots,V_n)$ given by (2.6), it can easily be shown that $(V_{(1)},\ldots,V_{(\ell)})$ and $(Z_{n^*_{i-1}+1},\ldots, Z_{n^*_i-1})$, $i=1,\ldots,\ell$, are independently distributed. Further, $(V_{(1)},\ldots,V_{(\ell)}) \sim KD^{II}_p(\alpha_{(1)},\ldots,\alpha_{(\ell)},\gamma,\Xi)$ and $(Z_{n^*_{i-1}+1},\ldots,Z_{n^*_i-1}) \sim D^{I}_p(\alpha_{n^*_{i-1}+1},\ldots, \alpha_{n^*_i-1};\alpha_{n^*_i})$, $i=1,\ldots,\ell$. □

When $\ell=1$, the distribution of $\sum_{i=1}^n V_i$ is Kummer-Gamma with parameters $\sum_{i=1}^n \alpha_i$, γ and Ξ. From the joint density of $U_1,\ldots,U_n$, we have

$$\begin{aligned} E&\left[\prod_{i=1}^n \det\left(U_i\right)^{r_i}\right] \\ &= K_1\left(\alpha_1,\ldots,\alpha_n,\beta,\Lambda\right) \underset{\substack{0<\sum_{i=1}^n U_i<I_p \\ 0<U_i<I_p}}{\int\cdots\int} \operatorname{etr}\left(-\Lambda\sum_{i=1}^n U_i\right) \\ &\quad\times \prod_{i=1}^n \det\left(U_i\right)^{\alpha_i+r_i-(p+1)/2} \det\left(I_p-\sum_{i=1}^n U_i\right)^{\beta-(p+1)/2} \prod_{i=1}^n dU_i \\ &= \frac{K_1\left(\alpha_1,\ldots,\alpha_n,\beta,\Lambda\right)}{K_1\left(\alpha_1+r_1,\ldots,\alpha_n+r_n,\beta,\Lambda\right)}, \quad \operatorname{Re}\left(\alpha_i+r_i\right) > \frac{p-1}{2} \\ &= \frac{\prod_{i=1}^n \Gamma_p\left(\alpha_i+r_i\right)\Gamma_p\left(\sum_{i=1}^n \alpha_i+\beta\right)}{\prod_{i=1}^n \Gamma_p\left(\alpha_i\right)\Gamma_p\left[\sum_{i=1}^n\left(\alpha_i+r_i\right)+\beta\right]} \\ &\quad\times \frac{{}_1F_1\left(\sum_{i=1}^n\left(\alpha_i+r_i\right);\sum_{i=1}^n\left(\alpha_i+r_i\right)+\beta;-\Lambda\right)}{{}_1F_1\left(\sum_{i=1}^n \alpha_i;\sum_{i=1}^n \alpha_i+\beta;-\Lambda\right)}, \\ &\qquad\qquad \operatorname{Re}\left(\alpha_i+r_i\right) > \frac{p-1}{2}, \end{aligned} \tag{3.32}$$

where the last step has been obtained using (2.7). Further

$$
\begin{aligned}
&E\left[\det\left(I_p-\sum_{i=1}^{n}U_i\right)^{h}\right]\\
&\quad=K_1(\alpha_1,\ldots,\alpha_n,\beta,\Lambda)\int\cdots\int_{\substack{0<\sum_{i=1}^{n}U_i<I_p\\ U_i>0}}\operatorname{etr}\left(-\Lambda\sum_{i=1}^{n}U_i\right)\\
&\qquad\times\prod_{i=1}^{n}\det(U_i)^{\alpha_i-(p+1)/2}\det\left(I_p-\sum_{i=1}^{n}U_i\right)^{\beta+h-(p+1)/2}\prod_{i=1}^{n}dU_i\\
&\quad=\frac{K_1(\alpha_1,\ldots,\alpha_n,\beta,\Lambda)}{K_1(\alpha_1,\ldots,\alpha_n,\beta+h,\Lambda)},\quad \operatorname{Re}(h)>-\beta+\frac{p-1}{2}\\
&\quad=\frac{\Gamma_p(\beta+h)\Gamma_p\left(\sum_{i=1}^{n}\alpha_i+\beta\right)}{\Gamma_p(\beta)\Gamma_p\left(\sum_{i=1}^{n}\alpha_i+\beta+h\right)}\frac{{}_1F_1\left(\sum_{i=1}^{n}\alpha_i;\sum_{i=1}^{n}\alpha_i+\beta+h;-\Lambda\right)}{{}_1F_1\left(\sum_{i=1}^{n}\alpha_i;\sum_{i=1}^{n}\alpha_i+\beta;-\Lambda\right)},\\
&\qquad\qquad\operatorname{Re}(h)>-\beta+\frac{p-1}{2}.
\end{aligned}
\tag{3.33}
$$

Alternately, the above moment expression can be obtained by noting that $\sum_{i=1}^{n}U_i$ has Kummer-Beta distribution. Similarly, for Kummer-Dirichlet type II matrices

$$
\begin{aligned}
&E\left[\prod_{i=1}^{n}\det(V_i)^{r_i}\right]\\
&\quad=\prod_{i=1}^{n}\frac{\Gamma_p(\alpha_i+r_i)}{\Gamma_p(\alpha_i)}\frac{\Psi\left(\sum_{i=1}^{n}(\alpha_i+r_i),\sum_{i=1}^{n}(\alpha_i+r_i)-\gamma+(p+1)/2;\Xi\right)}{\Psi\left(\sum_{i=1}^{n}\alpha_i,\sum_{i=1}^{n}\alpha_i-\gamma+(p+1)/2;\Xi\right)},\\
&\qquad\qquad\operatorname{Re}(\alpha_i+r_i)>\frac{p-1}{2},\\
&E\left[\det\left(I_p+\sum_{i=1}^{n}V_i\right)^{-h}\right]=\frac{\Psi\left(\sum_{i=1}^{n}\alpha_i,\sum_{i=1}^{n}\alpha_i-\gamma-h+(p+1)/2;\Xi\right)}{\Psi\left(\sum_{i=1}^{n}\alpha_i,\sum_{i=1}^{n}\alpha_i-\gamma+(p+1)/2;\Xi\right)}.
\end{aligned}
\tag{3.34}
$$

Next two results give certain asymptotic distributions for the Kummer-Dirichlet type I and type II distributions (see Javier and Gupta [4]).

Theorem 3.11. *Let* $(U_1,\ldots,U_n)\sim KD_p^{I}(\alpha_1,\ldots,\alpha_n,\beta,\beta\Lambda)$ *and* $W=(W_1,\ldots,W_n)$ *be defined by* $W_i=\beta U_i$, $i=1,\ldots,n$. *Then* W *is asymptotically distributed as a product of independent matrix variate gamma densities*;

more specifically

$$\lim_{\beta\to\infty} f(W) = \prod_{i=1}^{n} \frac{\det\left(W_i\right)^{\alpha_i-(p+1)/2} \operatorname{etr}\left[-\left(I_p+\Lambda\right)W_i\right]}{\det\left(I_p+\Lambda\right)^{\alpha_i}\Gamma_p\left(\alpha_i\right)}, \tag{3.35}$$

where $f(W)$ *denotes the density of the matrix* W.

Proof. In the joint density of $(U_1,\ldots,U_n)$ given by (2.5) transform $W_i = \beta U_i$, $i = 1,\ldots,n$ with the Jacobian $J(U_1,\ldots,U_n \to W_1,\ldots,W_n) = \beta^{-np(p+1)/2}$. The density of $W = (W_1,\ldots,W_n)$ is given by

$$\begin{aligned} f(W) = {} & \frac{\Gamma_p\left(\sum_{i=1}^n \alpha_i+\beta\right)}{\Gamma_p(\beta)} \beta^{-p\sum_{i=1}^n \alpha_i} \left\{ {}_1F_1\left(\sum_{i=1}^{n}\alpha_i;\sum_{i=1}^{n}\alpha_i+\beta;-\beta\Lambda\right)\right\}^{-1} \\ & \times \operatorname{etr}\left(-\Lambda\sum_{i=1}^{n} W_i\right)\left\{\prod_{i=1}^{n}\frac{\det\left(W_i\right)^{\alpha_i-(p+1)/2}}{\Gamma_p\left(\alpha_i\right)}\right\} \\ & \times \det\left(I_p-\frac{1}{\beta}\sum_{i=1}^{n} W_i\right)^{\beta-(p+1)/2}. \end{aligned} \tag{3.36}$$

The result follows, since

$$\begin{aligned} & \lim_{\beta\to\infty} \frac{\Gamma_p\left(\sum_{i=1}^n \alpha_i+\beta\right)}{\Gamma_p(\beta)} \beta^{-p\sum_{i=1}^n \alpha_i} = 1, \\ & \lim_{\beta\to\infty} {}_1F_1\left(\sum_{i=1}^{n}\alpha_i;\sum_{i=1}^{n}\alpha_i+\beta;-\beta\Lambda\right) \\ & \quad = {}_1F_0\left(\sum_{i=1}^{n}\alpha_i;-\Lambda\right) = \det\left(I_p+\Lambda\right)^{-\sum_{i=1}^n \alpha_i}, \\ & \lim_{\beta\to\infty} \det\left(I_p-\frac{1}{\beta}\sum_{i=1}^{n} W_i\right)^{\beta-(p+1)/2} = \operatorname{etr}\left(-\sum_{i=1}^{n} W_i\right). \end{aligned} \tag{3.37}$$

□

An analogous result for Kummer-Dirichlet type II distribution is shown to be the following.

Theorem 3.12. *Let* $(V_1,\ldots,V_n) \sim KD_p^{II}(\alpha_1,\ldots,\alpha_n,\gamma,|\gamma|\Xi)$, $\gamma \neq 0$, *and* $W = (W_1,\ldots,W_n)$ *be defined by* $W_i = |\gamma|V_i$, $i = 1,\ldots,n$. *Then*, W *is asymptotically distributed as a product of independent matrix variate gamma* *densities; more specifically*

$$\lim_{|\gamma|\to\infty} g(W)=\prod_{i=1}^{n}\frac{\det(W_i)^{\alpha_i-(p+1)/2}\operatorname{etr}\left[-(I_p+\Xi)W_i\right]}{\det(I_p+\Xi)^{\alpha_i}\Gamma_p(\alpha_i)}, \tag{3.38}$$

where $g(W)$ *denotes the density of the matrix* W*.*

Proof. We prove the result for $\gamma>0$. The proof for $\gamma<0$ follows similar steps. The density of $W=(W_1,\dots,W_n)$ is given by

$$\begin{aligned} g(W)=\gamma^{-p\sum_{i=1}^{n}\alpha_i}&\left\{\Psi\left(\sum_{i=1}^{n}\alpha_i,\sum_{i=1}^{n}\alpha_i-\gamma+\frac{p+1}{2};\gamma\Xi\right)\right\}^{-1}\\ &\times\operatorname{etr}\left(-\Xi\sum_{i=1}^{n}W_i\right)\left\{\prod_{i=1}^{n}\frac{\det(W_i)^{\alpha_i-(p+1)/2}}{\Gamma_p(\alpha_i)}\right\}\\ &\times\det\left(I_p+\frac{1}{\gamma}\sum_{i=1}^{n}W_i\right)^{-\gamma}. \end{aligned} \tag{3.39}$$

The result follows, since

$$\begin{aligned} \lim_{\gamma\to\infty}\gamma^{p\sum_{i=1}^{n}\alpha_i}\Psi\left(\sum_{i=1}^{n}\alpha_i,\sum_{i=1}^{n}\alpha_i-\gamma+\frac{p+1}{2};\gamma\Xi\right)&=\det(I_p+\Xi)^{-\sum_{i=1}^{n}\alpha_i},\\ \lim_{\gamma\to\infty}\det\left(I_p+\frac{1}{\gamma}\sum_{i=1}^{n}W_i\right)^{-\gamma}&=\operatorname{etr}\left(-\sum_{i=1}^{n}W_i\right). \end{aligned} \tag{3.40}$$

□

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References

[1] C. Armero and M. J. Bayarri, *A Bayesian analysis of a queueing system with unlimited service*, J. Statist. Plann. Inference **58** (1997), no. 2, 241–261. CMP 1 450 015. Zbl 880.62025.

[2] M. B. Gordy, *Computationally convenient distributional assumptions for common-value auctions*, Comput. Econom. **12** (1998), 61–78. Zbl 912.90093.

[3] A. K. Gupta and D. K. Nagar, *Matrix Variate Distributions*, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, vol. 104, Chapman & Hall/CRC, Florida, 2000. MR 2001d:62055. Zbl 935.62064.

[4] W. R. Javier and A. K. Gupta, *On generalized matric variate beta distributions*, Statistics **16** (1985), no. 4, 549–557. MR 86m:62023. Zbl 602.62039.
[5] D. K. Nagar and L. Cardeño, *Matrix variate Kummer-Gamma distribution*, Random Oper. Stochastic Equations **9** (2001), no. 3, 207–218.
[6] D. K. Nagar and A. K. Gupta, *Matrix variate Kummer-Beta distribution*, to appear in J. Austral. Math. Soc.
[7] K. W. Ng and S. Kotz, *Kummer-Gamma and Kummer-Beta univariate and multivariate distributions*, Research report, Department of Statistics, The University of Hong Kong, Hong Kong, 1995.

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