ATTRACTORS OF SEMIGROUPS ASSOCIATED WITH NONLINEAR SYSTEMS FOR DIFFUSIVE PHASE SEPARATION

NOBUYUKI KENMOCHI

ABSTRACT. We consider a model for diffusive phase transitions, for instance, the component separation in a binary mixture. Our model is described by two functions, the absolutete temperature $\theta := \theta(t,x)$ and the order parameter $w := w(t,x)$, which are governed by a system of two nonlinear parabolic PDEs. The order parameter w is constrained to have double obstacles $\sigma_* \le w \le \sigma^*$ (i.e., σ_* and σ^* are the threshold values of w). The objective of this paper is to discuss the semigroup $\{S(t)\}$ associated with the phase separation model, and construct its global attractor.

1. INTRODUCTION

This paper is concerned with a system of nonlinear parabolic PDEs of the form, referred to as (PSC),

$$[\rho(u) + \lambda(w)]_t - \Delta u + \nu\rho(u) = f(x) \text{ in } Q := (0,+\infty) \times \Omega, \tag{1.1}$$

$$w_t - \Delta\{-\kappa\Delta w + \xi + g(w) - \lambda'(w)u\} = 0 \text{ in } Q, \tag{1.2}$$

$$\xi \in \beta(w) \text{ in } Q, \tag{1.3}$$

$$\frac{\partial u}{\partial n} + n_o u = h(x) \text{ on } \Sigma := (0,+\infty) \times \Gamma, \tag{1.4}$$

$$\frac{\partial w}{\partial n} = 0,\ \frac{\partial}{\partial n}\{-\kappa\Delta w + \xi + g(w) - \lambda'(w)u\} = 0 \text{ on } \Sigma, \tag{1.5}$$

$$u(0,\cdot) = u_o,\ w(0,\cdot) = w_o \text{ in } \Omega. \tag{1.6}$$

Here Ω is a bounded domain in R^N $(1 \le N \le 3)$ with smooth boundary $\Gamma := \partial\Omega$; ρ is an increasing function such as $\rho(u) = -\frac{1}{u}$ for $-\infty < u < 0$;

1991 *Mathematics Subject Classification.* Primary: 35Q55.

Key words and phrases. System of parabolic equations, diffusive phase separation, semigroup, attractor.

Received: May 23, 1996.

$\lambda,\ g$ are smooth functions on $\mathbf{R}$ and λ' is the derivative of λ; β is a maximal monotone graph in $\mathbf{R}\times\mathbf{R}$ with bounded domain $D(\beta)$ in $\mathbf{R}$; $\kappa>0,\ n_o>0$ and $\nu\geq 0$ are constants; $f,\ h,\ u_o,\ w_o$ are prescribed data.

This system arises in the non-isothermal diffusive phase separation in a binary mixture. In such a context, $\theta:=\rho(u)$ is the (absolute) temperature and w is the local concentration of one of the components; physically, (1.1) is the energy balance equation, where $\rho(u)+\lambda(w)$ is the internal energy, and (1.2) is the mass balance equation with constraint (1.3) for w, where $-\kappa\Delta w+\xi+g(w)-\lambda'(w)u$ can be interpreted as the (generalized) chemical potential difference. The details of modeling are referred, for instance, to [1, 2, 7, 10, 12].

In the one-dimensional case, i.e. $N=1$, the existence and uniqueness of a global solution of (PSC) was proved in [10], and in [14] for the case without constraint (1.3). In the higher dimensional case ($N=2$ or 3), any uniqueness result has not been noticed in the general setting; for a model in which the mass balance equation includes a viscosity term $-\mu\Delta w_t$, the uniqueness was obtained in [10]. Recently, in [6] a uniqueness result was established in a very wide space of distributions under the additional assumption that

$$\lambda \text{ is convex on } \overline{D(\beta)} \text{ and } D(\rho)\subset(-\infty,0]. \tag{1.7}$$

So far as the large time behaviour of solutions is concerned, we have noticed a few papers (e.g. [5, 9, 10, 14]) including some results about the ω-limit set of each single solution as time t goes to $+\infty$, but no results, except [12], on attractors so far for non-isothermal phase separation models; in [12] the regular case of ρ was treated, so this result is not applicable to (PSC) including a singular function ρ.

In this paper, assuming (1.7), we shall give a new existence result for problem (PSC) with initial data $[u_o,w_o]$ in a larger class than that in [10]. Also, based on our existence result, we shall consider a semigroup $\{S(t)\}_{t\geq 0}$ consisting of operators $S(t)$ which assign to each initial data $[u_o,w_o]$ the element $[u(t),w(t)]$, $\{u,w\}$ being the solution. Moreover we shall construct the global attractor for $\{S(t)\}$ in the product space $L^2(\Omega)\times H^1(\Omega)$. Unfortunately, the mapping $t\to S(t)[u_o,w_o]$ lacks the continuity at $t=0$ in $L^2(\Omega)\times H^1(\Omega)$ for bad initial data $[u_o,w_o]$, which comes from the singularity of $\rho(u)$. Therefore the general theory on attractors (cf. [4, 17]) cannot be directly applied to our case. However, the construction of the global attractor will be done by introducing a Lyapunov-like functional and by appropriately modified versions of some results in [4, 17]. Especially, the term $\nu\rho(u)$ with positive ν in (1.1) is very important in order to find an absorbing set.

Notation. In general, for a (real) Banach space W we denote by $|\cdot|_W$ the norm and by W^* its dual space endowed with the dual norm. For any compact time interval $[t_o,t_1]$ we denote by $C_w([t_o,t_1];W)$ the space of all weakly continuous functions from $[t_o,t_1]$ into W, and mean by "$u_n\to u$ in $C_w([t_o,t_1];W)$ as $n\to+\infty$" that for each $z^*\in W^*$, $\langle z^*,u_n(t)-u(t)\rangle_{W^*,W}\to 0$ uniformly on $[t_o,t_1]$ as $n\to+\infty$, where $\langle\cdot,\cdot\rangle_{W^*,W}$ stands for the duality pairing between W^* and W.

For two real valued functions $u,\ v$ we define

$$u \wedge v := \min\{u, v\},\ u \vee v := \max\{u, v\}.$$

Throughout this paper, let Ω be a bounded domain in $\mathbf{R}^N$ $(1 \le N \le 3)$ with smooth boundary $\Gamma := \partial\Omega$, and for simplicity fix some notation as follows:

$H := L^2(\Omega),\ H_o := \left\{ z \in L^2(\Omega);\ \int_\Omega z dx = 0 \right\}.$

$V := H^1(\Omega),\ V_o := \left\{ z \in H^1(\Omega);\ \int_\Omega z dx = 0 \right\}.$

$(\cdot,\cdot):$ inner product in H.

$\langle \cdot,\cdot \rangle:$ duality pairing between V^* and V.

$\langle \cdot,\cdot \rangle_o:$ duality pairing between V_o^* and V_o.

$(\cdot,\cdot)_\Gamma:$ inner product in $L^2(\Gamma)$.

$a(v, w) := \int_\Omega \nabla v \cdot \nabla w dx$ for $v,\ w \in V$.

$\pi_o:$ projection from H onto H_o, *i.e.* $[\pi_o z](x) := z(x) - \frac{1}{|\Omega|}\int_\Omega z(y)dy,$
$z \in H.$

Also, we define $|\cdot|_V$ and $|\cdot|_{V_o}$ by

$$|v|_V := \left\{ \int_\Omega |\nabla v|^2 dx + n_o \int_\Gamma v^2 d\Gamma \right\}^{\frac{1}{2}},\ v \in V,$$

and

$$|v|_{V_o} := \left\{ \int_\Omega |\nabla v|^2 dx \right\}^{\frac{1}{2}},\ v \in V_o;$$

clearly we have standard relations

$$V \subset H \subset V^*,\ V_o \subset H_o \subset V_o^*,$$

in which all the injections are compact and densely defined. Associated with the above norms, the duality mappings $F:\ V \to V^*$ and $F_o:\ V_o \to V_o^*$ are defined in the following manner:

$$\langle Fv, z \rangle = a(v, z) + n_o(v, z)_\Gamma \text{ for } v, z \in V, \tag{1.8}$$

$$\langle F_o v, z \rangle_o = a(v, z) \text{ for } v, z \in V_o. \tag{1.9}$$

Clearly, if $\ell := Fv \in H$, then $v \in H^2(\Omega)$ and it is a unique solution of

$$\begin{cases} -\Delta v = \ell \text{ in } \Omega, \\ \dfrac{\partial v}{\partial n} + n_o v = 0 \text{ on } \Gamma; \end{cases}$$

if $\ell := F_o v \in H_o$, then $v \in H^2(\Omega)$ and it is a unique solution of

$$\begin{cases} -\Delta v = \ell \text{ in } \Omega, \\ \dfrac{\partial v}{\partial n} = 0 \text{ on } \Gamma, \ \displaystyle\int_\Omega v dx = 0. \end{cases}$$

2. Existence and Uniqueness Result for (PSC)

Throughout this paper we suppose that $\rho,\ \beta,\ \lambda,\ g,\ \kappa,\ n_o$ and ν satisfy the following hypotheses (H1) - (H5):

- : (H1) ρ is a single-valued maximal monotone graph in $\mathbf{R} \times \mathbf{R}$, its domain $D(\rho)$ and range $R(\rho)$ are open in $\mathbf{R}$ and it is locally bi-Lipschitz continuous as a function from $D(\rho)$ onto $R(\rho)$; we denote by ρ^{-1} the inverse of ρ and by $\hat{\rho^{-1}}$ a proper l.s.c. convex function on $\mathbf{R}$ whose subdifferential coincides with ρ^{-1} in $\mathbf{R}$.
- : (H2) β is a maximal monotone graph in $\mathbf{R} \times \mathbf{R}$ which is the subdifferential of a non-negative proper l.s.c. convex function $\hat{\beta}$ on $\mathbf{R}$ such that
$$\overline{D(\hat{\beta})} = [\sigma_*, \sigma^*]$$
for finite numbers $\sigma_*,\ \sigma^*$ with $\sigma_* < \sigma^*$.
- : (H3) $\lambda :\ \mathbf{R} \to \mathbf{R}$ is of C^2-class, convex on $[\sigma_*, \sigma^*]$ and

$$\lambda''(w)u \le 0 \text{ for all } w \in [\sigma_*, \sigma^*] \text{ and } u \in D(\rho). \tag{2.1}$$

- : (H4) $g :\ \mathbf{R} \to \mathbf{R}$ is of C^2-class; we denote a primitive of g, which is non-negative on $[\sigma_*, \sigma^*]$, by $\hat{g}$.
- : (H5) $\kappa > 0,\ n_o > 0$ and $\nu \ge 0$ are constants.

Now we give a variational formulation for (PSC).

Definition 2.1. *Let $f \in H,\ h \in L^2(\Gamma),\ u_o$ be a measurable function on Ω with $\rho(u_o) \in H$ and $w_o \in V$ with $\hat{\beta}(w_o) \in L^1(\Omega)$. Then, for any finite $T > 0$, a couple $\{u, w\}$ of functions $u :\ [0,T] \to V$ and $w :\ [0,T] \to H^2(\Omega)$ is called a (weak) solution of (PSC):=(PSC;f, h, u_o, w_o) on $[0,T]$, if the following conditions (w1) - (w4) are satisfied.*

- : *(w1) $u \in L^2(0,T;V),\ \rho(u) \in C_w([0,T];H),\ \rho(u)' \in L^1(0,T;V^*)$, $w \in L^2(0,T;H^2(\Omega)) \cap C_w([0,T];V)$ with $\hat{\beta}(w) \in L^\infty(0,T;L^1(\Omega))$, $w' \in L^2(0,T;V^*)$ and $\lambda(w)' \in L^1(0,T;V^*)$.*
- : *(w2) $\rho(u)(0) = \rho(u_o)$ and $w(0) = w_o$.*
- : *(w3) For a.e. $t \in [0,T]$ and all $z \in V$,*

$$\begin{aligned} &\frac{d}{dt}(\rho(u(t)) + \lambda(w(t)), z) + a(u(t), z) \\ &\qquad\qquad + (n_o u(t) - h, z)_\Gamma + \nu(\rho(u(t)), z) \\ &\qquad = (f, z). \end{aligned} \tag{2.2}$$

- : *(w4) $\frac{\partial w(t)}{\partial n} = 0$ a.e. on Γ for a.e. $t \in [0,T]$, and there is a function $\xi \in L^2(0,T;H)$ such that $\xi(t) \in \beta(w(t))$ a.e. on Ω for a.e. $t \in [0,T]$*

and

$$(2.3)\quad \frac{d}{dt}(w(t),\eta) + \kappa(\Delta w(t),\Delta\eta) - (g(w(t)) + \xi(t) - \lambda'(w(t))u(t),\Delta\eta) = 0$$

for all $\eta \in H^2(\Omega)$ *with* $\frac{\partial\eta}{\partial n} = 0$ *a.e. on* Γ *and for a.e.* $t \in [0,T]$.

A couple $\{u,w\}$ *of functions* $u: \mathbf{R}_+ \to V$ *and* $w: \mathbf{R}_+ \to H^2(\Omega)$ *is called a global solution of (PSC) (or a solution of (PSC) on* $\mathbf{R}_+$*), if it is a solution of (PSC) on* $[0,T]$ *for every finite* $T > 0$.

As easily understood from the above definition, since $\sigma_* \le w \le \sigma^*$ for any solution $\{u,w\}$ of (PSC), the behaviour of $g,\ \lambda$ on the outside of $[\sigma_*,\sigma^*]$ gives no influence to the solution and we may assume without loss of generality that

(2.4)
the support of g is compact in $\mathbf{R}$ and λ is linear on the outside of $[\sigma_*,\sigma^*]$.

Remark 2.1. From the above definition we easily observe (1)-(3) below.

: (1) For any solution $\{u,w\}$ of (PSC) on $[0,T]$ we see that

$$\frac{d}{dt}\int_\Omega w(t)dx = 0 \text{ for a.e. } t \in [0,T],$$

so that

$$\frac{1}{|\Omega|}\int_\Omega w(t)dx = \frac{1}{|\Omega|}\int_\Omega w_o dx =: m_o \text{ for all } t \in [0,T].$$

This implies that $w - m_o \in C_w([0,T];V_o)$ and $w' \in L^2(0,T;V_o^*)$.

: (2) In terms of the duality mapping $F: V \to V^*$ the variational identity (2.2) is written in the form

$$(2.5)\qquad \frac{d}{dt}(\rho(u(t)) + \lambda(w(t))) + Fu(t) + \nu\rho(u(t)) = f^* \text{ in } V^*$$

for a.e. $t \in [0,T]$, where $f^* \in V^*$ is given by

$$\langle f^*, z\rangle = (f,z) + (h,z)_\Gamma \text{ for all } z \in V.$$

: (3) In terms of the duality mapping $F_o: V_o \to V_o^*$ variational identity (2.3) is written in the form

$$(2.6)\quad F_o^{-1}w'(t) + \kappa F_o(\pi_o w(t)) + \pi_o[\xi(t) + g(w(t)) - \lambda'(w(t))u(t)] = 0 \text{ in } H_o$$

for a.e. $t \in [0,T]$.

We now introduce some functions and spaces in order to formulate an existence-uniqueness result. Let u^∞ be the unique solution of

$$(2.7)\quad \begin{cases} u^\infty \in V; \\ a(u^\infty,z) + (n_o u^\infty - h, z)_\Gamma + \nu(\rho(u^\infty),z) = (f,z) \text{ for all } z \in V; \end{cases}$$

clearly (2.7) has one and only one solution $u^\infty \in V$ for given $f \in H$ and $h \in L^2(\Gamma)$. If $\nu > 0$, then

$$(2.8)\qquad \rho(u^\infty) \in H.$$

In case of $\nu = 0$ we suppose (2.8) holds.

Next we define a functional $J(\cdot,\cdot)$ on the set $\rho^{-1}(H)\times V := \{[z,v]; \rho(z)\in H, v\in V\}$, by putting

$$J(z,v) := J_o(z,v) + J_1(z,v)$$

with

$$J_o(z,v) := \varepsilon_o|\rho(z)+\lambda(v)|_H^2$$

and

$$J_1(z,v) := \int_\Omega \hat{\rho^{-1}}(\rho(z))dx - (\rho(z)+\lambda(v), u^\infty) + \frac{\kappa}{2}|\nabla v|_H^2 + \int_\Omega (\hat\beta(v)+\hat g(v))dx + C_o, \tag{2.9}$$

where ε_o is a (small) positive number determined later and C_o is a constant so that $J_1(\cdot,\cdot)$ is non-negative; in fact, such a constant C_o exists, since

$$\hat{\rho^{-1}}(r) - ru^\infty(x) \ge \hat{\rho^{-1}}(\rho(u^\infty(x))) - \rho(u^\infty(x))u^\infty(x) \tag{2.10}$$

for all $r\in \mathrm{R}$ and a.e. $x\in\Omega$. With the functional J_1 and a number m_o with $\sigma_*\le m_o\le\sigma^*$, we put

$$D(m_o) := \left\{[z,v]\in\rho^{-1}(H)\times V; J_1(z,v)<+\infty, \frac{1}{|\Omega|}\int_\Omega v dx = m_o\right\} \tag{2.11}$$

and

$$D^M(m_o) := \{[z,v]\in D(m_o); J_1(z,v)\le M\} \text{ for each } M>0. \tag{2.12}$$

Also, for a number m_o with $\sigma_*\le m_o\le\sigma^*$, we put

(2.13)

$$D_o(m_o) := \left\{ [z,v] \;;\; \begin{array}{c} z\in V,\ \rho(z)\in H,\ v\in H^2(\Omega),\ v-m_o\in V_o, \\ \dfrac{\partial v}{\partial n}=0 \text{ a.e. on } \Gamma,\ \text{there is } \xi\in H \text{ such that} \\ \xi\in\beta(v) \text{ a.e. on } \Omega,\ -\kappa\Delta v+\xi\in V \end{array} \right\},$$

$$D_o^M(m_o) := \{[z,v]\in D_o(m_o); J_1(z,v)\le M\} \text{ for each } M>0. \tag{2.14}$$

Clearly, $D_o^M(m_o)\subset D^M(m_o)$, $D(m_o)=\cup_{M>0}D^M(m_o)$ and $D_o(m_o) = \cup_{M>0}D_o^M(m_o)$.

First we recall the following theorem which guarantees the uniqueness of the solution of (PSC).

Theorem 2.1. ([5; Theorem 2.1]) *Assume that (H1)-(H5) hold, and let $f\in H$, $h\in L^2(\Gamma)$ and m_o be a number with $\sigma_*\le m_o\le\sigma^*$. Let $[u_{oi},w_{oi}]$, $i=1,2$, be initial data in $D(m_o)$, and $\{u_i,w_i\}$ be any solution of $(PSC)_i := (PSC; f,h,u_{oi},w_{oi})$ on $[0,T]$, $T>0$, for $i=1,2$. Then, with notation $e_i(t) := \rho(u_i(t))+\lambda(w_i(t))$ for $t\in[0,T]$, and for all $s,\ t\in[0,T]$, $s\le t$,*

$$|e_1(t)-e_2(t)|_{V^*}^2 + |w_1(t)-w_2(t)|_{V_o^*}^2 \le e^{R_o(t-s)}(|e_1(s)-e_2(s)|_{V^*}^2 + |w_1(s)-w_2(s)|_{V_o^*}^2), \tag{2.15}$$

where $R_o := R_o(\kappa,n_o,\lambda,g)$ is a positive constant dependent only on κ, n_o and the Lipschitz constants of λ and g.

Hypothesis (2.1) of (H3) is essential for the proof of inequality (2.15).
An existence result is stated as follows.

Theorem 2.2. *Assume that (H1)-(H5) hold as well as (2.8), and let m_o be any number with $\sigma_* < m_o < \sigma^*$. Also, let $f \in H$ and $h \in L^\infty(\Gamma)$. Assume that*

$$n_o \sup D(\rho) \geq h(x) \geq n_o \inf D(\rho) \text{ for a.e. } x \in \Gamma \tag{2.16}$$

and there are constants A_1, A_1' such that

$$\rho(r)(n_o r - h(x)) \geq -A_1|r| - A_1' \text{ for a.e. } x \in \Gamma \text{ and all } r \in D(\rho). \tag{2.17}$$

Then, for each $[u_o, w_o] \in D(m_o)$, problem (PSC) admits a (unique) global solution $\{u, w\}$. Moreover, the following inequalities (2.18) - (2.20) hold.

$$J_1(u(t), w(t)) + \int_s^t |u(\tau) - u^\infty|_V^2 d\tau + \int_s^t |w'(\tau)|_{V_o^*}^2 d\tau \leq J_1(u(s), w(s)) \tag{2.18}$$

for $0 \leq s \leq t$;

$$\begin{aligned} |\rho(u(t)) + \lambda(w(t))|_H^2 \leq M_1(T)\{&|\rho(u(s)) + \lambda(w(s))|_H^2 \\ &+ J_1(u(s), w(s)) + 1\} \end{aligned} \tag{2.19}$$

for $0 \leq s \leq t \leq s + T$, where $M_1(T)$ is an increasing function of $T \in \mathbf{R}_+$, independent of initial data $[u_o, w_o] \in D(m_o)$;

$$\begin{aligned} (t-s)\{|u(t) - u^\infty|_V^2 + |w'(t)|_{V_o^*}^2 + \nu|\int_\Omega \hat{\rho}(u(t))dx|\} \\ + \kappa \int_s^t (\tau - s)|w'(\tau)|_{V_o}^2 d\tau \\ \leq M_2(T)\{J_1(u(s), w(s)) \\ + |\rho(u(s)) + \lambda(w(s)) - \rho(u^\infty)|_{V^*}^2 + 1\} \end{aligned} \tag{2.20}$$

for all $s \geq 0$ and a.e. $t \in [s, s+T]$, where $M_2(T)$ is an increasing function of $T \in \mathbf{R}_+$ independent of initial data $[u_o, w_o] \in D(m_o)$.

Remark 2.2. From (2.18) and (2.20) of Theorem 2.2 we further derive an estimate of the form

$$\begin{aligned} &(t-s)\{|w(t)|_{H^2(\Omega)}^2 + |\xi(t)|_H^2\} \\ &\quad \leq M_3(T)\{J_1(u(s), w(s)) + |\rho(u(s)) + \lambda(w(s)) - \rho(u^\infty)|_{V^*}^2 + 1\} \end{aligned} \tag{2.21}$$

for $0 \leq s < t \leq s + T$, where $\xi \in L^2_{loc}(\mathbf{R}_+; H)$ is the function as in condition (w4) of Definition 2.1 and $M_3(\cdot)$ is a function having the same properties as $M_i(\cdot)$, $i = 1, 2$. In fact, since

$$\kappa F_o(\pi_o w(t)) + \pi_o \xi(t) = -F_o^{-1} w'(t) + \pi_o[\lambda'(w(t))u(t) - g(w(t))] =: \ell(t),$$

it follows from a regularity result in [3] that

$$|w(t)|_{H^2(\Omega)}^2 + |\xi(t)|_H^2 \leq C_1(|\ell(t)|_H^2 + 1) \text{ for a.e. } t \geq 0 \tag{2.22}$$

with a constant C_1 independent of initial data $[u_o, w_o] \in D(m_o)$ and ℓ. Combining the above inequality with (2.18) and (2.20) we conclude an estimate of the form (2.21) for all $0 \leq s < t \leq s + T$.

Remark 2.3. Let $\{u, w\}$ be the global solution of (PSC) which is given by Theorem 2.2. Then, we have by estimates (2.18) - (2.21) that

: (i) u is a bounded and weakly continuous function from $[\delta, +\infty)$ into V for each $\delta > 0$;
: (ii) w is a bounded and weakly continuous function from $[\delta, +\infty)$ into $H^2(\Omega)$ for each $\delta > 0$;
: (iii) ξ is a bounded function from $[\delta, +\infty)$ into H for each $\delta > 0$, and satisfies that

$$\xi(t) \in \beta(w(t)) \text{ a.e. on } \Omega \text{ for all } t > 0.$$

Also, we have

$$[u(t), w(t)] \in D_o(m_o) \text{ for all } t > 0, \tag{2.23}$$

which is nothing else but the smoothing effect for solutions.

Remark 2.4. In case $m_o = \sigma_*$ (resp. $m_o = \sigma^*$), it follows that $w \equiv m_o$ for any solution $\{u, w\}$ of (PSC) on $[0, T]$, since $w \geq m_o$ (resp. $w \leq m_o$) and $\int_\Omega w(t)dx = |\Omega| m_o$, and moreover u satisfies

$$\langle \rho(u)'(t), z\rangle + a(u(t), z) + (n_o u(t) - h(t), z)_\Gamma + \nu(\rho(u(t)), z) = (f(t), z)$$

$$\text{for all } z \in H^1(\Omega) \text{ and a.e. } t \in [0, T]$$

and $u(0) = u_o$.

3. Approximate Problems and Estimates for Their Solutions

The solution of (PSC) will be constructed as a limit of solutions $\{u_{\mu\varepsilon\eta}, w_{\mu\varepsilon\eta}\}$ of approximate problems $(\text{PSC})_{\mu\varepsilon\eta}$, defined below, as $\mu, \varepsilon, \eta \to 0$; parameters ε, η concern with approximation $\rho_{\varepsilon\eta}$ of function ρ, while parameter μ concerns with the coefficient of viscosity term, i.e. $-\mu\Delta w_t$, added in the mass balance equation.

The main idea for approximation is found in [7, 10, 15], and uniform estimates for approximate solutions with respect to parameters are quite similar to those in the above cited papers. Therefore, we mention very briefly some estimates for approximate solutions. In the rest of this section, we make all the assumptions of Theorem 2.2 as well as (2.4).

If λ is linear, i.e. $\lambda'' = 0$, on $[\sigma_*, \sigma^*]$, then the proof of Theorem 2.2 is very simple. Therefore, in the rest of this section, we assume that

$$\lambda'' > 0 \text{ somewhere on } [\sigma_*, \sigma^*],\ D(\rho) \subset (-\infty, 0). \tag{3.1}$$

Given real parameters $\varepsilon, \eta \in (0, 1]$, we consider approximations ρ_ε and $\rho_{\varepsilon\eta}$ of ρ as follows. Putting

$$D(\rho) = (r_*, r^*) \text{ for } -\infty \leq r_* < r^* \leq 0,$$

choose two families of numbers $\{a_\varepsilon; 0 \leq \varepsilon \leq 1\}$ with $a_o = r_*$ and $\{b_\eta; 0 \leq \eta \leq 1\}$ with $b_o = r^*$ such that

$$r_* < a_\varepsilon < a_{\varepsilon'} < a_1 < r_o < b_1 < b_\eta < b_{\eta'} < r^*$$

if $0 < \varepsilon < \varepsilon' < 1$ and $0 < \eta' < \eta < 1$, where r_o is a fixed number in $D(\rho)$, and

$$a_\varepsilon \downarrow r_* \text{ as } \varepsilon \to 0,\ \ b_\eta \uparrow r^* \text{ as } \eta \to 0.$$

For each ε and η we define

$$\rho_\varepsilon(r) = \begin{cases} \rho(r) & \text{for } r \geq a_\varepsilon, \\ \rho(a_\varepsilon) + r - a_\varepsilon & \text{for } r < a_\varepsilon, \end{cases}$$

and

$$\rho_{\varepsilon\eta}(r) = \begin{cases} \rho(b_\eta) + r - b_\eta & \text{for } \ r > b_\eta, \\ \rho(r) & \text{for } a_\varepsilon \leq r \leq b_\eta, \\ \rho(a_\varepsilon) + r - a_\varepsilon & \text{for } r < a_\varepsilon. \end{cases}$$

Note that $\rho_{\varepsilon\eta}$ is bi-Lipschitz continuous on $\mathbf{R}$ and

$$\rho_{\varepsilon\eta} \to \rho_\varepsilon \text{ in the graph sense as } \eta \to 0 \text{ for each fixed } \varepsilon,$$

$$\rho_\varepsilon \to \rho \text{ in the graph sense as } \varepsilon \to 0,$$

and moreover there is a positive constant $C(\varepsilon)$ for each $\varepsilon \in (0, 1]$ such that

$$\frac{d}{dr}\rho_{\varepsilon\eta}(r) \geq C(\varepsilon),\ \ \frac{d}{dr}\rho_\varepsilon(r) \geq C(\varepsilon). \tag{3.2}$$

We write sometimes ρ_o or ρ_{oo} for ρ and $\rho_{\varepsilon o}$ for ρ_ε.

Besides, for $\varepsilon, \eta \in [0, 1]$, let $u^\infty_{\varepsilon\eta}$ be the solution of

$$\begin{cases} u^\infty_{\varepsilon\eta} \in V; \\ a(u^\infty_{\varepsilon\eta}, z) + (n_o u^\infty_{\varepsilon\eta} - h, z)_\Gamma + \nu(\rho_{\varepsilon\eta}(u^\infty_{\varepsilon\eta}), z) = (f, z) \text{ for all } z \in V. \end{cases}$$

Clearly, $\{u^\infty_{\varepsilon\eta}\}$ is bounded in V, $\{\rho_{\varepsilon\eta}(u^\infty_{\varepsilon\eta})\}$ is bounded in H,

$$u^\infty_{\varepsilon\eta} \to u^\infty_{\varepsilon 0} \text{ in } V \text{ as } \eta \to 0 \text{ for each fixed } \varepsilon \in [0, 1],$$

and

$$\rho_{\varepsilon\eta}(u^\infty_{\varepsilon\eta}) \to \rho(u^\infty_{\varepsilon 0}) \text{ weakly in } H \text{ as } \eta \to 0 \text{ for each fixed } \varepsilon \in [0, 1],\ \text{ if } \nu > 0.$$

Also, we define functionals $J_{1\varepsilon\eta}(\cdot, \cdot)$ by

$$\begin{aligned} J_{1\varepsilon\eta}(z, v) := &\int_\Omega \hat{\rho}^{-1}_{\varepsilon\eta}(\rho_{\varepsilon\eta}(z))dx - (\rho_{\varepsilon\eta}(z) + \lambda(v), u^\infty_{\varepsilon\eta}) \\ &+ \frac{\kappa}{2}|\nabla v|^2_H + \int_\Omega \{\hat{\beta}(v) + \hat{g}(v)\}dx + C_o \end{aligned} \tag{3.3}$$

for $[z.v] \in H \times V$, where $\hat{\rho}^{-1}_{\varepsilon\eta}$ is the primitive of $\rho^{-1}_{\varepsilon\eta}$ with $\hat{\rho}^{-1}_{\varepsilon\eta}(r_o) = \hat{\rho}^{-1}(r_o)$ and C_o is a sufficiently large positive constant so that $J_{1\varepsilon\eta} \geq 0$ on $H \times V$ for all $\varepsilon, \eta \in [0, 1]$; of course, $J_{1oo} = J_1$, $u^\infty_{00} = u^\infty$ and C_o is supposed to be the same constant as in expression (2.9) of J_1.

Now, let us consider approximate problems including $\rho_{\varepsilon\eta}, 0 < \varepsilon \leq 1, 0 \leq \eta \leq 1$, and the viscosity term $-\mu\Delta w_t, 0 < \mu \leq 1$, which is formulated below and referred as $(\mathrm{PSC})_{\mu\varepsilon\eta}$.

Definition 3.1. *For $0 < T < +\infty$ we say that a couple of functions $u := u_{\mu\varepsilon\eta} : [0,T] \to V$ and $w := w_{\mu\varepsilon\eta} : [0,T] \to H^2(\Omega)$ is a "solution" of (PSC)$_{\mu\varepsilon\eta}$ on $[0,T]$, if the following properties $(w1)_{\mu\varepsilon\eta} - (w4)_{\mu\varepsilon\eta}$ are fulfilled:*

: $(w1)_{\mu\varepsilon\eta}$ $u \in W^{1,2}(0,T;H) \cap C_w([0,T];V)$, $w \in W^{1,2}(0,T;H) \cap C([0,T];V) \cap L^2(0,T;H^2(\Omega))$;

: $(w2)_{\mu\varepsilon\eta}$ $u(0) = u_{o\varepsilon\eta} := \{u_o \vee a_\varepsilon\} \wedge b_\eta$ *and* $w(0) = w_o$;

: $(w3)_{\mu\varepsilon\eta}$ *for a.e.* $t \in [0,T]$ *and* $z \in V$,

$$(3.4)\qquad \begin{aligned}&(\rho_{\varepsilon\eta}(u)'(t) + \lambda(w)'(t), z) + a(u(t), z)\\ &\qquad + (n_o u(t) - h, z)_\Gamma + \nu(\rho_{\varepsilon\eta}(u(t)), z) = (f(t), z);\end{aligned}$$

: $(w4)_{\mu\varepsilon\eta}$ $\dfrac{\partial w(t)}{\partial n} = 0$ *a.e. on* Γ *for a.e.* $t \in [0,T]$, *and there is a function* $\xi =: \xi_{\mu\varepsilon\eta} \in L^2(0,T;H)$ *such that*

$$\xi(t) \in \beta(w(t)) \text{ a.e. in } \Omega \text{ for a.e. } t \in [0,T]$$

and

$$(3.5)\qquad \begin{aligned}&(w'(t), z - \mu\Delta z) + \kappa(\Delta w(t), \Delta z)\\ &\qquad - (\xi(t) + g(w(t)) - \lambda'(w(t))u(t), \Delta z) = 0\end{aligned}$$

for a.e. $t \in [0,T]$ *and all* $z \in H^2(\Omega)$ *with* $\dfrac{\partial z}{\partial n} = 0$ *a.e. on* Γ.

Clearly, (3.4) and (3.5) are respectively written in the forms (cf. (2.5), (2.6) in Remark 2.1)

$$(3.6)\qquad \rho_{\varepsilon\eta}(u)'(t) + \lambda(w)'(t) + Fu(t) + \nu\rho_{\varepsilon\eta}(u(t)) = f^*(t) \text{ in } V^*$$

for a.e. $t \in [0,T]$, and

$$(3.7)\quad (F_o^{-1} + \mu I)w'(t) + \kappa F_o(\pi_o w(t)) + \pi_o[\xi(t) + g(w(t)) - \lambda'(w(t))u(t)] = 0 \quad \text{in } H_o \text{ for a.e. } t \in [0,T].$$

With functions $u^\infty_{\varepsilon\eta}$, (3.6) is also written in the form

$$(3.8)\quad \rho_{\varepsilon\eta}(u)'(t) + \lambda(w)'(t) + F(u(t) - u^\infty_{\varepsilon\eta}) + \nu(\rho_{\varepsilon\eta}(u(t)) - \rho_{\varepsilon\eta}(u^\infty_{\varepsilon\eta})) = 0 \quad \text{in } V^* \text{ for a.e. } t \in [0,T].$$

According to an existence-uniqueness result in [8, Theorem 2.2], for each $\mu, \varepsilon, \eta \in (0,1]$ problem (PSC)$_{\mu\varepsilon\eta}$ has one and only one solution $\{u_{\mu\varepsilon\eta}, w_{\mu\varepsilon\eta}\}$ on $[0,T]$, if the initial data u_o and w_o are given so that $[u_o, w_o] \in D_o(m_o)$ (hence $[u_{o\varepsilon\eta}, w_o] \in D_o(m_o)$ for every $\varepsilon, \eta \in (0,1]$); moreover, $w_{\mu\varepsilon\eta}$ has regularity properties (cf. [10; Lemmas 5.2, 6.2])

$$(3.9)\qquad \begin{cases} w_{\mu\varepsilon\eta} \in C_w([0,T];H^2(\Omega)),\ w'_{\mu\varepsilon\eta} \in L^\infty(0,T;H) \cap L^2(0,T;V),\\ \xi_{\mu\varepsilon\eta} \in L^\infty(0,T;H).\end{cases}$$

Now we give some estimates for $\{u_{\mu\varepsilon\eta}, w_{\mu\varepsilon\eta}\}$.

Estimate (I)

By regularity (3.9) we can compute rigorously

$$(3.8) \times (u_{\mu\varepsilon\eta} - u^\infty_{\varepsilon\eta}) + (3.7) \times w'_{\mu\varepsilon\eta}$$

to get

$$\begin{aligned} \frac{d}{d\tau} J_{1\varepsilon\eta}(u_{\mu\varepsilon\eta}(\tau), w_{\mu\varepsilon\eta}(\tau)) + |u_{\mu\varepsilon\eta}(\tau) - u^\infty_{\varepsilon\eta}|^2_V \\ + |w'_{\mu\varepsilon\eta}(\tau)|^2_{V_o^*} + \mu |w'_{\mu\varepsilon\eta}(\tau)|^2_H \\ \le 0 \end{aligned} \tag{3.10}$$

for a.e. $\tau \in [0, T]$. For details, see [10, Lemma 5.1]. The integration of (3.10) over $[0, t]$ yields

$$\begin{aligned} J_{1\varepsilon\eta}(u_{\mu\varepsilon\eta}(t), w_{\mu\varepsilon\eta}(t)) + \int_0^t |u_{\mu\varepsilon\eta} - u^\infty_{\varepsilon\eta}|^2_V d\tau \\ + \int_0^t (|w'_{\mu\varepsilon\eta}|^2_{V_o^*} + \mu |w'_{\mu\varepsilon\eta}|^2_H) d\tau \\ \le J_{1\varepsilon\eta}(u_{o\varepsilon\eta}, w_o) \text{ for all } t \in [0, T]. \end{aligned} \tag{3.11}$$

Estimates (II)

We observe from hypothesis (2.17) that (cf. [7; Lemma 3.1])

$$\rho_{\varepsilon\eta}(r)(n_o r - h(x)) \ge -A_2 |r| - A'_2 \text{ for all } r \in \mathbf{R} \text{ and a.e. } x \in \Gamma,$$

where $A_2,\ A'_2$ are positive constants independent of $\varepsilon,\ \eta \in (0, 1]$. By using this inequality we compute

$$(3.6) \times \{\rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta}) + \lambda(w_{\mu\varepsilon\eta})\}$$

to get

$$\begin{aligned} \frac{d}{d\tau} |\rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta}(\tau)) + \lambda(w_{\mu\varepsilon\eta}(\tau))|^2_H \\ + (\nu - \nu_1)|\rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta}(\tau)) + \lambda(w_{\mu\varepsilon\eta}(\tau))|^2_H \\ \le k_1(\nu_1)\{|u_{\mu\varepsilon\eta}(\tau)|^2_V + |w_{\mu\varepsilon\eta}(\tau)|^2_V + 1\} \end{aligned} \tag{3.12}$$

for a.e. $\tau \in [0, T]$, where ν_1 is an arbitrary positive number and $k_1(\nu_1)$ is a positive constant depending on ν_1 but neither of $\mu, \varepsilon, \eta \in (0, 1]$ nor initial data $[u_o, w_o] \in D_o(m_o)$. From (3.12) with (3.11) it follows that

$$\begin{aligned} |\rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta}(t)) + \lambda(w_{\mu\varepsilon\eta}(t))|^2_H \\ \le R_1(T)\{|\rho(u_{o\varepsilon\eta}) + \lambda(w_o)|^2_H + J_{1\varepsilon\eta}(u_{o\varepsilon\eta}, w_o) + 1\} \end{aligned} \tag{3.13}$$

for all $t \in [0, T]$, where $R_1(\cdot) : \mathbf{R}_+ \to \mathbf{R}_+$ is an increasing function independent of $\mu, \varepsilon, \eta \in (0, 1]$ and initial data $[u_o, w_o] \in D_o(m_o)$.

In particular, if $\nu > 0$, then we obtain, by taking $\nu_1 = \frac{\nu}{2}$ in (3.12),

$$\begin{aligned} \frac{d}{d\tau} |\rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta}(\tau)) + \lambda(w_{\mu\varepsilon\eta}(\tau))|^2_H + \frac{\nu}{2} |\rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta}(\tau)) + \lambda(w_{\mu\varepsilon\eta}(\tau))|^2_H \\ \le k_2(\nu)\{|u_{\mu\varepsilon}(\tau) - u^\infty_{\varepsilon\eta}|^2_V + |w_{\mu\varepsilon\eta}(\tau)|^2_V + 1\} \end{aligned} \tag{3.14}$$

for a.e. $\tau \in [0, T]$, where $k_2(\nu)$ is a positive constant depending on ν but neither of $\mu, \varepsilon, \eta \in (0.1]$ nor initial data $[u_o, w_o] \in D_o(m_o)$.

Estimates (III)

Next, compute

$$(3.8) \times F^{-1}(\rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta}) + \lambda(w_{\mu\varepsilon}) - \rho_{\varepsilon\eta}(u^\infty_{\varepsilon\eta}))$$

to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{d\tau}&|\rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta}(\tau)) + \lambda(w_{\mu\varepsilon\eta}(\tau)) - \rho_{\varepsilon\eta}(u^\infty_{\varepsilon\eta})|^2_{V^*}\\ &+ (u_{\mu\varepsilon\eta}(\tau) - u^\infty_{\varepsilon\eta}, \rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta}(\tau)) - \rho_{\varepsilon\eta}(u^\infty_{\varepsilon\eta}))\\ &+ (u_{\mu\varepsilon\eta}(\tau) - u^\infty_{\varepsilon\eta}, \lambda(w_{\mu\varepsilon\eta}(\tau))) + \frac{\nu}{2}|\rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta}(\tau))\\ &+ \lambda(w_{\mu\varepsilon\eta}(\tau)) - \rho_{\varepsilon\eta}(u^\infty_{\varepsilon\eta})|^2_{V^*}\\ \le\ & \frac{\nu}{2}|\lambda(w_{\mu\varepsilon\eta}(\tau))|^2_H\end{aligned}$$

for a.e. $\tau \in [0, T]$. Since

$$\begin{aligned}(u_{\mu\varepsilon\eta}(\tau) - u^\infty_{\varepsilon\eta}, \rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta}(\tau))) &\ge \int_\Omega \hat\rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta}(\tau))dx - \int_\Omega \hat\rho_{\varepsilon\eta}(u^\infty_{\varepsilon\eta})dx\\ &\ge (u_{\mu\varepsilon\eta}(\tau) - u^\infty_{\varepsilon\eta}, \rho_{\varepsilon\eta}(u^\infty_{\varepsilon\eta})),\end{aligned} \tag{3.15}$$

it follows from the above inequality that

$$\begin{aligned}\frac{d}{d\tau}&|\rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta}(\tau)) + \lambda(w_{\mu\varepsilon\eta}(\tau)) - \rho_{\varepsilon\eta}(u^\infty_{\varepsilon\eta})|^2_{V^*}\\ &+ \nu|\rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta}(\tau)) + \lambda(w_{\mu\varepsilon\eta}(\tau)) - \rho_{\varepsilon\eta}(u^\infty_{\varepsilon\eta})|^2_{V^*}\\ &+ 2|\int_\Omega \hat\rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta}(\tau))dx|\\ \le\ & k_3\{|u_{\mu\varepsilon\eta}(\tau) - u^\infty_{\varepsilon\eta}|^2_H + 1\}\end{aligned} \tag{3.16}$$

for a.e. $\tau \in [0, T]$, where k_3 is a positive constant independent of $\mu, \varepsilon, \eta \in (0, 1]$ and initial data $[u_o, w_o] \in D_o(m_o)$. Therefore the integration of (3.16) over $[0, t]$ yields

$$\begin{aligned}&|\rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta}(t)) + \lambda(w_{\mu\varepsilon\eta}(t)) - \rho_{\varepsilon\eta}(u^\infty_{\varepsilon\eta})|^2_{V^*} + |\int_0^t \int_\Omega \hat\rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta})dxd\tau|\\ &\le R_2(T)\{J_{1\varepsilon\eta}(u_{o\varepsilon\eta}, w_o) + |\rho(u_{o\varepsilon\eta}) + \lambda(w_o) - \rho_{\varepsilon\eta}(u^\infty_{\varepsilon\eta})|^2_{V^*} + 1\}\end{aligned} \tag{3.17}$$

for all $t \in [0, T]$, where $R_2(\cdot) : \mathbf{R}_+ \to \mathbf{R}_+$ is an increasing function independent of $\mu, \varepsilon, \eta \in (0, 1]$ and initial data $[u_o, w_o] \in D_o(m_o)$.

Estimate (IV)

Finally, compute the following items (1) - (4):

(1): Multiply (3.8) by $u'_{\mu\varepsilon\eta}$ and integrate over $[0, \tau] \times \Omega$ for each $0 \le \tau \le t$.

(2): Multiply $\frac{d}{dt}$(3.7) by $w'_{\mu\varepsilon\eta}$ and integrate over $[0,\tau]\times\Omega$ for each $0\le\tau\le t$.
(3): Add the results of (1) and (2).
(4): Multiply the result of (3) by τ and integrate in τ over $[0,t]$.

Then we have

$$
\begin{aligned}
&\frac{1}{2}\{t|u_{\mu\varepsilon\eta}(t)-u^{\infty}_{\varepsilon\eta}|^2_V+t|w'_{\mu\varepsilon\eta}(t)|^2_{V_o^*}+t\mu|w'_{\mu\varepsilon\eta}|^2_H\}\\
&\quad+\nu t\int_\Omega\hat\rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta}(t))dx-\nu t(\rho_{\varepsilon\eta}(u^{\infty}_{\varepsilon\eta}),u_{\mu\varepsilon\eta}(t))\\
&\quad+\kappa\int_0^t\tau|\nabla w'_{\mu\varepsilon\eta}|^2_H d\tau+\int_0^t\tau(\rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta})',u'_{\mu\varepsilon\eta})d\tau\\
&\le L_g\int_0^t\tau|w'_{\mu\varepsilon\eta}|^2_H d\tau+\frac{1}{2}\int_0^t\{|u_{\mu\varepsilon\eta}-u^{\infty}_{\varepsilon\eta}|^2_V\\
&\quad+|w'_{\mu\varepsilon\eta}|^2_{V_o^*}+\mu|w'_{\mu\varepsilon\eta}|^2_H\}d\tau\\
&\quad+\nu\int_0^t\int_\Omega\hat\rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta})dxd\tau-\nu\int_0^t(\rho_{\varepsilon\eta}(u^{\infty}_{\varepsilon\eta}),u_{\mu\varepsilon\eta})d\tau\\
&\quad+\int_0^t\int_\Omega\tau\lambda''(w_{\mu\varepsilon\eta})|w'_{\mu\varepsilon\eta}|^2u_{\mu\varepsilon\eta}dxd\tau
\end{aligned}
\tag{3.18}
$$

for all $t\in[0,T]$. For the rigorous derivation of (3.18), we refer to [10, Lemma 5.2].

Here, we use the interpolation inequality

$$
|z|^2_H\le\frac{\kappa}{2}|\nabla z|^2_H+C_\kappa|z|^2_{V_o^*}\ \text{for all } z\in V_o,
\tag{3.19}
$$

where C_κ is a positive constant depending only on κ. Applying (3.19) to the first term of the right hand side of (3.18), we derive from (3.18) with (3.11) and (3.17) that

$$
\begin{aligned}
&t|u_{\mu\varepsilon\eta}(t)-u^{\infty}_{\varepsilon\eta}|^2_V+t|w'_{\mu\varepsilon\eta}(t)|^2_{V_o^*}+t\mu|w'_{\mu\varepsilon\eta}(t)|^2_H\\
&\quad+\nu t\left|\int_\Omega\hat\rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta}(t))dx\right|+\kappa\int_0^t\tau|\nabla w'_{\mu\varepsilon\eta}|^2_H d\tau\\
&\quad+2\int_0^t\tau(\rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta})',u')d\tau\\
&\le R_3(T)\{J_{1\varepsilon\eta}(u_{o\varepsilon\eta},w_o)+|\rho(u_{o\varepsilon\eta})+\lambda(w_o)-\rho_{\varepsilon\eta}(u^{\infty}_{\varepsilon\eta})|^2_{V^*}+1\}\\
&\quad+2\int_0^t\int_\Omega\tau\lambda''(w_{\mu\varepsilon\eta})|w'_{\mu\varepsilon\eta}|^2u_{\mu\varepsilon\eta}dxd\tau
\end{aligned}
\tag{3.20}
$$

for all $t\in[0,T]$, where $R_3(\cdot):\mathbf{R}_+\to\mathbf{R}_+$ is a function having the same properties as $R_i(\cdot), i=1,2$.

Remark 3.1. In Estimates (IV), if (4) is replaced by the following (4)′:

(4)′ Integrate the result of (3) in τ over $[0,t]$, then we have, instead of (3.20),

$$
\begin{aligned}
&|u_{\mu\varepsilon\eta}(t)-u^{\infty}_{\varepsilon\eta}|^2_V+|w'_{\mu\varepsilon\eta}(t)|^2_{V^*_o}\\
&\qquad +\mu|w'_{\mu\varepsilon\eta}(t)|^2_H+\nu\left|\int_\Omega\hat{\rho}_{\varepsilon\eta}(u_{\mu\varepsilon\eta}(t))dx\right|\\
&\qquad +\kappa\int_0^t|w'_{\mu\varepsilon\eta}|^2_{V_o}d\tau+2\int_0^t(\rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta})',u')d\tau\\
&\quad\le K_o(T,|\rho(u_{o\varepsilon\eta})|_H,|u_{o\varepsilon\eta}|_V,|w_o|_{H^2(\Omega)},|-\kappa\Delta w_o+\xi_o|_V)\\
&\qquad +2\int_0^t\int_\Omega\lambda''(w_{\mu\varepsilon\eta})|w'_{\mu\varepsilon\eta}|^2u_{\mu\varepsilon\eta}dxd\tau
\end{aligned}
\tag{3.21}
$$

for a.e. $t\in[0,T]$ and $\mu,\varepsilon,\eta\in(0,1]$, where ξ_o is a function in H satisfying that

$$-\kappa\Delta w_o+\xi_o\in V \text{ and } \xi_o\in\beta(w_o) \text{ a.e. } \Omega$$

and $K_o(\cdot,\cdots,\cdot)$ is an increasing function on $\mathbf{R}^5_+$ with respect to all the arguments. Moreover, just as in Remark 2.2, we note that $|w_{\mu\varepsilon\eta}(t)|^2_{H^2(\Omega)}+|\xi_{\mu\varepsilon\eta}(t)|^2_H$ is estimated from above by the H-norm of

$$\ell_{\mu\varepsilon\eta}(t):=-(F_o^{-1}+\mu I)w'_{\mu\varepsilon\eta}(t)+\pi_o[\lambda'(w_{\mu\varepsilon\eta}(t))u_{\mu\varepsilon\eta}(t)-g(w_{\mu\varepsilon\eta}(t))],$$

namely,

$$
\begin{aligned}
|w_{\mu\varepsilon\eta}(t)|^2_{H^2(\Omega)}+|\xi_{\mu\varepsilon\eta}(t)|^2_H&\le C_1(|\ell_{\mu\varepsilon\eta}(t)|^2_H+1)\ (cf.(2.22))\\
&\le C_2(|w'_{\mu\varepsilon\eta}(t)|^2_{V^*_o}+\mu^2|w'_{\mu\varepsilon\eta}(t)|^2_H+|u_{\mu\varepsilon\eta}(t)|^2_H+1)\\
&\quad\text{for a.e. } t\in[0,T] \text{ and all } \mu,\varepsilon,\eta\in(0,1],
\end{aligned}
$$

where C_1 and C_2 are positive constants. Therefore, it follows from (3.21) that

$$
\begin{aligned}
&|w_{\mu\varepsilon\eta}(t)|^2_{H^2(\Omega)}+|\xi_{\mu\varepsilon\eta}(t)|^2_H\\
&\quad\le K_1(T,|\rho(u_{o\varepsilon\eta})|_H,|u_{o\varepsilon\eta}|_V,|w_o|_{H^2(\Omega)},|-\kappa\Delta w_o+\xi_o|_V)\\
&\qquad +k_4\int_0^T\int_\Omega\lambda''(w_{\mu\varepsilon\eta})|w'_{\mu\varepsilon\eta}|^2u_{\mu\varepsilon\eta}dxd\tau
\end{aligned}
\tag{3.22}
$$

for all $t\in[0,T]$, where $K_1(\cdot,\cdots,\cdot)$ is a function on $\mathbf{R}^5_+$ having the same properties as $K_o(\cdot)$ and k_4 is a positive constant independent of $\mu,\varepsilon,\eta\in(0,1]$.

4. Convergence of Approximate Solutions and Proof of the Existence Result

The solution of (PSC) is constructed in two steps of limiting process as $\eta\to 0$ and $\varepsilon,\mu\to 0$.

In the first step, parameters μ and ε are fixed, and parameter η goes to 0. For each $\varepsilon\in(0,1]$, we write $J_{1\varepsilon}$ for $J_{1\varepsilon 0}$ and u^{∞}_{ε} for $u^{\infty}_{\varepsilon 0}$.

Lemma 4.1. *Let $\mu, \varepsilon \in (0,1]$, $0 < T < +\infty$ and $[u_o, w_o] \in D_o(m_o)$. Then problem $(PSC)_{\mu\varepsilon} := (PSC)_{\mu\varepsilon 0}$ has one and only one solution $\{u_{\mu\varepsilon}, w_{\mu\varepsilon}\}$ on $[0,T]$. Moreover it satisfies the following energy inequalities (4.1) - (4.3):*

$$\tag{4.1} \begin{aligned} J_{1\varepsilon}(u_{\mu\varepsilon}(t), w_{\mu\varepsilon}(t)) + \int_0^t |u_{\mu\varepsilon} - u_\varepsilon^\infty|_V^2 d\tau \\ + \int_0^t (|w'_{\mu\varepsilon}|^2_{V_o^*} + \mu|w^\infty_{\mu\varepsilon}|^2_H) d\tau \\ \leq J_{1\varepsilon}(u_{0\varepsilon}, w_o) \end{aligned}$$

for all $t \in [0,T]$, where $u_{o\varepsilon} := u_o \wedge a_\varepsilon$;

$$\tag{4.2} |\rho_\varepsilon(u_{\mu\varepsilon}(t)) + \lambda(w_{\mu\varepsilon}(t))|^2_H \leq R_1(T)\{|\rho(u_{o\varepsilon}) + \lambda(w_o)|^2_H + J_{1\varepsilon}(u_{o\varepsilon}, w_o) + 1\}$$

for all $t \in [0,T]$;

$$\tag{4.3} \begin{aligned} t|u_{\mu\varepsilon}(t) - u_\varepsilon^\infty|^2_V + t|w'_{\mu\varepsilon}(t)|^2_{V_o^*} + t\mu|w'_{\mu\varepsilon}(t)|^2_H \\ + \nu t\left|\int_\Omega \hat\rho_\varepsilon(u_{\mu\varepsilon}(t))dx\right| + \kappa\int_0^t \tau|w'_{\mu\varepsilon}|^2_{V_o^*} d\tau \\ \leq R_3(T)\{J_{1\varepsilon}(u_{o\varepsilon}, w_o) + |\rho(u_{o\varepsilon}) + \lambda(w_o) - \rho_\varepsilon(u_\varepsilon^\infty)|^2_{V^*} + 1\} \end{aligned}$$

for all $t \in [0,T]$. Here $R_1(\cdot)$ and $R_3(\cdot)$ are the same ones as in Estimates (II), (IV).

Proof. Let $\{u_{\mu\varepsilon\eta}, w_{\mu\varepsilon\eta}\}$, $\mu, \varepsilon, \eta \in (0,1]$, be the approximate solutions considered in section 3. Then we now recall such an estimate, essentially due to [16], for the integral

$$I_{\mu\varepsilon\eta}(t) := \int_0^t \int_\Omega \lambda''(w_{\mu\varepsilon\eta})|w'_{\mu\varepsilon\eta}|^2 u_{\mu\varepsilon\eta} dx d\tau$$

in [10; the proof of Lemma 5.2] that

$$\tag{4.4} I_{\mu\varepsilon\eta}(t) \leq \delta_1 \sup_{0\leq\tau\leq t} |w'_{\mu\varepsilon\eta}(\tau)|^2_H + \delta_1 \int_0^t |w'_{\mu\varepsilon\eta}|^2_{V_o} d\tau + k_5(\mu, \delta_1)$$

for all $t \in [0,T]$, where δ_1 is an arbitrary (small) positive number and $k_5(\mu, \delta_1)$ is a positive constant depending only on μ, δ_1, but not on $\varepsilon, \eta \in (0,1]$.

It is easy to see from (3.13), (3.21), (3.22) in Remark 3.1 and (4.4) that

$$\tag{4.5} \begin{aligned} |u_{\mu\varepsilon\eta}|^2_{L^\infty(0,T;V)} + |\rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta})|^2_{L^\infty(0,T;H)} \\ + |w_{\mu\varepsilon\eta}|^2_{L^\infty(0,T;H^2(\Omega))} + |\xi_{\mu\varepsilon\eta}|^2_{L^\infty(0,T;H)} \\ + |w'_{\mu\varepsilon\eta}|^2_{L^2(0,T;V_o)} + |w'_{\mu\varepsilon\eta}|^2_{L^\infty(0,T;H)} + \int_0^T (\rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta})', u'_{\mu\varepsilon\eta}) d\tau \\ \leq k_6 := k_6(T, \mu, u_o, w_o) \end{aligned}$$

for $\varepsilon, \eta \in (0,1]$, where k_6 is a positive constant depending on T, μ and initial data $[u_o, w_o] \in D_o(m_o)$, but not on $\varepsilon, \eta \in (0,1]$. Also, by (3.2) we have

$$\tag{4.6} \int_0^T (\rho_{\varepsilon\eta}(u_{\mu\varepsilon\eta})', u'_{\mu\varepsilon\eta}) d\tau \geq C(\varepsilon)|u'_{\mu\varepsilon\eta}|^2_{L^2(0,T;H)}.$$

Therefore, for a sequence $\{\eta_n\}$ with $\eta_n \downarrow 0$ (as $n \to +\infty$), some functions $u_{\mu\varepsilon} \in W^{1,2}(0,T;H) \cap C_w([0,T];V)$, $w_{\mu\varepsilon} \in C_w([0,T];H^2(\Omega)) \cap W^{1,2}(0,T;V) \cap W^{1,\infty}(0,T;H)$ and $\xi_{\mu\varepsilon\eta} \in L^\infty(0,T;H)$ it follows from (4.5) and (4.6) that

$$u_{\mu\varepsilon\eta_n} \to u_{\mu\varepsilon} \text{ in } C([0,T];H) \cap C_w([0,T];V) \text{ and weakly in } W^{1,2}(0,T;H), \tag{4.7}$$

$$w_{\mu\varepsilon\eta_n} \to w_{\mu\varepsilon} \text{ in } C([0,T];V) \cap C_w([0,T];H^2(\Omega)) \text{ and weakly in } W^{1,2}(0,T;V) \tag{4.8}$$

and

$$\xi_{\mu\varepsilon\eta_n} \to \xi_{\mu\varepsilon} \text{ weakly}^* \text{ in } L^\infty(0,T;H); \tag{4.9}$$

these imply that $\rho_{\varepsilon\eta_n}(u_{\mu\varepsilon\eta_n}) \to \rho_\varepsilon(u_{\mu\varepsilon})$ weakly* in $L^\infty(0,T;H)$ and that $\xi_{\mu\varepsilon} \in \beta(w_{\mu\varepsilon})$ a.e. on $[0,T] \times \Omega$, since $\xi_{\mu\varepsilon\eta_n} \in \beta(w_{\mu\varepsilon\eta_n})$ a.e. on $[0,T] \times \Omega$. Besides, it is easily inferred from the above convergences (4.7) - (4.9) that $\{u_{\mu\varepsilon}, w_{\mu\varepsilon}\}$ is a (unique) solution of (PSC)$_{\mu\varepsilon}$ on $[0,T]$. By the way, since $u_{\mu\varepsilon} \le 0$ a.e. on $[0,T] \times \Omega$, condition (H3) implies that $\lambda''(w_{\mu\varepsilon\eta_n})|w'_{\mu\varepsilon\eta_n}|^2 u_{\mu\varepsilon} \le 0$ a.e. on $[0,T] \times \Omega$, so that

$$\begin{aligned}
&\limsup_{n\to+\infty} \int_0^T \int_\Omega \tau\lambda''(w_{\mu\varepsilon\eta_n})|w'_{\mu\varepsilon\eta_n}|^2 u_{\mu\varepsilon\eta_n} dx d\tau \\
&\le \limsup_{n\to+\infty} \int_0^T \int_\Omega \tau\lambda''(w_{\mu\varepsilon\eta_n})|w'_{\mu\varepsilon\eta_n}|^2 (u_{\mu\varepsilon\eta_n} - u_{\mu\varepsilon}) dx d\tau \\
&\le \text{const.} \limsup_{n\to+\infty} |u_{\mu\varepsilon\eta_n} - u_{\mu\varepsilon}|_{C([0,T];H)} \cdot |w'_{\mu\varepsilon\eta_n}|^2_{L^2(0,T;L^4(\Omega))} \\
&= 0.
\end{aligned} \tag{4.10}$$

Now, passing to the limit in (3.11) and (3.13) and (3.20) as $\eta_n \to 0$, we have by (4.10) inequalities (4.1), (4.2) and (4.3). ■

Remark 4.1. By combining (3.21) and (3.22) in Remark 3.1 with (4.10) for the solution $\{u_{\mu\varepsilon}, w_{\mu\varepsilon}\}$ of (PSC)$_{\mu\varepsilon}$ we have the following estimate:

$$\begin{aligned}
&|u_{\mu\varepsilon}(t) - u_\varepsilon^\infty|_V^2 + |w'_{\mu\varepsilon}(t)|^2_{V_o^*} + \mu|w'_{\mu\varepsilon}(t)|_H^2 + \nu\left|\int_\Omega \hat\rho_\varepsilon(u_{\mu\varepsilon}(t))dx\right| \\
&\qquad + \kappa\int_0^t |\nabla w'_{\mu\varepsilon}|_H^2 d\tau + 2\int_0^t (\rho_\varepsilon(u_{\mu\varepsilon})', u'_{\mu\varepsilon}) d\tau \\
&\quad \le \overline{K}_o(T, |\rho(u_{o\varepsilon})|_H, |u_{o\varepsilon}|_V, |w_o|_{H^2(\Omega)}, |-\Delta w_o + \xi_o|_V)
\end{aligned} \tag{4.11}$$

for a.e. $t \in [0,T]$;

$$\begin{aligned}
&|w_{\mu\varepsilon}(t)|^2_{H^2(\Omega)} + |\xi_{\mu\varepsilon}(t)|_H^2 \\
&\qquad \le \overline{K}_1(T, |\rho(u_{o\varepsilon})|_H, |u_{o\varepsilon}|_V, |w_o|_{H^2(\Omega)}, |-\Delta w_o + \xi_o|_V)
\end{aligned} \tag{4.12}$$

for all $t \in [0,T]$, where $\overline{K}_i(\cdot, \cdots, \cdot) : \mathbf{R}_+^5 \to \mathbf{R}_+$, $i = 1,2$, are increasing functions with respect to all the arguments.

Proof of Theorem 2.2. First assume that $[u_o, w_o] \in D_o(m_o)$. In this case we can use the estimates (4.11) and (4.12) in addition to (4.1) - (4.3). Therefore, for suitable sequences $\{\mu_n\}$ and $\{\varepsilon_n\}$ with $\mu_n \downarrow 0$ and $\varepsilon_n \downarrow 0$ (as $n \to +\infty$) and some functions $u \in L^\infty(0,T;V)$, $\tilde{\rho} \in L^\infty(0,T;H)$, $w \in C_w([0,T];H^2(\Omega))$ and $\xi \in L^\infty(0,T;H)$, the solution $\{u_n, w_n\}$ of $(\text{PSC})_n$ $:= (\text{PSC})_{\mu_n\varepsilon_n}$ converges to the couple $\{u, w\}$ in the sense that

$$u_n \to u \text{ weakly}^* \text{ in } L^\infty(0,T;V), \tag{4.13}$$

$$\rho_n := \rho_{\varepsilon_n}(u_n) \to \tilde{\rho} \text{ in } C_w([0,T];H) \text{ and weakly in } W^{1,2}(0,T;V^*), \tag{4.14}$$

$$w_n \to w \text{ in } C([0,T];H) \cap C_w([0,T];H^2(\Omega)), \tag{4.15}$$

$$w_n' \to w' \text{ weakly}^* \text{ in } L^\infty(0,T;V_o^*), \tag{4.16}$$

$$\xi_n := \xi_{\mu_n\varepsilon_n} \to \xi \text{ weakly}^* \text{ in } L^\infty(0,T;H). \tag{4.17}$$

By the standard techniques of the maximal monotone operator theory, it follows from (4.13) and (4.14) that $\tilde{\rho} = \rho(u)$ and $u_n \to u$ in $C_w([0,T];V)$ (cf. [10; the proof of Theorem 2.3] and [15;Remark 1.3]), and also from (4.15) and (4.17) that $\xi \in \beta(w)$ a.e. on $[0,T] \times \Omega$. Now, it is easy to see that the limit $\{u, w\}$ is a solution of (PSC) on $[0,T]$, and estimates (2.18) - (2.20) hold for $s = 0 < t \le T$ and $M_1(T) = R_1(T), M_2(T) = R_3(T)$, by Lemma 4.1.

Secondly, consider the general case of $[u_o, w_o] \in D(m_o)$. In this case, choose a sequence $\{[u_{ok}, w_{ok}]\}$ in $D_o(m_o)$ such that

$$\rho(u_{ok}) \to \rho(u_o) \text{ in } H,\ w_{ok} \to w_o \text{ in } V,\ J_1(u_{ok}, w_{ok}) \to J_1(u_o.w_o)$$

as $k \to +\infty$. As was shown above, for each k problem $(\text{PSC};f,h,u_{ok},w_{ok})$ has one and only one solution $\{u_k, w_k\}$ on $[0,T]$ and it satisfies inequalities (2.18) - (2.20) and (2.21) for $s = 0 < t \le T$. Therefore, there is a subsequence of $\{u_k, w_k\}$, denoted by $\{u_k, w_k\}$ again, with some functions $u, \tilde{\rho}, w$ and ξ such that

$$\begin{aligned} u_k \to u &\text{ weakly in } L^2(0,T;V) \\ &\text{and weakly}^* \text{ in } L^\infty(\delta,T;V) \text{ for every } 0 < \delta \le T, \end{aligned} \tag{4.18}$$

$$\rho(u_k) \to \tilde{\rho} \text{ weakly}^* \text{ in } L^\infty(0,T;H), \tag{4.19}$$

$$\begin{aligned} w_k \to w \text{ in } C([0,T];H) \cap C_w(([0,T];V),\ &\text{weakly in } L^2(0,T;H^2(\Omega)) \\ &\text{and in } C_w([\delta,T];H^2(\Omega)) \text{ for every } 0 < \delta \le T, \end{aligned} \tag{4.20}$$

$$\begin{aligned} w_k' \to w' &\text{ weakly in } L^2(0,T;V_o^*) \\ &\text{and weakly}^* \text{ in } L^\infty(\delta,T;V_o^*) \text{ for every } 0 < \delta \le T, \end{aligned} \tag{4.21}$$

$$\begin{aligned} \xi_k \to \xi &\text{ weakly in } L^2(0,T;H) \\ &\text{and weakly}^* \text{ in } L^\infty(\delta,T;H) \text{ for every } 0 < \delta \le T. \end{aligned} \tag{4.22}$$

Just as in the first step, (4.18) and (4.19) imply that $\tilde{\rho} = \rho(u)$ and

$$u_k \to u \text{ in } C_w([\delta, T]; V) \text{ for every } 0 < \delta \leq T, \tag{4.23}$$

and (4.20) and (4.22) imply that $\xi \in \beta(w)$ a.e. on $[0, T] \times \Omega$. Also, convergences (4.18) - (4.22) are enough to show that $\{u, w\}$ is a solution of our problem (PSC) on $[0, T]$ and satisfies (2.18) - (2.20) for $s = 0 < t \leq T$. We finally get a global solution $\{u, w\}$ of (PSC), since T is arbitrary.

Inequalities (2.18) - (2.20) for $s > 0$ and $T > 0$ are immediately obtained by taking s as the initial time. ∎

5. The Attractor for the Semigroup Associated With (PSC)

Based on Theorem 2.2, for each $t \geq 0$ we can define a mapping $S(t) : D(m_o) \to D(m_o)$ by

$$S(t)[u_o, w_o] = [u(t), w(t)],\ [u_o, w_o] \in D(m_o), \tag{5.1}$$

where $\{u, w\}$ is the global solution of (PSC;f, h, u_o, w_o).

Theorem 5.1. *Assume that (H1) - (H5), (2.16) and (2.17) are satisfied. Let $\sigma_* < m_o < \sigma^*$. Then the family $\{S(t)\} := \{S(t); t \geq 0\}$, defined by (5.1), satisfies the following properties:*

: *(a) $\{S(t)\}$ is a semigroup defined on $D(m_o)$, i.e.*

$$S(0) = I \text{ on } D(m_o),\ S(t+s) = S(t)S(s) \text{ on } D(m_o) \text{ for any } s, t \geq 0.$$

: *(b) Let $0 < \delta < T < +\infty$. Then $S(\cdot)[z, v] \in C_w([\delta, T]; V \times H^2(\Omega))$ for any $[z, v] \in D(m_o)$. Moreover, if $[z_n, v_n] \in D(m_o), n = 1, 2, \cdots, [z, v] \in D(m_o)$, $\{J_1(z_n, v_n)\}$ is bounded, $\rho(z_n) \to \rho(z)$ weakly in H and $v_n \to v$ weakly in V as $n \to +\infty$, then*

$$S(\cdot)[z_n, v_n] \to S(\cdot)[z, v] \text{ in } C_w([\delta, T]; V \times H^2(\Omega)) \tag{5.2}$$

as $n \to +\infty$.

: *(c) For each $M > 0$, $D^M(m_o)$ is positively invariant for $\{S(t)\}$ (i.e. $S(t)D^M(m_o) \subset D^M(m_o)$ for all $t \geq 0$), and in particular,*

$$S(t)D^M(m_o) \subset D^M_o(m_o) \text{ for all } t > 0. \tag{5.3}$$

Proof. Assertion (a) is a direct consequence of the global existence and uniqueness result (cf. Theorems 2.1 and 2.2) for (PSC), and assertion (c) follows from (2.18) - (2.21) and (2.23). The proof of assertion (b) is exactly same as that of the second step in the proof of Theorem 2.2; see (4.20) and (4.23). ∎

The main result of this paper is stated in the following theorem.

Theorem 5.2. *Assume that (H1) - (H5) with $\nu > 0$, (2.16) and (2.17) are satisfied. Let $\sigma_* < m_o < \sigma^*$ and let $\{S(t)\}$ be the semigroup on $D(m_o)$ defined by (5.1). Then there exists a subset A of $D_o(m_o)$ such that*

: *(i) A is compact and connected in $H \times V$, and is bounded in $V \times H^2(\Omega)$;*

: *(ii) A is invariant for $\{S(t)\}$, i.e. $A = S(t)A$ for all $t \geq 0$;*

: *(iii) for each subset B of $D(m_o)$ with* $\sup_{[z,v]\in B}\{J_1(z,v)+|\rho(z)|_H^2\} < +\infty$, *and for each $\varepsilon > 0$ there exists a finite time $T_{B,\varepsilon} > 0$ such that*

$$\mathrm{dist}_{H\times V}(S(t)[z,v],A) < \varepsilon \text{ for all } [z,v]\in B \text{ and all } t \geq T_{B,\varepsilon},$$

where $\mathrm{dist}_{H\times V}(\cdot,\cdot)$ *stands for the distance in $H \times V$.*

In this paper, we say that a set A is a global attractor for the semigroup $\{S(t)\}$, if it has properties (i), (ii) and (iii) of Theorem 5.2. The key for the proof of Theorem 5.2 is to find an absorbing set B_o for the semigroup $\{S(t)\}$. To do so we prove a lemma under the same assumptions of Theorem 5.2.

Lemma 5.1. *(1) There is a positive constant N_o such that*

$$\kappa|\pi_o w(t)|_{V_o}^2 + \int_\Omega \hat\beta(w(t))dx \leq N_o\{|w'(t)|_{V_o^*} + |u(t)|_H + 1\}, \quad a.e.\ t \geq 0, \tag{5.4}$$

for all global solutions $\{u,w\}$ with initial data $[u_o,w_o]\in D(m_o)$.

(2) There are positive constants $\varepsilon_o, \varepsilon_1$ and N_1 such that

$$\begin{aligned}&\frac{d}{dt}\{J_1(u(t),w(t)) + \varepsilon_o|\rho(u(t))+\lambda(w(t))|_H^2\}\\ &\quad + \varepsilon_1\{J_1(u(t),w(t)) + \varepsilon_o|\rho(u(t))+\lambda(w(t))|_H^2\}\\ &\leq N_1, \quad a.e.\ t\geq 0,\end{aligned} \tag{5.5}$$

for all global solutions $\{u,w\}$ with initial data $[u_o,w_o]\in D(m_o)$.

Proof. (1) We multiply (2.6) by $\pi_o w(t) (= w(t) - m_o)$ to get

$$\begin{aligned}(F_o^{-1}w'(t),\pi_o w(t)) + \kappa|\pi_o w(t)|_{V_o}^2 + (\xi(t), w(t)-m_o) + (g(w(t)), w(t)-m_o)\\ = (\lambda'(w(t))u(t), w(t)-m_o) \text{ for a.e. } t\geq 0.\end{aligned}$$

This implies that (5.4) holds for some constant N_o independent of initial data $[u_o,w_o]\in D(m_o)$, since

$$(\xi(t), w(t)-m_o) \geq \int_\Omega \hat\beta(w(t))dx - |\Omega|\hat\beta(m_o),$$

$$|(F_o^{-1}w'(t),\pi_o w(t))| \leq \text{const.}|w'(t)|_{V_o^*}$$

and

$$|(\lambda'(w(t))u(t), w(t)-m_o)| \leq \text{const.}|u(t)|_H.$$

(2) By the definition of subdifferential ρ^{-1} of $\hat{\rho^{-1}}$, we observe that

$$\begin{aligned}&\int_\Omega \hat{\rho^{-1}}(\rho(u(t)))dx - (\rho(u(t))+\lambda(w(t)), u^\infty)\\ &\leq (\rho(u(t)) - \rho(u^\infty), u(t)) + \int_\Omega \hat{\rho^{-1}}(\rho(u^\infty))dx\\ &\qquad - (\rho(u(t))+\lambda(w(t)), u^\infty)\\ &= (\rho(u(t))+\lambda(w(t)), u(t)-u^\infty)\\ &\qquad - (\rho(u^\infty)+\lambda(w(t)), u(t)) + \int_\Omega \hat{\rho^{-1}}(\rho(u^\infty))dx\\ &\leq k_7\{|u(t)-u^\infty|_V^2 + |\rho(u(t))+\lambda(w(t))|_H^2 + 1\} \text{ for all } t > 0,\end{aligned} \tag{5.6}$$

where k_7 is a positive constant independent of any solution $\{u, w\}$ of (PSC) with $[u_o, w_o] \in D(m_o)$. Combining (5.4) and (5.6), we see that

$$\tag{5.7} \begin{aligned} J_1(u(t), w(t)) \le k_8\{|u(t) - u^\infty|_V^2 + |w'(t)|_{V_o^*}^2 \\ + |\rho(u(t)) + \lambda(w(t))|_H^2 + 1\} \text{ for a.e. } t \ge 0, \end{aligned}$$

where k_8 is a positive constant independent of any solution $\{u, w\}$ of (PSC) with $[u_o, w_o] \in D(m_o)$. Besides, recall the inequalities

$$\tag{5.8} \frac{d}{dt} J_1(u(t), w(t)) + |u(t) - u^\infty|_V^2 + |w'(t)|_{V_o^*}^2 \le 0 \text{ for a.e. } t \ge 0,$$

and

$$\tag{5.9} \begin{aligned} \frac{d}{dt}|\rho(u(t)) + \lambda(w(t))|_H^2 + \frac{\nu}{2}|\rho(u(t)) + \lambda(w(t))|_H^2 \\ \le k_2\{|u(t) - u^\infty|_V^2 + |w(t)|_V^2 + 1\} \end{aligned}$$

for a.e. $t \ge 0$, which are derived from (2.18) and (3.14), respectively. In (5.9), note from (5.4) that $|w(t)|_V^2$ can be estimated by const.$\{|w'(t)|_{V_o^*} + |u(t)|_H + 1\}$, so that

$$\tag{5.10} \begin{aligned} \frac{d}{dt}|\rho(u(t)) + \lambda(w(t))|_H^2 + \frac{\nu}{2}|\rho(u(t)) + \lambda(w(t))|_H^2 \\ \le k_9\{|u(t) - u^\infty|_V^2 + |w'(t)|_{V_o^*}^2 + 1\} \end{aligned}$$

for a.e. $t \ge 0$, where k_9 is a positive constant independent of all solutions $\{u, w\}$ of (PSC) with $[u_o, w_o] \in D(m_o)$.

Now, compute (5.8) + $\varepsilon_o \times$ (5.10) with $\varepsilon_o = \frac{1}{2k_9}$ to obtain

$$\tag{5.11} \begin{aligned} &\frac{d}{dt}\{J_1(u(t), w(t)) + \varepsilon_o|\rho(u(t)) + \lambda(w(t))|_H^2\} \\ &\quad + \frac{1}{2}\{|u(t) - u^\infty|_V^2 + |w'(t)|_{V_o^*}^2 + 1\} \\ &\quad + \frac{\nu\varepsilon_o}{2}|\rho(u(t)) + \lambda(w(t))|_H^2 \\ &\le 1 \end{aligned}$$

for a.e. $t \ge 0$. Therefore, putting

$$\varepsilon_1 = \min\{\frac{1}{2k_8}, \frac{\nu\varepsilon_o}{4k_8}, \frac{\nu}{4}\} \text{ and } N_1 = 1,$$

we obtain (5.5) immediately from (5.11). ∎

Remark 5.1. In general, for a solution $\{u, w\}$ of (PSC) we do not know the absolute continuity of $J_1(u(t), w(t))$ in time t. But it follows from (2.18) of Theorem 2.2 that $J_1(u(t), w(t))$ is of bounded variation (hence almost everywhere differentiable in t) and its derivative $\frac{d}{dt}J_1(u, w)$ is integrable on each bounded interval of $\mathbf{R}_+$, and

$$J_1(u(t), w(t)) - J_1(u(s), w(s)) \le \int_s^t \frac{d}{d\tau} J_1(u, w) d\tau \text{ for any } 0 \le s \le t < +\infty.$$

Therefore inequality (5.5) makes sense. Also, inequality (5.9) similarly makes sense.

Now we fix the functional

$$J(z,v) := J_1(z,v) + \varepsilon_o|\rho(z)+\lambda(v)|_H^2$$

with ε_o in (2) of Lemma 5.1. For the simplicity of notation we write $J(S(t)[u_o,w_o])$ for $J(u(t),w(t))$, $\{u,w\}$ being the global solution of (PSC) with initial data $[u_o,w_o] \in D(m_o)$.

Lemma 5.2. *There exists a subset B_o of $D(m_o)$ such that*

: *(a)* $\sup_{[z,v]\in B_o} J(z,v) < +\infty$,

: *(b) for any subset B with $\sup_{[z,v]\in B} J(z,v) < +\infty$, there exists a time $t_B > 0$ such that*

$$S(t)B \subset B_o \textit{ for all } t \geq t_B.$$

Proof. From (5.5) of Lemma 5.1 it follows that

$$J(S(t)[z,v]) \leq e^{-\varepsilon_1 t}J(z,v) + \frac{N_1}{\varepsilon_1} \text{ for all } t \geq 0 \text{ and all } [z,v] \in D(m_o).$$

Now, take a subset

$$B_o := \{[z,v] \in D(m_o); J(z,v) \leq 1 + \frac{N_1}{\varepsilon_1}\}.$$

Then, B_o clearly satisfies properties (a) and (b). ∎

Lemma 5.3. *Let B be any subset of $D(m_o)$ with* $\sup_{[z,v]\in B} J(z,v) < +\infty$, *and δ be any positive number. Then $B_\delta := \overline{\cup_{t\geq\delta} S(t)B}^{H\times V}$ is in $D_o(m_o)$, compact in $H \times V$, bounded in $V \times H^2(\Omega)$ and*

$$\sup_{[z,v]\in B_\delta} J(z,v) < +\infty.$$

Proof. By Lemma 5.2, there is a finite time $t_o > 0$ such that $S(t)B \subset B_o$ and $S(t)B_o \subset B_o$ for all $t \geq t_o$. Hence, if $t \geq 2t_o+\delta$, then $S(t)B = S(\delta)S(t-t_o-\delta)S(t_o)B \subset S(\delta)B_o$. By (2.18) - (2.23) of Theorem 2.2 and Remarks 2.2, 2.3, $S(\delta)B_o$ is in $D_o(m_o)$, bounded in $V \times H^2(\Omega)$ and $\sup_{[z,v]\in S(\delta)B_o} J(z,v) < +\infty$. Therefore, $B_{2t_o+\delta} := \overline{\cup_{t\geq 2t_o+\delta} S(t)B}^{H\times V}$ is in $D_o(m_o)$, compact in $H \times V$, bounded in $V \times H^2(\Omega)$ and $\sup_{[z,v]\in B_{2t_o+\delta}} J(z,v) < +\infty$. Applying again (2.18) - (2.23) with $s = 0$ and $T = 2t_o + \delta$, we see that the set

$$B_{\delta,2t_o+\delta} := \overline{\bigcup_{\delta\leq t\leq 2t_o+\delta} S(t)B}^{H\times V}$$

has the same properties as $B_{2t_o+\delta}$. Consequently B_δ $(\subset B_{2t_o+\delta} \cup B_{\delta,2t_o+\delta})$ has the required properties. ∎

Proof of Theorem 5.2. The construction of A is quite standard. In fact, we shall show that the set

$$A := \bigcap_{s>0} \overline{\bigcup_{t\geq s} S(t)B_o}^{H\times V} \tag{5.12}$$

is the required one, where B_o is the absorbing set found by Lemma 5.2. By (5.12) it is clear that

$$(5.13)\qquad \begin{cases} [z,v] \in A \text{ if and only if there are sequences } \{t_n\} \text{ with} \\ t_n \uparrow +\infty \text{ as } n \to +\infty \\ \text{and } \{[z_n,v_n]\} \subset B_o \text{ such that} \\ S(t_n)[z_n,v_n] \to [z,v] \text{ in } H \times V. \end{cases}$$

Moreover, on account of Lemmas 5.2 and 5.3, we see that $A \subset D_o(m_o) \cap B_o$, and in (5.13) the sequences $\{t_n\}$ and $\{[z_n,v_n]\}$ can be chosen so as to satisfy further that for some $[\tilde{z},\tilde{v}] \in D_o(m_o)$

$$(5.14)\qquad \begin{cases} S(t_n)[z_n,v_n] \to [z,v] \text{ weakly in } V \times H^2(\Omega), \\ [z_n,v_n] \to [\tilde{z},\tilde{v}] \text{ weakly in } V \times H^2(\Omega), \\ \sup_{n\ge 1} J(z_n,v_n) < +\infty,\ \sup_{n\ge 1} J(S(t_n)[z_n,v_n]) < +\infty. \end{cases}$$

(i) By Lemma 5.3, for each $s > 0$, $B_s := \overline{\cup_{t\ge s} S(t)B_o}^{H\times V}$ is compact in $H \times V$ and bounded in $V \times H^2(\Omega)$, so is A.

Next, by contradiction we show the connectedness of A. Assume that A is not connected in $H \times V$. Then there would exist two compact sets A_1 and A_2 in $H \times V$ such that

$$A_1 \cup A_2 = A,\ A_1 \cap A_2 = \emptyset,\ A_i \ne \emptyset,\ i = 1,2.$$

Let $X_i := [z^i, v^i] \in A_i$ for $i = 1,2$, and choose by (5.13) and (5.14) sequences $\{t_n^i\}$ with $t_n^i \uparrow +\infty$ and $X_n^i := [z_n^i, v_n^i] \in B_o \cap D_o(m_o)$ such that

$$(5.15)\qquad \begin{aligned} &S(t_n^i)X_n^i \to X^i \text{ weakly in } V \times H^2(\Omega), \\ &\qquad \sup_{n\ge 1} J(S(t_n^i)X_n^i) < +\infty,\ i = 1,2. \end{aligned}$$

Without loss of generality we may assume that

$$t_n^1 < t_n^2 < t_{n+1}^1,\ t_n^2 - t_n^1 \ge 1,\ \text{for all } n = 1,2,\cdots.$$

Putting $\tilde{X}_n^2 := S(t_n^2 - t_n^1)X_n^2$, we have

$$(5.16)\qquad S(t_n^1)\tilde{X}_n^2\ (= S(t_n^2)X_n^2)\ \to X^2 \text{ weakly in } V \times H^2(\Omega)$$

and by Lemma 5.3

$$(5.17)\qquad \sup_{n\ge 1} J(\tilde{X}_n^2) < +\infty.$$

Consider the segment

$$L_n(\tau) := \tau X_n^1 + (1-\tau)\tilde{X}_n^2,\ \tau \in [0,1].$$

Then, we easily observe from (5.15) - (5.17) that

$$(5.18)\qquad L_n(\tau) \in D_o(m_o),\ \tau \in [0,1],\ \sup_{\tau\in[0,1],n\ge 1} J(L_n(\tau)) < +\infty.$$

This implies by (b) of Theorem 5.1 that $S(t_n^1)L_n(\tau)$, $\tau \in [0,1]$, is a continuous curve combining $S(t_n^1)X_n^1$ and $S(t_n^1)\tilde{X}_n^2$ in $H \times V$. Now, take ε-neighborhoods U_ε^i of A_i for $i = 1,2$, for a sufficiently small number $\varepsilon > 0$ so that $U^1_\varepsilon \cap U^2_\varepsilon = \emptyset$. In this case, $S(t^1_n)X^1_n \in U^1_\varepsilon$ and $S(t^1_n)\tilde{X}^2_n \in U^2_\varepsilon$ for all sufficiently large n (cf. (5.15),(5.16)). Hence there is a sequence $\{\tau_n\} \subset (0,1)$ such that

$$S(t^1_n)L_n(\tau_n) \notin U^1_\varepsilon \cup U^2_\varepsilon. \tag{5.19}$$

Besides, by Lemma 5.2, choose a positive number t_o such that

$$S(t_o)L_n(\tau) \in B_o \text{ for all } \tau \in [0,1],\ n = 1,2,\cdots .$$

Since $S(t^1_n)L_n(\tau) = S(t^1_n - t_o)S(t_o)L_n(\tau) \subset S(t^1_n - t_o)B_o$, it follows from Lemma 5.3 that $\{S(t^1_n)L_n(\tau); \tau \in [0,1], n \geq n_o\}$ is bounded in $V \times H^2(\Omega)$ for all sufficiently large n_o. Therefore, by (5.13), any accumulation point of $\{S(t^1_n)L_n(\tau_n); n \geq n_o\}$ in $H \times V$ belongs to A. This contradicts (5.19).

(ii) First we prove $S(t)A \subset A$ for all $t > 0$. Let $X := [z,v]$ be any element of A. Then, there are sequences $\{t_n\}$ with $t_n \uparrow +\infty$ and $\{X_n := [z_n, v_n]\}$ satisfying the properties in (5.13) and (5.14). We note by (b) of Theorem 5.1 that for $t > 0$

$$S(t + t_n)X_n = S(t)S(t_n)X_n \to S(t)X \text{ weakly in } V \times H^2(\Omega).$$

Hence $S(t)X \in A$ by (5.13) again. Thus $S(t)A \subset A$.

Conversely, we show $A \subset S(t)A$ for $t > 0$. Let $X := [z,v]$ be any element of A. Then, there are sequences $\{t_n\}$ and $\{X_n := [z_n, v_n]\} \subset D_o(m_o) \cap B_o$ satisfying (5.13) and (5.14). Now, by Lemmas 5.2 and 5.3, $\{Y_n := S(t_n - t)X_n; n \geq n_o\}$ is bounded in $V \times H^2(\Omega)$ and $\sup_{n \geq n_o} J(Y_n) < +\infty$ (if n_o is sufficiently large), so we may assume that $Y_n \to Y$ weakly in $V \times H^2(\Omega)$ for some $Y \in A$. Applying (b) of Theorem 5.1 we see that

$$S(t)Y_n \to S(t)Y \text{ weakly in } V \times H^2(\Omega),$$

so that $X = S(t)Y$, since $S(t)Y_n = S(t_n)X_n \to X$ weakly in $V \times H^2(\Omega)$ by (5.13). Thus $A \subset S(t)A$.

(iii) By Lemma 5.2 it is enough to show that

$$\mathrm{dist}_{H \times V}(S(t)[z,v], A) \to 0 \text{ uniformly in } [z,v] \in B_o \text{ as } t \to +\infty.$$

But this is evident from the definition (5.12) of A. ∎

Acknowledgment. The author would like to thank Professor J. K. Hale for his valuable comments on the approach of this paper to the construction of global attractors. He kindly pointed out that a similar idea was already used in some papers dealing with construction of global attractors of dynamical systems associated with damped forced wave or KdV equations.

References

[1] H. W. Alt and I. Pawlow, *Existence of solutions for non-isothermal phase separation*, Adv. Math. Sci. Appl. **1** (1992), 319-409.

[2] J. F. Blowey and C. M. Elliott, *The Cahn-Hilliard gradient theory for phase separation with non-smooth free energy*, European J. Appl. Math. **I**, **2** (1991), 233-280.

[3] H. Brézis, M. Crandall and A. Pazy, *Perturbations of nonlinear maximal monotone sets*, Comm. Pure Appl. Math. **23** (1970), 123-144.

[4] J. K. Hale, *Asymptotic Behaviour of Dissipative Systems*, Math. Surveys Monographs, **25**, Amer. Math. Soc., Providence, Rhode Island, 1988.

[5] A. Ito and N. Kenmochi, *Asymptotic behaviour of solutions to non-isothermal phase separation model with constraints in one-dimensional space*, Tech. Rep. Math. Sci., Chiba Univ., **9**, No. 12, 1993.

[6] N. Kenmochi, *Uniqueness of the solution to a nonlinear system arising in phase transition*, in Nonlinear Analysis and Applications, GAKUTO Inter. Ser. Math. Sci. Appl. **7**, Gakkōtosho, Tokyo, 1995, 261-271.

[7] N. Kenmochi and M. Niezgódka, *Systems of nonlinear parabolic equations for phase change problems*, Adv. Math. Sci. Appl. **3** (1993/94), 89-117.

[8] N. Kenmochi and M. Niezgódka, *Nonlinear system for non-isothermal diffusive phase separation*, J. Math. Anal. Appl. **188** (1994), 651-679.

[9] N. Kenmochi and M. Niezgódka, *Large time behaviour of a nonlinear system for phase separation*, in Progress in PDEs: the Metz Surveys 2, Pitman Res. Notes Math. Ser., **296**, Longman Sci. Tech., New York, 1993, 12-22.

[10] N. Kenmochi and M. Niezgódka, *Viscosity approach for modeling non-isothermal diffusive phase separation*, Japan J. Industrial Appl. Math. **13** (1996), 135-169.

[11] N. Kenmochi, M. Niezgódka and I. Pawlow, *Subdifferential operator approach to the Cahn-Hilliard equation with constraint*, J. Differential Equations **117** (1995), 320-356.

[12] N. Kenmochi, M. Niezgódka and S. Zheng, *Global attractor of a non-isothermal model for phase separation*, in Curvature Flows and Related Topics, GAKUTO Inter. Ser. Math. Sci. **5**, Gakkōtosho, Tokyo, 1995, 129-143.

[13] O. Penrose and P. C. Fife, *Thermodynamically consistent models of phase-field type for the kinetics of phase transitions*, Phys. D, **43** (1990), 44-62.

[14] W. Shen and S. Zheng, *On the coupled Cahn-Hilliard equations*, Comm. Partial Differential Equations **18** (1993), 701-727.

[15] J. Shirohzu, N. Sato and N. Kenmochi, *Asymptotic convergence in models for phase change problems*, in Nonlinear Analysis and Applications, GAKUTO Inter. Ser. Math. Sci. Appl. **7**, Gakkōtosho, Tokyo, 1995, 361-385.

[16] J. Sprekels and S. Zheng, *Global smooth solutions to a thermodynamically consistent model of phase-field type in higher space dimensions*, J. Math. Anal. Appl. **176** (1993), 200-223.

[17] R. Temam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, Springer-Verlag, Berlin, 1988.

Department of Mathematics, Faculty of Education, Chiba University, Inage-ku, Chiba, 263 Japan
E-mail address: kenmochi@math.e.chiba-u.ac.jp