SPECIES SURVIVAL VERSUS EIGENVALUES

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Mathematical models describing the behavior of hypothetical species in spatially hetero-
geneous environments are discussed and analyzed using the fibering method devised and
developed by S. I. Pohozaev.

1. Introduction

The purpose of this work is to analyze mathematical models describing the behavior of
individuals inhabiting an environment with spatial heterogeneity. The latter means that
the environment influences the reproductivity and mobility of the species. This effect is
obtained if we model the density function of these organisms by evolution partial differ-
ential equations with nonconstant coefficients. More precisely, we introduce a function
which corrects the reproductive capabilities of these species and characterizes the most
and least favorable regions for its development. With this at hand, we wish to know if
such an environment constitutes a refuge, that is, if this environment has good enough
conditions for the studied species persistence, which in mathematical terms implies the
existence of a stable nontrivial stationary solution—the limit density of these individuals
in the steady state.

Such a stationary solution is obtained by making use of the fibering method [7, 13,
14, 15, 16], introduced and developed by Pohozaev. This method implies that the funda-
mental eigenvalue of a certain spectral problem must be of a determined magnitude.

This paper is organized as follows. In the next section, we present the details of the
considered mathematical model. In Section 3, we apply the fibering method to a problem
studied by Cantrell and Cosner [3, 4, 5, 6] and discuss the relationship between their ap-
proach and the one used here. Section 4 contains concluding remarks and comments. In
the appendix, we adapt the method to a particular case considered by Ludwig et al. [10].

2. The mathematical model

We consider a species which inhabits a limited plane region $\Omega \subset \mathbb{R}^2$. We suppose that Ω
is a domain, that is, an open and connected set. Further, we admit that the population

dynamics is of reaction-diffusion type and movement and reproduction capacities depend on the position in Ω. More precisely, let $u = u(t, x, y)$ be a positive function which is the density of the individuals of the population at the moment $t \geq 0$ near the point with coordinates (x, y), then the initial/boundary value problem governing the density u is:

$$u_t = \nabla \cdot (D(x, y)\nabla u) + r\gamma(x, y)u - \frac{r}{K}u^2, \quad (x, y) \in \Omega, \ t \in (0, \infty),$$
$$u = 0, \quad (x, y) \in \partial\Omega, t > 0, \tag{2.1}$$
$$u = u_0(x, y), \quad (x, y) \in \Omega, \ t = 0.$$

Here, the positive and bounded function D denotes the diffusion coefficient, which is permitted to depend on the spatial variables (x, y). It is assumed that there exist positive constants c_1 and c_2 such that $0 < c_1 \leq D(x, y) \leq c_2$. The function γ, which measures the interference of the environment in the reproductive capacities of the studied population u, may also depend on (x, y). We suppose that γ is a bounded function such that $|\gamma| \leq 1$. Moreover, we admit that γ may assume negative values elsewhere, indicating in this way the existence of some absolutely improper subregions for the development of the population. However we will consider the environmental saturation as a constant K in Ω. This constant is known as the support capacity for the subregion where $\gamma > 0$. As in the classical Verhulst model, the constant $r > 0$ is the rate of intrinsic growth. Finally, the bounded and nonnegative function u_0 is the initial population distribution.

The analysis of species survival via models like the problem (2.1) is based on the existence and stability of *stationary solutions*, that is, solutions of the following problem:

$$-\nabla \cdot (D(x, y)\nabla u) = r\gamma(x, y)u - \frac{r}{K}u^2, \quad (x, y) \in \Omega, \ u = 0, \ (x, y) \in \partial\Omega. \tag{2.2}$$

The existence or nonexistence of nontrivial solutions to the above stationary problem determines the chances for success in the colonization of the environment.

In (2.2), we scale the independent variables:

$$\overline{x} := \frac{x}{\sqrt{|\Omega|}}, \tag{2.3}$$

$$\overline{y} := \frac{y}{\sqrt{|\Omega|}}, \tag{2.4}$$

$$\overline{t} := rt; \tag{2.5}$$

we also use the dimensionless unknown function

$$\overline{u} := \frac{u}{K}. \tag{2.6}$$

Introducing the new functions

$$\overline{D}(\overline{x}, \overline{y}) := \frac{1}{r|\Omega|}D\left(\sqrt{|\Omega|}\overline{x}, \sqrt{|\Omega|}\overline{y}\right), \qquad \overline{\gamma}(\overline{x}, \overline{y}) := \gamma\left(\sqrt{|\Omega|}\overline{x}, \sqrt{|\Omega|}\overline{y}\right), \tag{2.7}$$

where $|\Omega|$ denotes the area of Ω, we obtain the following problem to be analyzed:

$$-\nabla \cdot (D(x,y)\nabla u) = \gamma(x,y)u - u^2, \quad (x,y) \in U,$$
$$u = 0, \quad (x,y) \in \partial U. \tag{2.8}$$

Above, U denotes the image of Ω via the transformation (2.3), (2.4) with $|U| = 1$ (for notation simplicity we have omitted the bars).

We also introduce a control parameter $\alpha > 0$ in order to modulate the reproductivity differences in the environment quality without changing its saturation and sign. Therefore, we will study the following variant of our problem (2.8):

$$-\nabla \cdot (D(x,y)\nabla u) = \alpha\gamma(x,y)u - u^2, \quad (x,y) \in U,$$
$$u = 0, \quad (x,y) \in \partial U. \tag{2.9}$$

Since (2.9) is of divergence form without first-order terms, we will use variational techniques.

To begin with, let the Sobolev space $W := H_0^1(\Omega)$ be considered with the norm

$$\|u\| := \left(\iint_U D(x,y)|\nabla u|^2 dx\,dy \right)^{1/2}. \tag{2.10}$$

This norm is equivalent to the usual norm of $H_0^1(U)$. Indeed, this follows from the assumptions on D and from the Poincaré inequality. Then, $u \in W$ is called a *weak solution* of (2.9) if for all $v \in W$,

$$\iint_U D(x,y)\nabla u \cdot \nabla v\,dx\,dy = \alpha \iint_U \gamma(x,y)uv\,dx\,dy - \iint_U u^2 v\,dx\,dy. \tag{2.11}$$

The solutions of the variational problem will be identified with the critical points of the following *Euler functional*:

$$\mathcal{J}_\alpha(u) := \frac{1}{2} \iint_U D(x,y)|\nabla u|^2 dx\,dy - \frac{\alpha}{2} \iint_U \gamma(x,y)u^2 dx\,dy + \frac{1}{3} \iint_U |u|^3 dx\,dy$$
$$= \frac{1}{2}\|u\|^2 - \frac{\alpha}{2}G(u) - \frac{1}{3}F(u), \tag{2.12}$$

where

$$F(u) := -\iint_U |u|^3 dx\,dy, \qquad G(u) := \iint_U \gamma(x,y)u^2 dx\,dy. \tag{2.13}$$

Biological models in the terms of reaction-diffusion equations are well known for a long time (see for instance [9, 11, 12]). The present model was proposed by Cantrell and Cosner [2, 3, 4, 6] (see also [8]). However in the cited works, these authors emphasize its importance for the study of species persistence in heterogeneous environments. Moreover, they admit negative values of some of the parameters. Motivated by these papers, we obtain here similar results using other methods, namely, the fibering method of S. I. Pohozaev. This will be done in the next section.

3. The fibering method

In the study of the stationary problem (2.9), we will use the fibering method introduced and developed by Pohozaev in [13, 14, 15, 16]. This method provides a powerful tool for proving existence theorems, in particular for problems which obey certain kinds of homogeneity. In [7], Drábek and Pohozaev have applied the method to an equation involving the p-Laplacian operator. We will describe here an adapted version of the fibering method to our specific problem. The exposition will follow [7, 16] closely. We present enough details of the results in order to compare with those obtained in [2, 3, 4, 6] and discuss the relationship between them, in particular their importance to biology.

To begin with, we consider the Euler functional defined in (2.12). Clearly, $\mathscr{J}_\alpha(u) \in C^1$. Critical points of $\mathscr{J}_\alpha(u)$ are then weak solutions of the problem. Later we will associate to $\mathscr{J}_\alpha(u)$ another functional with additional properties. For this purpose, the magnitude α will be compared with the fundamental eigenvalue λ_1 of the problem

$$\iint_U D(x,y)\nabla\phi \cdot \nabla\psi\,dx\,dy = \lambda \iint_U \gamma(x,y)\phi\psi\,dx\,dy, \quad \phi = 0 \text{ in } \partial U, \qquad (3.1)$$

for any $\psi \in W$. We recall that the fundamental eigenvalue λ_1 of the problem (3.1) can be characterized as follows:

$$\lambda_1 = \min_{\phi \in W, \iint_U \gamma(x,y)\phi^2 dx\,dy > 0} \frac{\iint_U D(x,y)|\nabla\phi|^2 dx\,dy}{\iint_U \gamma(x,y)\phi^2 dx\,dy}, \qquad (3.2)$$

where λ_1 is simple and positive (see [1] and the references therein).

Further, following [13, 14, 15, 16], for $u \in W$, we set

$$u = tv, \qquad (3.3)$$

where $t \neq 0$ is a real number and $v \in W$ (since we look for nontrivial solutions, the assumption $t \neq 0$ is natural). Substituting (3.3) into (2.12), we obtain

$$\mathscr{J}_\alpha(tv) = \frac{t^2}{2}\|v\|^2 - \frac{\alpha t^2}{2}G(v) - \frac{|t|^3}{3}F(v). \qquad (3.4)$$

We choose as *fibering functional* the principal part of $\mathscr{J}_\alpha$, that is,

$$H_\alpha(v) := \iint_U |\nabla v|^2 dx\,dy - \alpha \iint_U \gamma(x,y)|v|^2 dx\,dy = \|v\|^2 - \alpha G(v). \qquad (3.5)$$

If $u \in W$ is a critical point of $\mathscr{J}_\alpha$, then

$$\frac{\partial}{\partial t}\mathscr{J}_\alpha(tv) = 0. \qquad (3.6)$$

In our specific case, (3.6) assumes the form

$$tH_\alpha(v) - |t|tF(v) = 0. \qquad (3.7)$$

We obtain

$$|t| = \frac{H_\alpha(v)}{F(v)}. \tag{3.8}$$

Obviously, the following conditions are necessary for (3.8) to make sense:

$$F(v) \neq 0, \qquad \frac{H_\alpha(v)}{F(v)} > 0. \tag{3.9}$$

Substituting (3.8) in (2.12), we obtain the *induced functional*

$$\hat{\mathcal{J}}_\alpha(v) := \mathcal{J}_\alpha(t(v)v) = \frac{1}{6}\left(\frac{H_\alpha(v)}{F(v)}\right)^3 F(v). \tag{3.10}$$

The induced functional $\hat{\mathcal{J}}_\alpha$ obeys the following properties:

(1) for any $\tau \in \mathbb{R} \setminus \{0\}$ and $v \in W$ such that $F(v) \neq 0$,

$$\hat{\mathcal{J}}_\alpha(\tau v) = \hat{\mathcal{J}}_\alpha(v), \tag{3.11}$$

that is, the functional $\hat{\mathcal{J}}_\alpha$ is homogeneous of degree 0. Moreover $\langle \hat{\mathcal{J}}'_\alpha(v), v \rangle = 0$, where $\langle \hat{\mathcal{J}}'_\alpha(v), v \rangle$ is the Gatêaux derivative of $\hat{\mathcal{J}}_\alpha$ at $v \in W$ in the direction of v;

(2) if $v_c \in W$ is a critical point of $\hat{\mathcal{J}}_\alpha$, then $|v_c|$ is also a critical point of $\hat{\mathcal{J}}_\alpha$, hence, as in [7], one can assume that the critical points of $\hat{\mathcal{J}}_\alpha$ are nonnegative. The next two properties are direct consequences of the general fibering method described in [13, 14, 15];

(3) let $v \in W$ be a critical point of $\hat{\mathcal{J}}_\alpha$ such that $F(v) \neq 0$ and $H_\alpha(v)/F(v) > 0$, then the function

$$u = tv, \tag{3.12}$$

where $t > 0$ is determined by (3.8), and is a critical point of $\mathcal{J}_\alpha$;

(4) we consider a constraint

$$\mathcal{F}(v) = c, \tag{3.13}$$

where $\mathcal{F}$ is a C^1 functional. If

$$\langle \mathcal{F}'(v), v \rangle \neq 0, \qquad \mathcal{F}(v) = c, \tag{3.14}$$

then every critical point of $\hat{\mathcal{J}}_\alpha$ with the constraint $\mathcal{F}(v) = c$ is a critical point of $\hat{\mathcal{J}}_\alpha$.

Our first aim is to prove the existence of a critical point of $\hat{\mathcal{J}}_\alpha$ with an appropriate condition $\mathcal{F}(v) = c$ which in turn will be an actual critical point of $\hat{\mathcal{J}}_\alpha$ and hence a critical point of the Euler functional $\mathcal{J}_\alpha$—the weak solution of (2.9). See [7, 16] for further details.

Applying the described method to the problem (2.9), one can obtain the existence of solutions whose multiplicity depends on the magnitude of the fundamental eigenvalue λ_1. There are two cases to be considered:

(1) $\alpha \leq \lambda_1$;

(2) $\alpha > \lambda_1$.

In the first case, we get the existence of a positive nontrivial solution of (2.9) choosing the constraint

$$H_\alpha(v) = 1, \tag{3.15}$$

which satisfies the nondegeneracy condition since if $H(v) = 1$, then

$$\langle H'_\alpha(v), v \rangle = 2H(v) = 2 \neq 0. \tag{3.16}$$

However, $F(v)$ must be positive for (3.8). Hence, the fibering method cannot be applied immediately in this case. This is compatible with the results of Cantrell and Cosner [3] which predict the nonexistence of positive solution if $\alpha \leq \lambda_1$.

With regard to the case $\alpha > \lambda_1$, the fibering method should give, in principle, two critical points for the functional (2.12) which are positive functions. However, in order to obtain one of these, we need the positivity of the functional F. Therefore, we cannot get it in this case. Why this occurs is commented in the next section.

Now, we will obtain the "other" positive solution of (2.9) with $\alpha > \lambda_1$. Let

$$F(v) = -1 \tag{3.17}$$

be a constraint. The nondegeneracy condition is satisfied since

$$\langle F'(v), v \rangle = -3F(v) = 3 \neq 0. \tag{3.18}$$

Then the induced functional $\widehat{\mathcal{J}}_\alpha$ becomes

$$\widehat{\mathcal{J}}_\alpha(v) = \frac{1}{6}(H_\alpha(v))^3. \tag{3.19}$$

With t determined by (3.8), we look for a critical point v of $\widehat{\mathcal{J}}_\alpha$ such that

$$H_\alpha(v) < 0. \tag{3.20}$$

For this purpose, we look for a function which attains the minimum m_α of the problem

$$m_\alpha = \inf_{v \in W}(H_\alpha(v) \mid F(v) = -1). \tag{3.21}$$

First, we must prove that

$$W^- := \{v \in W \mid F(v) = -1\} \neq \varnothing. \tag{3.22}$$

Let e_1 be the positive eigenfunction associated to the fundamental eigenvalue (it satisfies (3.1) for any $\psi \in W$—see [1]). Then

$$F(e_1) < 0, \tag{3.23}$$

and hence it is easy to see that we can find a constant t_1 such that $F(t_1 e_1) = -1$. Therefore W^- is not empty. Moreover, by the variational characterization of λ_1,

$$H(t_1 e_1) = |t_1|^2 (\lambda_1 - \alpha) G(e_1) < 0, \tag{3.24}$$

which implies that the infimum m_α is negative.

We are going to prove the existence of a positive solution to (2.9) for $\alpha \in (\lambda_1, \lambda_1 + L)$, where $L > 0$ is a constant by a contradiction argument. Suppose that this is not true, and that there exists a sequence $\varepsilon_k \to 0$ such that for any $\alpha^k := \lambda_1 + \varepsilon_k$, the minimization problem (3.21) does not have a solution. For any integer k, let $(v_n^k)_{n=1}^\infty$ be a minimizing sequence for the problem (3.21), that is,

$$F(v_n^k) = -1, \quad H_{\alpha^k}(v_n^k) \longrightarrow m_{\alpha^k}, \quad \text{where } n \longrightarrow \infty. \tag{3.25}$$

If $(v_n^k)_{n=1}^\infty$ would be bounded for an integer number k, we may assume without loss of generality that it converges weakly in W to some $\bar{v}^k$ when $n \to \infty$. By the weak continuity of the functionals F and G and the weak lower semicontinuity of the principal part H_α, one can deduce, letting $n \to \infty$, that

$$F(\bar{v}^k) = -1, \qquad H_{\alpha^k}(\bar{v}^k) \le m_{\alpha^k}. \tag{3.26}$$

By the definition of m_{α^k} in (3.21), the opposite inequality holds, which is a contradiction. Thus we may consider $(v_n^k)_{n=1}^\infty$ to be unbounded:

$$\|v_n^k\| \longrightarrow \infty \quad \text{if } n \longrightarrow \infty. \tag{3.27}$$

Let

$$w_n^k := \frac{v_n^k}{r_n^k}, \tag{3.28}$$

where

$$|r_n^k| := \|v_n^k\|. \tag{3.29}$$

Since $\|w_n^k\| = 1$, we may assume that w_n^k converges weakly in W to a function $\bar{w}^k \in W$ as $n \to \infty$. By $\|\bar{w}^k\| \le 1$, passing to a subsequence of $\bar{w}^k$ converging weakly to $\bar{w} \in W$ when $k \to \infty$, we have

$$\|\bar{w}\| \le 1. \tag{3.30}$$

By the definitions of v_n^k and w_n^k, we get the inequality

$$
\begin{aligned}
|r_n^k|^2 (1 - \alpha^k G(w_n^k)) &= |r_n^k|^2 - |r_n^k|^2 \alpha^k G(w_n^k) \\
&= \|v_n^k\| - \alpha^k(v_n^k) \\
&= H_{\alpha^k}(v_n^k) \longrightarrow m_{\alpha^k} < 0.
\end{aligned}
\tag{3.31}
$$

Hence and by (3.30), letting $n \to \infty$ and $k \to \infty$, we obtain

$$\|\overline{w}\| - \lambda_1 G(\overline{w}) \le 1 - \lambda_1 G(\overline{w}) \le 0. \tag{3.32}$$

However the variational nature of λ_1 (see the Rayleigh quotient (3.2)) implies that the opposite inequality holds. Therefore, we can find a constant $k_1 \in \mathbb{R}$ such that

$$\overline{w} = k_1 e_1. \tag{3.33}$$

By the 3-homogeneity of the functional F we conclude that

$$F(w_n^k) = \frac{1}{|r_n^k|^3} F(v_n^k) = -\frac{1}{|r_n^k|^3}. \tag{3.34}$$

Letting $n \to \infty$ and $k \to \infty$, we obtain by the weak continuity of F

$$F(\overline{w}) = 0, \tag{3.35}$$

and, by (3.33),

$$|k_1|^3 F(e_1) = 0, \tag{3.36}$$

which contradicts (3.23). Therefore, with $\alpha = 1$, we obtain a solution u of the problem (2.9) if $\lambda < 1$. Indeed, let v be the minimizer of the problem (3.21). Introduce u by

$$u = t(v)v, \tag{3.37}$$

where $t(v)$ is defined in (3.8). Then by the properties (3) and (4) of the functional $\hat{\mathcal{J}}_\alpha$, u is a weak solution of (2.9). The references in [7, 16] ensure the positivity and differentiability of the solution u. In this way we have proved the following theorem.

THEOREM 3.1. *The boundary value problem (2.9) has at least one positive solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega) \cap C_{\mathrm{loc}}^{1,\sigma}(\Omega)$.*

4. Comments and concluding remarks

In [7], Drábek and Pohozaev investigated the existence of positive solutions of the following quasilinear problem:

$$\begin{aligned}
-\Delta_p u &= \lambda a(x)|u|^{p-2}u + b(x)|u|^{s-2}u \quad \text{in } \Omega, \\
u &= 0 \quad \text{in } \partial\Omega, \\
u &> 0 \quad \text{in } \Omega.
\end{aligned} \tag{4.1}$$

Here λ, p, q are real numbers, $a(x)$, $b(x)$ are given functions of $x \in \mathbb{R}^n$, and Δ_p is the p-Laplace operator which for $p = 2$ coincides with the usual Laplacian. The p-Laplacian for $p \ne 2$ is much less important for biological modelling (if any) than the classical $p = 2$

which embodies the ubiquitous Fickian/random diffusion process. For this reason, we let $p = 2$ and obtain the semilinear boundary value problem

$$
\begin{aligned}
-\Delta u &= \lambda a(x)u + b(x)|u|^{s-2}u \quad \text{in } \Omega, \\
u &= 0 \quad \text{in } \partial\Omega, \\
u &> 0 \quad \text{in } \Omega.
\end{aligned}
\tag{4.2}
$$

We recall some of the assumptions on the parameters used in [7].

The functions $a(x)$ and $b(x)$ are supposed to be bounded in Ω;

$$
a, b \in L^{\infty}(\Omega),
\tag{4.3}
$$

$$
a(x) = a_1(x) - a_2(x); \quad a_1, a_2 \geq 0, \ a_1(x) \not\equiv 0,
\tag{4.4}
$$

$$
b(x) = b_1(x) - b_2(x); \quad b_1, b_2 \geq 0, \ b_1(x) \not\equiv 0,
\tag{4.5}
$$

$$
\int_{\Omega} b(x)\,|u_1(x)|^s dx < 0,
\tag{4.6}
$$

where $u_1(x)$ is the first positive eigenfunction of the p-Laplace operator. For the Cantrell-Cosner problem (2.9), we have $\alpha = \lambda$, $a = \gamma$, $b = -1$, and $s = 3$. Obviously, the conditions (4.3), (4.4), and (4.6) are fulfilled, but the condition (4.5) is not satisfied since

$$
b = -b_2 = -1, \qquad b_1 \equiv 0.
\tag{4.7}
$$

That is the reason why it was not possible to get a second solution in the previous section. However, if we consider the model (4.2) with (4.3), (4.4), (4.5), and (4.6), which is in the spirit of the already cited works of Cantrell and Cosner, it is clear that a straightforward application of the fibering method will give at least two different positive solutions, a result which begs for further biological interpretations.

Appendix

Application of the fibering method to a simple boundary value problem

In this section, we apply the fibering method to a simple boundary value problem in dimension one. This example serves as an elementary introduction to the essential points of the method, which is a powerful tool to be used in more complicated situations.

The fibering method, due to S. I. Pohozaev, gives us information about existence and multiplicity of nonnegative and nontrivial solutions to several types of boundary value problems, where the partial differential equation is generally *non linear*. The presence of nonlinearity comes from problems of biological nature: for instance, in the study of generation of spatial patterns and persistence of species in regions explored by these species [11].

The introductory example chosen concerns the existence and multiplicity of solutions to the stationary Fisher/KPP equation under Dirichlet boundary condition:

$$
-u'' = \ell^2 u(1-u) \quad \text{in } (0,1), \qquad u(0) = u(1) = 0.
\tag{A.1}
$$

The problem (A.1) has been studied by Ludwig et al. [10] using the first integral method. This model, which is appropriately dimensionless, serves to describe the steady state distribution of individuals which spread out randomly in a one-dimensional medium with a lethal boundary (extremal points of the interval $(0,1)$) condition of the Dirichlet type. It is also assumed that reproduction of these species obeys the Verhulst model of logistic growth.

In order to use the fibering method, we need to reformulate the problem (A.1) in its equivalent variational form, that is, we need to consider the function which solves (A.1) by minimizing the following Euler functional:

$$
\begin{aligned}
f(u) &= \frac{1}{2} \int_0^1 |u'|^2 dx - \frac{\ell^2}{2} \int_0^1 |u|^2 dx + \frac{\ell^2}{3} \int_0^1 |u|^3 dx \\
&= \frac{1}{2} \|u\|^2 - \frac{\ell^2}{2} G(u) - \frac{\ell^2}{3} F(u),
\end{aligned}
\tag{A.2}
$$

where

$$
G(u) := \int_0^1 |u|^2 dx,
\tag{A.3}
$$

$$
F(u) := - \int_0^1 |u|^3 dx,
\tag{A.4}
$$

which are both weakly continuous functionals. There exists an equivalence among weak solutions of (A.1) and critical points of (A.2), for when we compute the Gâteaux derivative of (A.2), we have

$$
\begin{aligned}
\langle f'(u), v \rangle &= \lim_{\varepsilon \to 0} \frac{d}{d\varepsilon} f(u + \varepsilon v) \\
&= \lim_{\varepsilon \to 0} \left(\int_0^1 u'v' + \varepsilon v'^2 dx - \ell^2 \int_0^1 uv + \varepsilon v^2 dx + \ell^2 \int_0^1 u^2 v + 2\varepsilon uv + \varepsilon^2 |v|^3 dx \right) \\
&= \int_0^1 (-u'' - \ell^2 u + \ell^2 u^2) v \, dx = 0, \quad \forall v \in H_0^1(0,1).
\end{aligned}
\tag{A.5}
$$

Hence, the expression inside the parentheses in the last integral must vanish if and only if u solves the differential equation in problem (A.1). We naturally choose as the function space the Sobolev space $H_0^1(0,1)$ which consists of functions belonging to $L^2(0,1)$ having a generalized derivative also in $L^2(0,1)$ and vanishing for $x = 0$ and $x = 1$. We use for simplicity the following norm:

$$
\|u\|^2 := \int_0^1 |u'|^2 dx,
\tag{A.6}
$$

which is equivalent to the usual norm in $H_0^1(0,1)$ by Poincaré's inequality. It is useful to note here that, by the embedding $W_1^2(0,1) \hookrightarrow C^{0,\sigma}(0,1)$ for some positive σ, the solution of (A.1) is a continuous function.

The aim of the fibering method is to associate the minimization problem of the functional $f(u)$ defined in (A.2) with another equivalent minimization problem with constraint(s) chosen appropriately. Once we find a critical point of the problem with constraint(s), we readily encounter a critical point of $f(u)$, as we will see hereafter.

Let t be a real number, which will be a posteriori determined, and we consider another class of functions $v \in H_0^1(0,1)$, which are related to the functions u from the domain of the functional $f(u)$ by the equality

$$u = tv. \tag{A.7}$$

The aim is to associate critical points u of $f(u)$ with critical points of another functional which will be obtained by the relation (A.7). Substituting (A.7) in (A.2), we have

$$\tilde{f}(t,v) := f(tv) = \frac{|t^2|}{2}\|v\|^2 - \frac{\ell^2|t^2|}{2}G(v) - \frac{\ell^2|t|^3}{3}F(v). \tag{A.8}$$

We choose as a *fibering functional* the main part of $f(u)$, that is,

$$H_\ell(v) := \int_0^1 v'^2 dx - \ell^2 \int_0^1 v^2 dx = \|v\|^2 - \ell^2 G(v). \tag{A.9}$$

Since u is a minimizer of the Euler Functional defined in (A.2), the following equality holds:

$$\frac{\partial}{\partial t}\tilde{f}(t,v) = 0. \tag{A.10}$$

Following Pohozaev, (A.10) is called the *bifurcation equation*. Applying (A.7) to our particular problem and noting that $t \neq 0$, we have

$$H_\ell(v) - |t|F(v) = 0, \tag{A.11}$$

which gives

$$|t| = |t(v)| = \frac{H_\ell(v)}{F(v)}. \tag{A.12}$$

We need to impose at this point that the denominator in the expression defined above does not vanish, which is a natural restriction, since we look for nontrivial solutions of the problem (A.1). We also need to know that the fibering functional $H_\ell(v)$ and the functional $F(v)$ should have the same sign, if we want (A.12) to make sense. Hence, we obtain the *induced functional* minimized by v once we substitute (A.12) in (A.8):

$$\hat{f}_\ell(v) := \tilde{f}(t(v),v) = \frac{1}{6}\left(\frac{H_\ell(v)}{F(v)}\right)^3 F(v). \tag{A.13}$$

It is worth noting some facts concerning $\hat{f}_\ell(v)$, which are consequences of the process we used above to obtain this functional.

(i) Considering any $0 \neq t \in \mathbb{R}$, we have

$$\hat{f}_\ell(tv) = \frac{1}{6}\left(\frac{|t|^2 H_\ell(v)}{|t|^3 F(v)}\right)^3 |t|^3 F(v) = \frac{1}{6}\frac{|t|^6 |t|^3}{|t|^9}\left(\frac{H_\ell(v)}{F(v)}\right)^3 F(v) = \hat{f}_\ell(v),$$ (A.14)

which implies the 0-homogeneity of this induced functional.

(ii) As a corollary of the above remark (i), the Gâteaux derivative of $\hat{f}(v)$ in the direction v is equal to zero because

$$\langle \hat{f}_\ell'(v), v \rangle = \lim_{\varepsilon \to 0} \frac{d}{d\varepsilon}\hat{f}_\ell(v + \varepsilon v) = \lim_{\varepsilon \to 0} \frac{d}{d\varepsilon}\hat{f}_\ell((1+\varepsilon)v) = \lim_{\varepsilon \to 0} \frac{d}{d\varepsilon}\hat{f}_\ell(v) = 0.$$ (A.15)

(iii) From the above construction the following property holds:

$$\hat{f}_\ell(v) = \hat{f}_\ell(|v|).$$ (A.16)

This result comes from the definitions of $H_\ell(v)$ and $F(v)$ (see (A.3) and (A.4)). Therefore, if v_c is a critical point of $\hat{f}_\ell(v)$, then $|v_c|$ is also a critical point of the same functional and we can consider the critical point(s) of the induced functional $\hat{f}_\ell(v)$ as positive functions in $(0,1)$.

(iv) The critical points of the induced functional $\hat{f}_\ell(v)$ are also critical points of the same functional but with a specific type of constraint, that is, if v is a critical point of $\hat{f}(v)$ then, it is also a critical point of $\hat{f}_\ell(v)$ subject to the *fibering constraint* $H(v) = c$, where $H(v)$ is any differentiable functional satisfying

$$\langle H'(v), v \rangle \neq 0 \quad \text{always when } H(v) = c,$$ (A.17)

which is named the *nondegeneracy condition*.

The existence and multiplicity of solutions of the induced functional $\hat{f}_\ell(v)$ is intimately related to the magnitude of the constant ℓ. We should consider the following two cases:

(i) $l \leq \pi$;
(ii) $l > \pi$.

Describing the first case with brevity, the fibering method indicates the existence of at least one positive weak solution of problem (A.1) under the restriction

$$H_\ell(v) = 1,$$ (A.18)

which satisfies the nondegeneracy condition in this case, because

$$\langle H_\ell'(v), v \rangle = 2H_\ell(v) = 2 \neq 0.$$ (A.19)

We should have, by (A.12) and (A.18), the positivity of the functional $F(v)$. But this is in contradiction to its own definition (cf. (A.4)). The method does not give us information about this case, which was expected since it is known that there are no positive solutions in this case [10]. According to Ludwig et al. [10], the interval $(0,1)$ is not a *refuge* in this case.

In the second case where $\ell > \pi$, the fibering method should provide the existence of at least two critical points of the same functional $\hat{f}_\ell(v)$. These critical points are characterized as being solutions of the following variational problems with constraints:

(i) find a maximizer $v_1 \in H_0^1(0,1)$ of the problem

$$M_\ell = \sup_{v \in H_0^1(0,1)} [F(v) \mid H_\ell(v) = 1]; \qquad (A.20)$$

(ii) find a minimizer $v_2 \in H_0^1(0,1)$ of the problem

$$m_\ell = \inf_{v \in H_0^1(0,1)} [H_\ell(v) \mid F(v) = -1]. \qquad (A.21)$$

The proof of the existence of solutions to the problem (i) above makes use of the hypothesis $F(v) > 0$. Therefore, (i) does not give information about positive weak solutions of problem (A.1).

Henceforth, we concentrate on the variational problem (ii). Note that in this case the fibering constraint, which is

$$F(v) = -1, \qquad (A.22)$$

satisfies the nondegeneracy condition since

$$\langle F'(v), v \rangle = \lim_{\varepsilon \to 0} \frac{d}{d\varepsilon} F(v + \varepsilon v) = \lim_{\varepsilon \to 0} 3(1 + \varepsilon)^2 F(v) = 3F(v) = -3 \neq 0. \qquad (A.23)$$

By (A.4), we have that the fibering functional $\hat{f}_\ell(v)$ with the constraint (A.12) assumes the following form:

$$\hat{f}_\ell(v) = -\frac{1}{6}\left(-H_\ell(v)\right)^3. \qquad (A.24)$$

We should have, according to (A.12), that

$$H_\ell(v) < 0. \qquad (A.25)$$

We will show below that there exists a weak solution of the problem (ii). We will assume, by contradiction, the nonexistence of an $\varepsilon > 0$ such that problem (ii) has a positive solution for a determined interval to the right of $\pi/2$ with length not exceeding ε. Properties of eigenvalues and eigenfunctions from $-u''$ will be fundamental for us to get the desired contradiction.

Note that the set

$$W := \{v \in H_0^1(0,1) \mid F(v) = -1\} \qquad (A.26)$$

is nonempty. In order to conclude this, it is enough to take the first eigenfunction of the following problem:

$$-u'' = \lambda u, \qquad u(0) = u(1) = 0, \qquad (A.27)$$

which is $e_1 = \sin(\pi x)$. We have

$$F(e_1) = \frac{2}{3\pi} - 2 < 0, \tag{A.28}$$

thus, once we take

$$t_1 = \left(2 - \frac{2}{3\pi}\right)^{-1/3}, \tag{A.29}$$

we have, by the 3-homogeneity of the functional $F(v)$, that $F(t_1 e_1) = -1$. We also have that

$$H_\ell(t_1 e_1) = t_1^2 \left(\frac{\pi^2 - \ell^2}{2}\right) < 0, \tag{A.30}$$

always when $\ell > \pi$. So we conclude that m_λ is negative.

We will prove that the minimization problem given in (ii) has a nonnegative minimizer provided $\pi^2 < \ell^2 < \pi^2 + \varepsilon$, for some $\varepsilon > 0$.

We suppose, by absurd, that there exists a sequence $\varepsilon_k \to 0^+$ such that for any $\ell_k^2 := \pi^2 + \varepsilon_k$, problem (ii) has no nonnegative solution. For an arbitrary natural number k, let $(v_n^k)_{n=1}^\infty$ be a minimizing sequence of (ii)$_k$, that is,

$$F(v_n^k) = -1, \quad H_{\ell_k}(v_n^k) \longrightarrow m_{\ell_k}, \quad \text{when } n \longrightarrow \infty. \tag{A.31}$$

Assuming that this sequence is bounded with respect to the $H_0^1(0,1)$, we may assume, since we are in a reflexive Banach space, that this sequence is weakly convergent in $H_0^1(0,1)$ when $n \to \infty$. We call $\bar{v}^k$ the corresponding weak limit. By the weak continuity of $F(v)$, we have

$$F(\bar{v}^k) = -1. \tag{A.32}$$

We also have from the lower semicontinuity of the $H_0^1(0,1)$ norm, and from the weak continuity of the functional $G(v)$, that

$$H_\ell(\bar{v}^k) \le \liminf_{n \to \infty} H_\ell(v_n^k) = m_{\ell_k}. \tag{A.33}$$

But as $\bar{v}^k \in W$, we should have $H_\ell(\bar{v}^k) \ge m_{\ell_k}$. Thus, one concludes that $H_\ell(\bar{v}^k) = m_{\ell_k}$ and $\bar{v}^k$ is a solution of (ii)$_k$, which is a contradiction. Then, the sequence $(v_n^k)_{n=1}^\infty$ must be unbounded.

We may assume, passing to a subsequence if necessary, that

$$\|v_n^k\| \longrightarrow \infty \quad \text{as } n \longrightarrow \infty. \tag{A.34}$$

Considering this sequence, we define another related sequence $(w_n^k)_{n=1}^\infty$ by the following relation:

$$v_n^k = r_n^k w_n^k, \tag{A.35}$$

where $|r_n^k| = \|v_n^k\|$, $\|w_n^k\| = 1$. As a result, we obtain another uniformly bounded sequence in $H_0^1(0,1)$; thus, we may assume that the following statement holds:

$$w_n^k \rightharpoonup \bar{w}^k \quad \text{in } H_0^1(0,1) \text{ when } n \longrightarrow \infty. \tag{A.36}$$

Since $\|w_n^k\| = 1$, we have that $\|\bar{w}^k\| \leq 1$. Hence, we may suppose that there is a subsequence of $\bar{w}^k$ which converges weakly to a $\bar{w} \in H_0^1(0,1)$, and thus

$$\|\bar{w}\| \leq 1. \tag{A.37}$$

On the other hand, by the definitions of v_n^k and w_n^k, we obtain

$$\begin{aligned}
|r_n^k|^2 (1 - l_k^2 G(w_n^k)) &= |r_n^k|^2 (\|w_n^k\| - l_k^2 G(w_n^k)) \\
&= \|r_n^k v_n^k\| - l_k^2 G(r_n^k v_n^k) \\
&= H_{\ell^k}(v_n^k) \longrightarrow m_{\ell^k} < 0.
\end{aligned} \tag{A.38}$$

At this point, we will let the limit $k \to \infty$ and $n \to \infty$ to obtain, by the weak continuity of $G(v)$ and (A.37),

$$\|\bar{w}\| - \pi^2 G(\bar{w}) \leq 1 - \pi^2 G(\bar{w}) \leq 0. \tag{A.39}$$

The minimizing characterization of the eigenvalue $\lambda_1 = \pi^2$ implies that the opposite inequality holds. Hence, we conclude that

$$\|\bar{w}\| - \pi^2 G(\bar{w}) = 0. \tag{A.40}$$

We know, by the uniqueness of the eigenfunction related to the first eigenvalue of the operator $-u''$, that $\bar{w}$ must be a scalar multiple of $e_1 = \sin(\pi x)$, that is,

$$\bar{w} = k_1 e_1. \tag{A.41}$$

As we have

$$F(w_n^k) = |r_n^k|^{-3} F(v_n^k) = -|r_n^k|^{-3}, \tag{A.42}$$

we obtain, by passing to the limit when $n, k \to \infty$ and by the weak continuity of the functional $F(v)$,

$$F(\bar{w}) = 0, \tag{A.43}$$

and by (A.41) we also have

$$|k_1|^3 F(e_1) = 0 \Longrightarrow F(e_1) = 0 \neq \frac{2}{3\pi} - 2, \tag{A.44}$$

which is a contradiction by (A.28). Thus one concludes that for some $\varepsilon > 0$, problem (ii) has at least one nonnegative solution when $\pi^2 < \ell^2 < \pi^2 + \varepsilon$.

We can finally state the following theorem.

Theorem A.1. *There exists $\varepsilon > 0$ such that when $\pi^2 < \ell^2 < \pi^2 + \varepsilon$, problem (A.1) has a positive solution $u \in H_0^1(0,1) \cap L^\infty(0,1) \cap C_{\mathrm{loc}}^{1,\alpha}(0,1)$.*

Proof. Considering the solution v obtained by the above construction, and by obtaining the constant t as determined in (A.12), the function

$$u = tv \qquad (A.45)$$

is a nonnegative weak solution of the considered problem. Then the corresponding arguments and references in [7, 16] ensure that $u \in H_0^1(0,1) \cap L^\infty(0,1) \cap C_{\mathrm{loc}}^{1,\alpha}(0,1)$. See [7, 16] for more details on these points.   □

Acknowledgments

References

[1]   A. Alberico, A. Ferone, and R. Volpicelli, *Some properties for eigenvalues and eigenfunctions of nonlinear weighted problems*, Rend. Mat. Appl. (7) **19** (1999), no. 1, 45–63.

[2]   R. S. Cantrell and C. Cosner, *Diffusive logistic equations with indefinite weights: population models in disrupted environments*, Proc. Roy. Soc. Edinburgh Sect. A **112** (1989), no. 3-4, 293–318.

[3]   ———, *Diffusive logistic equations with indefinite weights: Population models in disrupted environments. II*, SIAM J. Math. Anal. **22** (1991), no. 4, 1043–1064.

[4]   ———, *The effects of spatial heterogeneity in population dynamics*, J. Math. Biol. **29** (1991), no. 4, 315–338.

[5]   ———, *Should a park be an island?* SIAM J. Appl. Math. **53** (1993), no. 1, 219–252.

[6]   ———, *Spatial heterogeneity and critical patch size: area effects via diffusion in closed environments*, J. Theoret. Biol. **209** (2001), 161–171.

[7]   P. Drábek and S. I. Pohozaev, *Positive solutions for the p-Laplacian: Application of the fibering method*, Proc. Roy. Soc. Edinburgh Sect. A **127** (1997), no. 4, 703–726.

[8]   W. F. Fagan, R. S. Cantrell, and C. Cosner, *How habitat edges change species interactions*, Amer. Natur. **153** (1999), no. 2, 165–182.

[9]   A. Kolmogoroff, I. Petrovsky, and N. Piscunoff, *Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique*, Moscow Univ. Math. Bull. **1** (1937), 1–25 (French).

[10] D. Ludwig, D. G. Aronson, and H. F. Weinberger, *Spatial patterning of the spruce budworm*, J. Math. Biol. **8** (1979), no. 3, 217–258.

[11] D. Ludwig, D. D. Jones, and C. S. Holling, *Qualitative analysis of insect outbreak systems: the spruce budworm and forest*, J. Animal Ecology **47** (1978), 315–332.

[12] J. D. Murray, *Mathematical Biology*, Biomathematics, vol. 19, Springer-Verlag, Berlin, 1989.

[13] S. I. Pohozaev, *An approach to nonlinear equations*, Dokl. Akad. Nauk SSSR **247** (1979), no. 6, 1327–1331 (Russian), translated in Soviet Math. Dokl. **20** (1979), 912–916.

[14] ———, *On a constructive method of the calculus of variations*, Dokl. Akad. Nauk SSSR **298** (1988), no. 6, 1330–1333 (Russian), translated in Soviet Math. Dokl. **37** (1988), 274–277.

[15] ———, *The fibration method for solving nonlinear boundary value problems*, Trudy Mat. Inst. Steklov. **192** (1990), 146–163 (Russian).

[16] ———, *The fibering method and its applications to nonlinear boundary value problem*, Rend. Istit. Mat. Univ. Trieste **31** (1999), no. 1-2, 235–305.

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