POSITIVE PERIODIC SOLUTIONS OF NONAUTONOMOUS FUNCTIONAL DIFFERENTIAL EQUATIONS DEPENDING ON A PARAMETER

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This article investigates the existence of positive periodic solutions for a first-order functional differential equations of the form

$$y'(t) = -a(t)y(t) + \lambda h(t) f\big(y\big(t-\tau(t)\big)\big), \tag{1}$$

where $a = a(t)$, $h = h(t)$, and $\tau = \tau(t)$ are continuous T-periodic functions. We will also assume that $T > 0$, $\lambda > 0$, $f = f(t)$ as well as $h = h(t)$ are positive, $\int_0^T a(t)\,dt > 0$.

Functional differential equations with periodic delays appear in a number of ecological models. In particular, our equation can be interpreted as the standard Malthus population model $y' = -a(t)y$ subject to perturbation with periodical delay. One important question is whether these equations can support positive periodic solutions. Such questions have been studied extensively by a number of authors (cf. [1, 2, 3, 4, 6, 7] and the references therein). In this paper, we are concerned with the existence and nonexistence of periodic solutions when the parameter λ varies. For this purpose, we call a continuously differentiable and T-periodic function a periodic solution of (1) associated with λ^* if it satisfies (1) when $\lambda = \lambda^*$. We show that there exists $\lambda^* > 0$ such that (1) has at least one positive T-periodic solution for $\lambda \in (0,\lambda^*]$ and does not have any T-periodic positive solutions for $\lambda > \lambda^*$. Our technique is based on the well-known upper and lower solutions method (cf. [5]).

We proceed from (1) and obtain

$$\left[y(t)\exp\left(\int_0^t a(s)\,ds\right)\right]' = \lambda \exp\left(\int_0^t a(s)\,ds\right) h(t) f\big(y\big(t-\tau(t)\big)\big). \tag{2}$$

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After integration from t to $t+T$, we obtain

$$y(t) = \lambda \int_t^{t+T} G(t,s)h(s)f\big(y(s-\tau(s))\big)\,ds, \tag{3}$$

where

$$G(t,s) = \frac{\exp\left(\int_t^s a(u)\,du\right)}{\exp\left(\int_0^T a(u)\,du\right) - 1}. \tag{4}$$

Note that the denominator in $G(t,s)$ is not zero since we have assumed that $\int_0^T a(t)\,dt > 0$.

It is not difficult to check that any T-periodic function $y(t)$ that satisfies (3) is also a T-periodic solution of (1). Note further that

$$\begin{gathered} 0 < N \equiv \min_{0\le s,t\le T} G(t,s) \le G(t,s) \le \max_{0\le t,s\le T} G(t,s) \equiv M, \quad t \le s \le t+T, \\ 1 \ge \frac{G(t,s)}{\max_{0\le s,t\le T} G(t,s)} \ge \frac{\min_{0\le s,t\le T} G(t,s)}{\max_{0\le s,t\le T} G(t,s)} = \frac{N}{M} > 0. \end{gathered} \tag{5}$$

Now let X be the set of all real T-periodic continuous functions, endowed with the usual linear structure as well as the norm

$$\|y\| = \sup_{0\le t\le T} |y(t)|. \tag{6}$$

Then X is a Banach space with cones

$$\begin{aligned} \Phi &= \{y(t) \in X : y(t) \ge 0\}, \\ \Omega &= \{y(t) : y(t) \ge \sigma\|y\|,\ t \in R\}, \end{aligned} \tag{7}$$

where $\sigma = N/M$.

Define a mapping $F : X \to X$ by

$$(Fy)(t) = \lambda \int_t^{t+T} G(t,s)h(s)f\big(y(s-\tau(s))\big)\,ds. \tag{8}$$

Then it is easily seen that F is completely continuous on bounded subsets of Ω and for $y \in \Phi$,

$$(Fy)(t) \le \lambda M \int_0^T h(s)f\big(y(s-\tau(s))\big)\,ds \tag{9}$$

so that

$$(Fy)(t) \ge \lambda N \int_0^T h(s)f\big(y(s-\tau(s))\big)\,ds \ge \sigma\|Fy\|. \tag{10}$$

That is, $F\Phi$ is contained in Ω.

Lemma 1. *The mapping F maps Φ into Ω.*

Lemma 2. *Suppose that*

$$\lim_{u\to+\infty}\frac{f(u)}{u}=+\infty. \tag{11}$$

Let I be a compact subset of $(0,+\infty)$. Then there exists a constant $b_I>0$ such that $\|u\|<b_I$ for all $\lambda\in I$ and all possible T-periodic positive solutions u of (1) associated with λ.

Proof. Suppose to the contrary that there is a sequence $\{u_n\}$ of T-periodic positive solutions of (1) associated with $\{\lambda_n\}$ such that $\lambda_n\in I$ for all n and $\|u_n\|\to+\infty$ as $n\to\infty$. Since $u_n\in\Omega$,

$$\min_{0\le t\le T}u_n(t)\ge\sigma\|u_n\|. \tag{12}$$

By (11), we may choose $R_f>0$ such that $f(u)\ge\eta u$ for all $u\ge R_f$, and there exists n_0 such that $\sigma\|u_{n_0}\|\ge R_f$, where η satisfies

$$\sigma\eta N\lambda_{n_0}\int_0^T h(s)\,ds>1. \tag{13}$$

Thus, we have

$$\begin{aligned}\|u_{n_0}\|\ge u_{n_0}(t)&=\lambda_{n_0}\int_t^{t+T}G(t,s)h(s)f\left(u_{n_0}\left(s-\tau(s)\right)\right)ds\\&\ge\sigma\eta N\lambda_{n_0}\int_0^T h(s)\|u_{n_0}\|\,ds>\|u_{n_0}\|.\end{aligned} \tag{14}$$

This is a contradiction. The proof is complete. □

Lemma 3. *Suppose that*

$$f \text{ is nondecreasing on } [0,+\infty) \text{ and } f(0)>0. \tag{15}$$

Let (1) have a T-periodic positive solution $y(t)$ associated with $\bar\lambda>0$. Then (1) also has a positive T-periodic solution associated with $\lambda\in(0,\bar\lambda)$.

Proof. In view of (3) and (15), we have

$$\begin{aligned}y(t)&=\bar\lambda\int_t^{t+T}G(t,s)h(s)f\left(y\left(s-\tau(s)\right)\right)ds\\&\ge\lambda\int_t^{t+T}G(t,s)h(s)f\left(y\left(s-\tau(s)\right)\right)ds,\quad 0<\lambda\int_t^{t+T}G(t,s)h(s)f(0)\,ds.\end{aligned} \tag{16}$$

Let $\overline{y}_0(t) = y(t)$,

$$\overline{y}_{k+1}(t) = \lambda \int_t^{t+T} G(t,s)h(s)f\left(\overline{y}_k(s-\tau(s))\right)ds, \quad k = 0,1,2,\ldots, \tag{17}$$

$\underline{y}_0(t) = 0$, and

$$\underline{y}_{k+1}(t) = \lambda \int_t^{t+T} G(t,s)h(s)f\left(\underline{y}_k(s-\tau(s))\right)ds, \quad k = 0,1,2,\ldots. \tag{18}$$

Clearly, we have

$$\overline{y}_0(t) \geq \overline{y}_1(t) \geq \cdots \geq \overline{y}_k(t) \geq \underline{y}_k(t) \geq \cdots \geq \underline{y}_1(t) \geq \underline{y}_0(t). \tag{19}$$

If we now let $y(t) = \lim_{k\to\infty} \overline{y}_k(t)$, then $y(t)$ satisfies (3). Clearly, we have

$$y(t) \geq \underline{y}_1(t) = \lambda \int_t^{t+T} G(t,s)h(s)f(0)\,ds > 0. \tag{20}$$

This completes our proof. □

Lemma 4. *Suppose that (11) and (15) hold. Then there exists $\lambda_* > 0$ such that (1) has a T-periodic positive solution.*

Proof. Let

$$\beta(t) = \int_t^{t+T} G(t,s)h(s)\,ds, \qquad M_f = \max_{0\leq t\leq T} f\left(\beta(t-\tau(t))\right), \qquad \lambda_* = \frac{1}{M_f}. \tag{21}$$

We have

$$\begin{gathered} \beta(t) = \int_t^{t+T} G(t,s)h(s)\,ds \geq \lambda_* \int_t^{t+T} G(t,s)h(s)f\left(\beta(s-\tau(s))\right)ds, \\ 0 < \lambda_* \int_t^{t+T} G(t,s)h(s)f(0)\,ds. \end{gathered} \tag{22}$$

Let $\overline{y}_0(t) = \beta(t)$,

$$\overline{y}_{k+1}(t) = \lambda_* \int_t^{t+T} G(t,s)h(s)f\left(\overline{y}_k(s-\tau(s))\right)ds, \quad k = 0,1,2,\ldots, \tag{23}$$

$\underline{y}_0(t) = 0$, and

$$\underline{y}_{k+1}(t) = \lambda_* \int_t^{t+T} G(t,s)h(s)f\left(\underline{y}_k(s-\tau(s))\right)ds, \quad k = 0,1,2,\ldots. \tag{24}$$

Clearly, we have

$$\overline{y}_0(t) \geq \overline{y}_1(t) \geq \cdots \geq \overline{y}_k(t) \geq \underline{y}_k(t) \geq \cdots \geq \underline{y}_1(t) \geq \underline{y}_0(t). \tag{25}$$

If we now let $y(t) = \lim_{k\to\infty} \overline{y}_k(t)$, then $y(t)$ satisfies (3). Clearly, we have

$$y(t) \geq \underline{y}_1(t) = \lambda_* \int_t^{t+T} G(t,s)h(s)f(0)\,ds > 0. \tag{26}$$

The proof is complete. □

Theorem 5. *Suppose that (11) and (15) hold. Then there exists $\lambda^* > 0$ such that (1) has at least one positive T-periodic solution for $\lambda \in (0,\lambda^*]$ and does not have any T-periodic positive solutions for $\lambda > \lambda^*$.*

Proof. Suppose to the contrary that there is a sequence $\{u_n\}$ of T-periodic positive solutions of (1) associated with $\{\lambda_n\}$ such that $\lim_{n\to\infty} \lambda_n = \infty$. Then either we have $\|u_{n_j}\| \to +\infty$ as $j \to \infty$ or there is $\tilde{M} > 0$ such that $\|u_n\| \leq \tilde{M}$. Assume the former case holds. Note that $u_n \in \Omega$ and thus

$$\min_{0\leq t\leq T} u_n(t) \geq \sigma \|u_n\|. \tag{27}$$

By (11), we may choose $R_f > 0$ and $\eta_1 > 0$ such that $f(u) \geq \eta_1 u$ when $\sigma u \geq R_f$. On the other hand, there exist $\{t_n\} \subset [0,T]$ such that $u_{n_j}(t_{n_j}) = \|u_{n_j}\|$ and $u'_{n_j}(t_{n_j}) = 0$ by the periodicity of $\{u_{n_j}(t)\}$. In view of (1), we have

$$\begin{aligned} a(t_{n_j})\|u_{n_j}\| &= a(t_{n_j})u(t_{n_j}) = \lambda_{n_j} h(t_{n_j}) f\big(u_{n_j}(t_{n_j} - \tau(t_{n_j}))\big) \\ &\geq \lambda_{n_j}\eta_1\sigma h(t_{n_j})\|u_{n_j}\| \end{aligned} \tag{28}$$

for all large j. That is, we have $\lambda_{n_j} \leq a(t_{n_j})/(\eta_1\sigma h(t_{n_j}))$. Note that $a(t)/h(t)$ is bounded. Thus, we obtain a contradiction.

Next, suppose that the latter case holds. In view of (15), there exists $\eta_2 > 0$ such that $f(0) \geq \eta_2 \tilde{M}$. Then as above, we will obtain

$$\begin{aligned} a(t_n)\|u_n\| &= a(t_n)u(t_n) = \lambda_n h(t_n) f\big(u_n(t_n - \tau(t_n))\big) \\ &\geq \lambda_n\eta_2 h(t_n)\tilde{M} \geq \lambda_n\eta_2 h(t_n)\|u_n\| \end{aligned} \tag{29}$$

for all n. A contradiction will again be reached.

Thus, there exists $\lambda^* > 0$ such that (1) has at least one positive T-periodic solution for $\lambda \in (0, \lambda^*)$ and no T-periodic positive solutions for $\lambda > \lambda^*$.

Finally, we assert that (1) has at least one T-periodic positive solution for $\lambda = \lambda^*$. Indeed, let $\{\lambda_n\}$ satisfy $0 < \lambda_1 < \cdots < \lambda_k < \lambda^*$ and $\lim_{k\to\infty} \lambda_k = \lambda^*$. Since $u_n(t)$ is T-periodic positive solution of (1) associated with λ_n and Lemma 2 implies that the set $\{u_n(t)\}$ of solutions is uniformly bounded in Ω, the sequence $\{u_n(t)\}$ has a subsequence converging to $u(t) \in \Omega$. We can now apply the Lebesgue convergence theorem to show that $u(t)$ is a T-periodic positive solution of (1) associated with $\lambda = \lambda^*$. The proof is complete. □

Example 6. Consider the equation

$$x'(t) + a(t)x(t) = \lambda h(t)\{x^{\gamma}(t - \tau(t)) + 1\}, \quad \gamma > 1, \tag{30}$$

where a, h, and τ satisfy the same assumptions stated for (1). In view of Theorem 5, there exists a $\lambda^* > 0$ such that (30) has at least one T-periodic positive solution for $\lambda \in (0, \lambda^*]$ and no T-periodic positive solution for $\lambda > \lambda^*$.

Example 7. Consider the equation

$$y'(t) = -ay(t) + \lambda b(y^2(t) + \varepsilon), \tag{31}$$

where $a, b, \varepsilon > 0$. Note that the function $f(x) = (x^2 + \varepsilon)$ satisfies (11) and (15) in Theorem 5. Therefore Theorem 5 may be applied. However, we may give a direct proof that, for $\lambda > a/(2b\sqrt{\varepsilon})$, this equation cannot have any positive 2π-periodic solutions associated with λ. Indeed, assume to the contrary that $y(t)$ is such a solution. Then $y'(\xi) = 0$ for some $\xi \in [0, 2\pi]$. Hence

$$-ay(\xi) + \lambda b y^2(\xi) + \lambda b\varepsilon = 0. \tag{32}$$

However, since the discriminant of the quadratic equation

$$\lambda b x^2 - ax + \lambda b\varepsilon = 0 \tag{33}$$

satisfies

$$a^2 - 4\lambda^2 b^2 \varepsilon < 0, \tag{34}$$

a contradiction is obtained. We remark that when $\varepsilon = 0$, our equation reduces to the well-known logistic equation.

Similarly, we can consider the equation

$$x'(t) = a(t)x(t) - \lambda h(t) f(x(t - \tau(t))), \tag{35}$$

where $a = a(t)$, $h = h(t)$, and $f = f(t)$ satisfy the same assumptions stated for (1). By (35), we have

$$x(t) = \int_t^{t+T} H(t,s)h(s)f(x(s-\tau(s)))\,ds, \tag{36}$$

where

$$H(t,s) = \frac{\exp\left(-\int_t^s a(u)\,du\right)}{1-\exp\left(-\int_0^T a(u)\,du\right)} = \frac{\exp\left(\int_s^{t+T} a(u)\,du\right)}{\exp\left(\int_0^T a(u)\,du - 1\right)} \tag{37}$$

which satisfies

$$M \geq H(t,s) \geq N, \quad t \leq s \leq t+T, \tag{38}$$

for some M and $N > 0$, and $\sigma = N/M \leq 1$.

Theorem 8. *Suppose that (11) and (15) hold. Then there exists $\lambda^* > 0$ such that (35) has at least one positive T-periodic solution for $\lambda \in (0,\lambda^*]$ and no T-periodic positive solution for $\lambda > \lambda^*$.*

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