GLOBAL SOLUTIONS OF A STRONGLY COUPLED REACTION-DIFFUSION SYSTEM WITH DIFFERENT DIFFUSION COEFFICIENTS

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We deal with a mathematical model for a four-component chemical reaction-diffusion process. The model is described by a system of strongly coupled reaction-diffusion equations with different diffusion rates. The existence of the global solution of this reaction-diffusion system in unbounded domain is proved by using semigroup theory and estimates on the growth of solutions.

1. Introduction

In this paper, we prove the existence of a global solution in an unbounded domain of the reaction-diffusion system

$$
\begin{aligned}
\frac{\partial u_1}{\partial t} &= a_1\Delta u_1 - f_1(u_1,u_2,v_1,v_2), \quad x\in\mathbb{R}^n,\ t>0,\\
\frac{\partial u_2}{\partial t} &= a_2\Delta u_2 - f_2(u_1,u_2,v_1,v_2), \quad x\in\mathbb{R}^n,\ t>0,\\
\frac{\partial v_1}{\partial t} &= b_1\Delta u_1 + d_1\Delta v_1 + f_3(u_1,u_2,v_1,v_2), \quad x\in\mathbb{R}^n,\ t>0,\\
\frac{\partial v_2}{\partial t} &= b_2\Delta u_2 + d_2\Delta v_2 + f_4(u_1,u_2,v_1,v_2), \quad x\in\mathbb{R}^n,\ t>0,
\end{aligned}
\tag{RDS1}
$$

with the initial conditions

$$
u_i(x,0)=u_i^0(x), \quad v_i(x,0)=v_i^0(x) \quad (i=1,2),\ x\in\mathbb{R}^n. \tag{IC1}
$$

Here $f_1(u_1,u_2,v_1,v_2) = m[k_2u_1^m v_2^r - k_1u_2^r v_1^m]$, $f_2(u_1,u_2,v_1,v_2) = -r[k_2u_1^m v_2^r - k_1u_2^r v_1^m]$, $k_1\geq 0$, $k_2\geq 0$, $m\geq 0$, $r\geq 1$, $f_3=\rho f_1$, and $f_4=\rho f_2$, $\rho>0$.

The constants a_i, b_i $(i=1,2)$ are such that $a_i>0$, $b_i\neq 0$ $(i=1,2)$, and $4a_id_i>b_i^2$ $(i=1,2)$ which reflects the parabolicity of the system. Δ is the Laplace operator in $\mathbb{R}^n$. Moreover we assume that the functions u_i^0 $(i=1,2)$ and v_i^0 $(i=1,2)$ are uniformly bounded, continuous, and nonnegative. This reaction-diffusion system is a mathematical model for

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a chemical reaction of the form

$$mA + r\bar{B} \underset{k_2}{\overset{k_1}{\rightleftharpoons}} rB + m\bar{A}. \tag{1.1}$$

u_1, u_2, v_1, and v_2 represent the concentrations of $\bar{A}$, $\bar{B}$, A, and B, respectively (see [3]).

We remark that the system

$$\begin{aligned} \frac{\partial u}{\partial t} &= a\Delta u - uh(v), \quad x \in \Omega,\ t > 0, \\ \frac{\partial v}{\partial t} &= b\Delta u + d\Delta v + uh(v), \quad x \in \Omega,\ t > 0, \end{aligned} \tag{RDS2}$$

with the initial conditions

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega \tag{IC2}$$

on a bounded domain $\Omega \subset \mathbb{R}^n$ with Neumann boundary conditions, $b > 0$, $a \neq d$, $v_0 \geq (b/(a-d))u_0 \geq 0$, and $h(s)$, a differentiable nonnegative function on $\mathbb{R}$, has been studied by Kirane [4]. The existence of global solutions for system (RDS2) on unbounded domains has been studied by Badraoui [1]. The existence of global solutions in $\mathbb{R}^n$ for (RDS2) with $h(s) = v^m$ has been studied by Collet and Xin [2].

The quasilinear system of reaction-diffusion equations

$$\begin{aligned} \frac{\partial u}{\partial t} &= \nabla \cdot (a(u)\nabla u) - uh(u)v, \quad x \in \Omega,\ t > 0, \\ \frac{\partial v}{\partial t} &= \nabla \cdot (b(v)\nabla v) + uh(u)v - \lambda v, \quad x \in \Omega,\ t > 0, \end{aligned} \tag{RDS3}$$

with the initial conditions

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega, \tag{IC3}$$

and with Neumann or Dirichlet boundary conditions, is studied by Kirane [5] where in particular the existence of a globally bounded solution is shown. Also he has discussed large time behavior of the solution.

Our aim is to investigate the existence of a global solution for system (RDS1)–(IC1) in an unbounded domain.

Throughout this paper the following notations are used.

(1) $\|\cdot\|$ is the supremum norm on $\mathbb{R}^n$, that is, $\|u\| = \sup_{x\in\mathbb{R}^n} |u(x)|$.
(2) $C_{\text{ub}}(\mathbb{R}^n)$ is the space of uniformly bounded continuous functions on $\mathbb{R}^n$ equipped with the supnorm.
(3) For any $f \in C_{\text{ub}}(\mathbb{R}^n)$, $\int f = \int_{\mathbb{R}^n} f(x)dx$ if the integral exists.
(4) For $f \in L^p(\mathbb{R}^n)$ $(p \geq 1)$, $\|f\|_p = (\int |f|^p)^{1/p}$.

2. Existence of a local solution

We convert system (RDS1)–(IC1) to an abstract first-order system in the Banach space $X = (C_{\text{ub}}(\mathbb{R}^n))^4$ of the form

$$\frac{d}{dt}(u(t)) = Au(t) + F(u(t)), \quad t > 0, \qquad u(0) = u_0 \in X, \tag{2.1}$$

where $u(t) = (u_1(t), u_2(t), v_1(t), v_2(t))^T$. The operator A is defined as

$$A = \begin{pmatrix} a_1\Delta & 0 & 0 & 0 \\ 0 & a_2\Delta & 0 & 0 \\ b_1\Delta & 0 & d_1\Delta & 0 \\ 0 & b_2\Delta & 0 & d_2\Delta \end{pmatrix} \tag{2.2}$$

with domain $D(A) = \{u = (u_1, u_2, v_1, v_2)^T \in X, \ (\Delta u_1, \Delta u_2, \Delta v_1, \Delta v_2)^T \in X\}$.

Moreover the function F is defined as

$$F(u(t)) = \big(- f_1(u(t)), - f_2(u(t)), f_3(u(t)), f_4(u(t))\big)^T, \tag{2.3}$$

where

$$f_i(u(t)) = f_i(u_1(t), u_2(t), v_1(t), v_2(t)), \quad i = 1, 2, 3, 4. \tag{2.4}$$

Note that for $\lambda > 0$ the operator $\lambda\Delta$ generates an analytic semigroup $G(t)$ in the space X given by

$$(G(t)u)(x) = \frac{1}{(4\pi\lambda t)^{n/2}} \int_{\mathbb{R}^n} \exp\left(- \frac{(x-y)^2}{4\lambda t}\right) u(y)\,dy, \quad t > 0, \ x \in \mathbb{R}^n. \tag{2.5}$$

Let $S_1(t)$, $S_2(t)$, $S_3(t)$, and $S_4(t)$ be the semigroups generated by $a_1\Delta$, $a_2\Delta$, $d_1\Delta$, and $d_2\Delta$, respectively. Then one can show that A generates an analytic semigroup $S(t)$ given by

$$S(t) = \begin{pmatrix} S_1(t) & 0 & 0 & 0 \\ 0 & S_2(t) & 0 & 0 \\ S_5(t) & 0 & S_3(t) & 0 \\ 0 & S_6(t) & 0 & S_4(t) \end{pmatrix}, \tag{2.6}$$

where

$$S_5(t) = \frac{b_1}{a_1 - d_1}(S_1(t) - S_3(t)), \tag{2.7}$$

$$S_6(t) = \frac{b_2}{a_2 - d_2}(S_2(t) - S_4(t)). \tag{2.8}$$

Assume that F is locally Lipschitz in u in the space X. Then there exist classical solutions on maximal existence interval $[0, T_0]$ (see [6]).

3. Existence of global solutions

For proving the existence of a global solution we assume that the solutions are nonnegative.

Theorem 3.1. *Consider the reaction-diffusion system (RDS1) with nonnegative initial conditions* $(u_1^0(x), u_2^0(x), v_1^0(x), v_2^0(x)) \in (C_{\text{ub}}(\mathbb{R}^n))^4$, $a_i > 0$, $d_i > a_i$, *and* $b_i < 0$. *Then there exist global in-time classical solutions such that*

$$(u_1, u_2, v_1, v_2) \in \left(C([0,\infty); C_{\text{ub}}(\mathbb{R}^n)) \cap C^1((0,\infty); C_{\text{ub}}(\mathbb{R}^n))\right)^4. \tag{3.1}$$

Lemma 3.2. *Let* (u_1, u_2, v_1, v_2) *be a classical solution of (RDS1). Define the functionals*

$$F^i(u_i, v_i) = (\alpha_i + u_i + u_i^2)e^{\epsilon_i v_i} \quad (i = 1,2) \text{ with } \epsilon_i > 0,\ \alpha_i > 0. \tag{3.2}$$

Then for any smooth nonnegative function $\psi = \psi(x,t)$ $(x \in \mathbb{R}^n)$ *with exponential spatial decay at infinity,*

$$\begin{aligned} \frac{d}{dt}\int \psi F^i = &\int (\psi_t + d_i \Delta\psi) F^i + \int ((d_i - a_i)F_1^i - b_i F_2^i)\nabla\psi\nabla u_i \\ &- \int \psi\Big[(a_i F_{11}^i + b_i F_{12}^i)|\nabla u_i|^2 + ((a_i + d_i)F_{12}^i + b_i F_{22}^i)\nabla u_i \nabla v_i \\ &+ d_i F_{22}^i (\nabla v_i)^2\Big] + \int \psi(\rho F_2^i f_i - F_1^i f_i), \quad i = 1,2, \end{aligned} \tag{3.3}$$

where

$$\begin{gathered} F_1^i = \frac{\partial F^i}{\partial u_i}, \qquad F_2^i = \frac{\partial F^i}{\partial v_i}, \qquad F_{11}^i = \frac{\partial^2 F^i}{\partial u_i^2}, \qquad F_{22}^i = \frac{\partial^2 F^i}{\partial v_i^2}, \\ F_{12}^i = \frac{\partial^2 F^i}{\partial u_i \partial v_i}, \quad i = 1,2. \end{gathered} \tag{3.4}$$

Proof. For $i = 1,2$, we have

$$\begin{aligned}
\frac{d}{dt}\int \psi F^i &= \int \psi_t F^i + \int \psi\left(\frac{\partial F^i}{\partial u_i}\frac{\partial u_i}{\partial t} + \frac{\partial F^i}{\partial v_i}\frac{\partial v_i}{\partial t}\right) \\
&= \int \psi_t F^i + \int \psi\left(F_1^i\frac{\partial u_i}{\partial t} + F_2^i\frac{\partial v_i}{\partial t}\right) \\
&= \int \psi_t F^i + \int \psi\left(F_1^i(a_i\Delta u_i - f_i) + F_2^i(b_i\Delta u_i + d_i\Delta v_i + \rho f_i)\right) \\
&= \int \psi_t F^i + a_i\int \psi F_1^i\Delta u_i + b_i\int \psi F_2^i\Delta u_i + d_i\int \psi F_2^i \Delta v_i \\
&\quad - \int \psi F_1^i f_i + \rho\int \psi F_2^i f_i.
\end{aligned} \tag{3.5}$$

However,

$$\begin{aligned}
\int \psi F_1^i\Delta u_i &= \int F_1^i\psi\Delta u_i \\
&= -\int \nabla(F^i\psi)\nabla u_i \\
&= -\int (F_1^i\nabla\psi + \psi\nabla F_1^i)\nabla u_i \\
&= -\int F_1^i\nabla\psi\nabla u_i - \int \psi\nabla F_1^i\nabla u_i \\
&= -\int F_1^i\nabla\psi\nabla u_i - \int \psi(F_{11}^i\nabla u_i + F_{12}^i\nabla v_i)\nabla u_i,
\end{aligned} \tag{3.6}$$

that is,

$$\int \psi F_1^i\Delta u_i = -\int F_1^i\nabla\psi\nabla u_i - \int \psi F_{11}^i\left|\nabla u_i\right|^2 - \int \psi F_{12}^i\nabla u_i\nabla v_i. \tag{3.7}$$

Similarly

$$\begin{aligned}
\int \psi F_2^i\Delta u_i &= -\int F_2^i\nabla\psi\nabla v_i - \int \psi F_{12}^i\left|\nabla u_i\right|^2 - \int \psi F_{22}^i\nabla u_i\nabla v_i, \\
\int \psi F_2^i\Delta v_i &= -\int F_2^i\nabla\psi\nabla v_i - \int \psi F_{22}^i\left|\nabla v_i\right|^2 - \int \psi F_{12}^i\nabla u_i\nabla v_i.
\end{aligned} \tag{3.8}$$

Also

$$\int F^i\Delta\psi = -\int \nabla F^i\nabla\psi = -\int F_1^i\nabla u_i\nabla\psi - \int F_2^i\nabla v_i\nabla\psi. \tag{3.9}$$

Using (3.5)–(3.9) we get

$$
\begin{aligned}
\frac{d}{dt}\int \psi F^i = & \int (\psi_t + d_i\Delta\psi)F^i - a_i\int F_1^i\nabla\psi\nabla u_i - a_i\int \psi F_{11}^i\,|\nabla u_i|^2 \\
& - a_i\int \psi F_{12}^i\nabla u_i\nabla v_i - b_i\int F_2^i\nabla\psi\nabla u_i - b_i\int \psi F_{12}^i\,|\nabla u_i|^2 \\
& - b_i\int \psi F_{22}^i\nabla u_i\nabla v_i + d_i\int F_1^i\nabla u_i\nabla\psi - d\int \psi F_{22}^i\,|\nabla v_i|^2 \\
& - d_i\int \psi F_{12}^i\nabla v_i\cdot\nabla v_i - \int \psi F_1^i f_i + \rho\int \psi F_2^i f_i \\
= & \int (\psi_t + d_i\Delta\psi)F^i + \int (d_iF_1^i - a_iF_1^i - b_iF_2^i)\nabla\psi\nabla u_i \\
& - \int (a_iF_{11}^i + b_iF_{12}^i)\,|\nabla u_i|^2 - \int \psi(a_iF_{12}^i + bF_{22}^i + d_iF_{12}^i)\nabla u_i\cdot\nabla v_i \\
& - d_i\int \psi F_{22}^i\,|\nabla v_i|^2 - \int \psi F_1^i f_i + \rho\int \psi F_2^i f_i \\
= & \int (\psi_t + d_i\nabla\psi)F^i + \int ((d_i - a_i)F_1^i - b_iF_2^i)\nabla\psi\nabla u_i \\
& - \int \psi\Big[(a_iF_{11}^i + b_iF_{12}^i)\,|\nabla u_i|^2 + ((a_i + d_i)F_{12}^i + b_iF_{22}^i)\nabla u_i\nabla v_i \\
& \qquad + d_iF_{22}^i\,|\nabla v_i|^2\Big] + \int \psi(\rho F_2^i f_i - F_1^i f_i).
\end{aligned}
\tag{3.10}
$$

□

Lemma 3.3. *There exist four positive constants* $\alpha_i = \alpha_i(a_i, b, d_i, \|u_i^0\|)$ $(i = 1,2)$ *and* $\epsilon_i = \epsilon_i(a_i, b_i, d_i, \|u_i^0\|)$ $(i = 1,2)$ *such that*

$$
\begin{aligned}
\frac{d}{dt}\int \psi F^i \le & \int (\psi_t + d_i\nabla\psi)F^i + \int ((d_i - a_i)F_1^i - b_iF_2^i)\nabla\psi\nabla u_i \\
& - \frac{1}{2}\int \psi\Big[\frac{a_i}{2}F_{11}^i\,|\nabla u_i|^2 + d_iF_{22}^i\,|\nabla v_i|^2\Big] - \frac{1}{2}\int \psi F_1^i f_i, \quad i = 1,2.
\end{aligned}
\tag{3.11}
$$

Proof. For any $(u_i, v_i) \in [0, \|u_i^0\|] \times \mathbb{R}^+$ $(i = 1,2)$, we choose α_i and ϵ_i in Lemma 3.2 such that

$$
\rho F_2^i \le \frac{1}{2}F_1^i, \quad i = 1,2,
\tag{3.12}
$$

$$
(a_i + d_i)^2(F_{12}^i)^2 + b_i^2(F_{22}^i)^2 + b_i(2a_i + d_i)(F_{12}^i)(F_{22}^i) - a_id_iF_{11}^iF_{22}^i \le 0, \quad i = 1,2,
\tag{3.13}
$$

$$
\frac{(F_1^i)^2}{F_{11}^i} \le F^i, \quad i = 1,2,
\tag{3.14}
$$

$$
F_{12}^i \le \frac{a_i}{2|b_i|}F_{11}^i, \quad i = 1,2.
\tag{3.15}
$$

We verify these conditions as follows

$$\begin{aligned}
F^i &= (\alpha_i + u_i + u_i^2)e^{\epsilon_i v_i}, \quad i = 1,2,\\
F_1^i &= (1+2u_i)e^{\epsilon_i v_i}, \quad i = 1,2,\\
F_{11}^i &= 2e^{\epsilon_i v_i}, \quad i = 1,2,\\
F_2^i &= \epsilon_i(\alpha_i + u_i + u_i^2)e^{\epsilon_i v_i}, \quad i = 1,2,\\
F_{22}^i &= \epsilon_i^2(\alpha_i + u_i + u_i^2)e^{\epsilon_i v_i}, \quad i = 1,2,\\
F_{12} = F_{21}^i &= \epsilon_i(1+2u_i)e^{\epsilon_i v_i}, \quad i = 1,2.
\end{aligned} \tag{3.16}$$

Denote

$$\begin{aligned}
\epsilon_i^{(1)} &= \frac{1}{2\rho\left(\alpha_i + ||u_i^0|| + ||u_i^0||^2\right)},\\
\epsilon_i^{(2)} &= \frac{1}{b_i\left(\alpha_i + ||u_i^0|| + ||u_i^0||^2\right)},\\
\epsilon_i^{(3)} &= \frac{1}{|b_i|(1+2||u_i^0||)},\\
\alpha_i^{(1)} &= \frac{(a_i+d_i)^2(1+2||u_i^0||)^2 + (2a_i+d_i)(1+2||u_i^0||)}{2a_id_i},\\
\alpha_i^{(2)} &= \frac{1+2||u_i^0||^2+2||u_i^0||}{2} = \frac{(1+||u_i^0||)^2 + ||u_i^0||^2}{2}.
\end{aligned} \tag{3.17}$$

If we choose ϵ_i $(i = 1,2)$ such that $\epsilon_i \le \epsilon_i^{(1)}$ $(i = 1,2)$ then condition (3.12) is satisfied.

Condition (3.13) is satisfied if

$$\begin{aligned}
&(a_i+d_i)^2\epsilon_i^2(1+2u_i)^2e^{2\epsilon_i v_i} + b_i^2\epsilon_i^4(\alpha_i+u_i+u_i^2)^2e^{2\epsilon_i v_i} + b_i(2a_i+d_i)\epsilon_i^3(1+2u_i)\\
&\quad\times(\alpha_i+u_i+u_i^2)^2e^{2\epsilon_i v_i} - 2a_id_i\epsilon_i^2(\alpha_i+u_i+u_i^2)e^{2\epsilon_i v_i} \le 0, \quad i = 1,2,
\end{aligned} \tag{3.18}$$

that is,

$$\begin{aligned}
&(a_i+d_i)^2(1+2u_i)^2 + b_i^2\epsilon_i^2(\alpha_i+u_i+u_i^2)^2\\
&\quad + b_i(2a_i+d_i)(1+2u_i)(\alpha_i+u_i+u_i^2)\epsilon_i - 2a_id_i(\alpha_i+u_i+u_i^2) \le 0.
\end{aligned} \tag{3.19}$$

If we choose ϵ_i $(i = 1,2)$ and α_i $(i = 1,2)$ such that $\epsilon_i \le \epsilon_i^{(2)}$ $(i = 1,2)$ and $\alpha_i \le \alpha_i^{(1)}$ $(i = 1,2)$, then (3.19) is satisfied. In other words, condition (3.13) is satisfied.

Condition (3.14) is satisfied if

$$\frac{(1+2u_i)^2}{2} \le (\alpha_i + u_i + u_i^2), \tag{3.20}$$

that is,

$$\alpha_i > \frac{1+2u_i^2+2u_i}{2}. \tag{3.21}$$

If we choose α_i $(i=1,2)$ such that $\alpha_i \le \alpha_i^{(2)}$ $(i=1,2)$, then (3.21) is satisfied. Hence condition (3.14) is satisfied.

Similarly we can show that condition (3.15) is satisfied if ϵ_i $(i=1,2)$ is chosen such that $\epsilon_i \le \epsilon_i^{(3)}$.

Now select

$$\begin{aligned} &\alpha_i \ge \max\left(\alpha_i^{(1)},\alpha_i^{(2)}\right) \quad (i=1,2),\\ &\epsilon_i \le \min\left(\epsilon_i^{(1)},\epsilon_i^{(2)},\epsilon_i^{(3)}\right) \quad (i=1,2). \end{aligned} \tag{3.22}$$

Then conditions (3.12)–(3.15) are satisfied.

Then, from (3.13) and (3.15), we get

$$\begin{aligned} &\left(a_iF_{11}^i + b_iF_{12}^i\right)|\nabla u_i|^2 + \left((a_i+d_i)F_{12}^i + b_iF_{22}^i\right)\nabla u_i\nabla v_i + d_iF_{22}^i|\nabla v_i|^2\\ &\quad \ge \frac{1}{2}\left[\left(a_iF_{11}^i + b_iF_{12}^i\right)|\nabla u_i|^2 + d_iF_{22}^i|\nabla v_i|^2\right]\\ &\quad \ge \frac{1}{2}\left[\frac{a_i}{2}F_{11}^i|\nabla u_i|^2 + d_iF_{22}^i|\nabla v_i|^2\right]. \end{aligned} \tag{3.23}$$

From (3.12), we get

$$\int \psi\left(\rho F_2^i f_i - F_1^i f_i\right) \le -\frac{1}{2}\int \psi F_1^i f_i, \quad i=1,2. \tag{3.24}$$

From (3.3), (3.23), (3.24), we get (3.11).

The proof of Lemma 3.3 is completed. □

Theorem 3.4. *If α_i, ϵ_i $(i=1,2)$ satisfy (3.22), then there exist a test function ψ and real positive constants β_i $(i=1,2)$ and σ_i such that*

$$\int \psi F_i \le \beta_i e^{\sigma_i t}, \quad \forall t>0,\ i=1,2. \tag{3.25}$$

Proof. We define the test function $\psi(x)$ as

$$\psi(x) = \frac{1}{\left(1+|x-x_0|^2\right)^n}, \quad x\in\mathbb{R}^n, \tag{3.26}$$

and $x_0 \in \mathbb{R}^n$ is an arbitrary point.

Then ψ is a smooth function with exponential decay at infinity and satisfies $|\Delta\psi| \le K_1\psi$, $|\nabla\psi| \le K_2\psi$. Let $K=\max(K_1,K_2)$ for some positive constant K.

Then from (3.11), we obtain

$$\begin{aligned}\frac{d}{dt}\int \psi F^i &\leq K d_i \int \psi F^i + K\left((d_i - a_i) + \frac{1}{2}|b_i|\right)\int F_1^i \psi |\nabla u_i| \\ &\quad - \frac{a_i}{4}\int \psi F_{11}^i |\nabla u|^2 - \frac{d_i}{2}\int \psi F_{22}^i |\nabla v_i|^2 - \frac{1}{2}\int \psi F_1^i f_i \\ &\leq K d_i \int \psi F^i + \frac{K^2}{a_i}\left((d_i - a_i) + \frac{1}{2}|b_i|\right)^2 \int \psi \frac{(F_1^i)^2}{F_{11}^i} \\ &\leq \left[K d_i + \frac{K^2}{a_i}\left((d_i - a_i) + \frac{1}{2}|b_i|^2\right)^2\right]\int \psi F^i, \quad i = 1,2.\end{aligned} \tag{3.27}$$

Let

$$\begin{aligned}\sigma_i &= K d_i + \frac{K^2}{a_i}\left(|d_i - a_i| + \frac{1}{2}|b_i|^2\right)^2, \quad i = 1,2, \\ \beta_i &\geq \left(\alpha_i + \|u_i^0\| + \|u_i^0\|^2\right) e^{\epsilon \|v_i^0\|} \|\psi\|_1, \quad i = 1,2.\end{aligned} \tag{3.28}$$

Then

$$\frac{d}{dt}\int \psi F^i \leq \sigma_i \int \psi F^i \quad \text{for } i = 1,2, \tag{3.29}$$

which implies (3.25). □

LEMMA 3.5. *For any unit cube Q and any finite $p \geq 1$,*

$$\int_Q |v_i|^p dx \leq 2^n \frac{\beta_i}{\alpha_i \epsilon_i^p} e^{\sigma_i t}(p+1)^{p+1} \quad \textit{for } i = 1,2. \tag{3.30}$$

Proof. Using the results in Theorem 3.4, for any nonnegative integer p we have

$$\beta_i e^{\sigma_i t} \geq \int \psi F^i \geq \alpha_i \int \psi e^{\epsilon_i v_i} \geq \alpha_i \epsilon_i^p \int_Q \psi \frac{v_i^p}{p!}, \quad t > 0,\ i = 1,2. \tag{3.31}$$

By taking x_0 at the center of Q, we get

$$\beta_i e^{\sigma_i t} \geq \frac{\alpha_i \epsilon_i^p}{p!}\int_Q \frac{v_i^p}{2^n} \geq \frac{\alpha_i \epsilon_i^p}{2^n (p+1)^{p+1}}\int_Q v_i^p, \quad i = 1,2. \tag{3.32}$$

This implies (3.30). □

Lemma 3.6. *There exist constants $c_i = c_i(n,\lambda,\|u_i^0\|,\|v_{3-i}^0\|,t)$, $i = 1,2$ such that*

$$\begin{aligned} G(t-s) * u_i^r(x,s)v_{3-i}^m(x,s) &= \frac{1}{\left(4\pi\lambda(t-s)\right)^{n/2}}\int e^{-|x-y|^2/4\lambda(t-s)}u_i^r(y,s)v_{3-i}^m(y,s)dy \\ &\leq c_i\left((t-s)^{n/2q}+(t-s)^{-n/2p}\right), \quad i=1,2, \end{aligned} \tag{3.33}$$

for any $p > \max\{1,n/2\}$, $1/p+1/q=1$. Here $G(t)$ is the semigroup generated by the operator $\lambda\Delta$, $(\lambda > 0)$ on the space $C_{\text{ub}}(\mathbb{R}^n)$.

Proof. Let $\{Q_j\}$, $j = 0,1,2,\ldots$, be the tilling of $\mathbb{R}^n$ by unit cubes Q_j's such that x is at the center of Q_0. Then

$$\int e^{-(x-y)^2/4\lambda(t-s)}u_i^r(y,s)v_{3-i}^m(y,s)dy = \sum_{Q_j}\int_{Q_j} e^{-(x-y)^2/4\lambda(t-s)}u_i^r(y,s)v_{3-i}^m(y,s)dy. \tag{3.34}$$

For $y \in Q_j$ we have the inequality

$$e^{-(x-y)^2/8\lambda(t-s)} \leq \sup_{y\in Q_j} e^{-(x-y)^2/8\lambda(t-s)} = e^{-\operatorname{dist}^2(x,Q_j)/8\lambda(t-s)}. \tag{3.35}$$

Also there exists a positive constant $c(n)$ such that if $y \in Q_j$, $j \neq 0$, we have

$$c(n)\operatorname{dist}^2(x,Q_j) \geq (x-y)^2. \tag{3.36}$$

Let $I_1 = \int e^{-(x-y)^2/8\lambda(t-s)}u_i^r(y,s)v_{3-i}^m(y,s)dy$. Then applying Hölder's inequality with $p \geq n/2$ and its conjugate q, we get

$$\begin{aligned} I_1 &\leq \left(\int_{Q_j} e^{-q(x-y)^2/8\lambda(t-s)}\right)^{1/q}\left(\int_{Q_j} u_i^{rp}(y,s)v_{3-i}^{mp}(y,s)dy\right)^{1/p} \\ &\leq \left(\int_{Q_j} e^{-q\operatorname{dist}^2(x,Q_j)/8\lambda(t-s)}\right)^{1/q}\left(\int_{Q_j} u_i^{rp}(y,s)v_{3-i}^{mp}(y,s)dy\right)^{1/p} \\ &\leq \frac{(8\pi\lambda)^{n/2q}(t-s)^{n/2q}}{q}\left(\int_{Q_j} u_i^{rp}(y,s)v_{3-i}^{mp}(y,s)dy\right)^{1/p} \\ &\leq \frac{(8\pi\lambda)^{n/2q}(t-s)^{n/2q}}{q}\left(\int_{Q_j} |u_i|^{rp}|v_{3-i}|^{mp}dy\right)^{1/p} \\ &\leq \frac{(8\pi\lambda)^{n/2q}(t-s)^{n/2q}}{q}||u_i^0||^r\left(\int_{Q_j} |v_{3-i}|^{mp}dy\right)^{1/p} \\ &\leq \frac{(8\pi\lambda)^{n/2q}(t-s)^{n/2q}}{q}||u_i^0||^r\left(2^{mp}\frac{\beta_i}{\alpha_i\epsilon_i^{mp}}e^{\sigma t}(mp+1)^{mp+1}\right)^{1/p} \\ &= \frac{(8\pi\lambda)^{n/2q}(t-s)^{n/2q}}{q}||u_i^0||^r 2^m\left(\frac{\beta_i}{\alpha_i}\right)^{1/p}e^{\sigma t/p}\epsilon_i^{-m}(mp+1)^{m+1/p}. \end{aligned} \tag{3.37}$$

That is

$$\int e^{-(x-y)^2/8\lambda(t-s)} u_i^r(y,s) v_{3-i}^m(y,s)\,dy \le c_i^1(n,\lambda,||u_i^0||,||v_{3-i}^0||,t)(t-s)^{n/2q}, \quad i=1,2, \tag{3.38}$$

where

$$c_i^1(n,\lambda,||u_i^0||,||v_{3-i}^0||,t) = \frac{(8\pi\lambda)^{n/2q}}{q}||u_i^0||^r 2^m \left(\frac{\beta_i}{\alpha_i}\right)^{1/p} e^{\sigma t/p}\epsilon_i^{-m}(mp+1)^{m+1/p}, \quad i=1,2. \tag{3.39}$$

Now we have

$$\begin{aligned} I_2 &= G(t-s) * u_i^r(x,s) v_{3-i}^m(x,s) \\ &\le \frac{1}{(4\pi\lambda(t-s))^{n/2}} \sum_{Q_j} e^{-\operatorname{dist}^2(x,Q_j)/8\lambda(t-s)} \int e^{-|x-y|^2/8\lambda(t-s)} u_i^r(y,s) v_{3-i}^m(y,s)\,dy \\ &\le \frac{1}{(4\pi\lambda(t-s))^{n/2}} c_i^1(n,\lambda,||u_i^0||,||v_{3-i}^0||,t)(t-s)^{n/2q} \sum_{Q_j} e^{-\operatorname{dist}^2(x,Q_j)/8\lambda(t-s)} \\ &\le c_i(n,\lambda,||u_i^0||,||v_{3-i}^0||,t)(t-s)^{-n/2p}\left(\int e^{-(x-y)^2/8\lambda c(n)(t-s)}\,dy + 1\right) \\ &\le c_i(n,\lambda,||u_i^0||,||v_{3-i}^0||,t)(t-s)^{-n/2p}\left(8\pi\lambda c(n)(t-s)^{n/2}+1\right) \\ &\le c_i(n,\lambda,||u_i^0||,||v_{3-i}^0||,t)\left((t-s)^{n/2q}+(t-s)^{-n/2p}\right), \end{aligned} \tag{3.40}$$

where $c_i(n,\lambda,\|u_i^0\|,\|v_{3-i}^0\|,t) = c_i^1(n,\lambda,\|u_i^0\|,\|v_{3-i}^0\|,t)(1/(4\pi\lambda)^{n/2})$. □

The integral formula for $u(t)$ is

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))\,ds. \tag{3.41}$$

From (3.41) we can deduce that

$$u_i(x,t) = S_i(t)u_i^0 - \int_0^t S_i(t-s)f_i(u(s))\,ds, \quad i=1,2, \tag{3.42}$$

$$v_1(x,t) = S_5(t)u_1^0 + S_3(t)v_1^0 - \int_0^t S_5(t-s)f_1(u(s))\,ds + \rho\int_0^t S_3(t-s)f_1(u(s))\,ds, \tag{3.43}$$

$$v_2(x,t) = S_6(t)u_2^0 + S_4(t)v_2^0 - \int_0^t S_6(t-s)f_1(u(s))\,ds + \rho\int_0^t S_4(t-s)f_1(u(s))\,ds. \tag{3.44}$$

From (2.7) and (3.43), we get

$$\begin{aligned} v_1(x,t) &= \frac{b_1}{a_1-d_1}(S_1(t)-S_3(t))u_1^0 + S_3(t)v_1^0 \\ &\quad - \int_0^t \frac{b_1}{a_1-d_1}(S_1(t-s)-S_3(t-s)) f_1(u(s))\,ds \\ &\quad + \rho\int_0^t S_3(t-s) f_1(u(s))\,ds \\ &= \frac{b_1}{a_1-d_1}S_1(t)u_1^0 + S_3(t)\left(v_1^0 - \frac{b_1}{a_1-d_1}u_1^0\right) - \frac{b_1}{a_1-d_1}\int_0^t S_1(t-s) f_1(u(s))\,ds \\ &\quad + \left(\rho + \frac{b_1}{a_1-d_1}\right)\int_0^t S_3(t-s) f_1(u(s))\,ds. \end{aligned} \tag{3.45}$$

Hence

$$\begin{aligned} v_1(x,t) &= \frac{b_1}{a_1-d_1}S_1(t)u_1^0 + S_3(t)\left(v_1^0 - \frac{b_1}{a_1-d_1}u_1^0\right) \\ &\quad + \frac{b_1}{a_1-d_1}\int_0^t S_1(t-s)\left(mk_1u_2^r(s)v_1^m(s)\right)ds \\ &\quad - \frac{b_1}{a_1-d_1}\int_0^t S_1(t-s)\left(mk_2u_1^m(s)v_2^r(s)\right)ds \\ &\quad - \left(\rho + \frac{b_1}{a_1-d_1}\right)\int_0^t S_3(t-s)\left(mk_1u_2^r(s)v_1^m(s)\right)ds \\ &\quad + \left(\rho + \frac{b_1}{a_1-d_1}\right)\int_0^t S_3(t-s)\left(mk_2u_1^m(s)v_2^r(s)\right)ds, \end{aligned} \tag{3.46}$$

that is,

$$\begin{aligned} \left|v_1(x,t)\right| &= \frac{b_1}{a_1-d_1}\left|S_1(t)u_1^0\right| + \left|S_3(t)\left(v_1^0 - \frac{b_1}{a_1-d_1}u_1^0\right)\right| \\ &\quad + \frac{b_1}{a_1-d_1}\left|\int_0^t S_1(t-s)\left(mk_1u_2^r(s)v_1^m(s)\right)ds\right| \\ &\quad + \frac{b_1}{a_1-d_1}\left|\int_0^t S_1(t-s)\left(mk_2u_1^m(s)v_2^r(s)\right)ds\right| \\ &\quad + \left(\rho + \frac{b_1}{a_1-d_1}\right)\left|\int_0^t S_3(t-s)\left(mk_1u_2^r(s)v_1^m(s)\right)ds\right| \\ &\quad + \left(\rho + \frac{b_1}{a_1-d_1}\right)\left|\int_0^t S_3(t-s)\left(mk_2u_1^m(s)v_2^r(s)\right)ds\right|. \end{aligned} \tag{3.47}$$

Using relation (3.33) and the property that $S_1(t)$ and $S_3(t)$ are contraction semigroups, we get

$$
\begin{aligned}
||v_1(x,t)|| \le{} & \frac{b_1}{a_1-d_1}||u_1^0||+||v_1^0|| \\
& + \frac{mb_1}{a_1-d_1}\left(k_1c_1(n,a_1,||u_2^0||,||v_1^0||,t)+k_2c_2(n,a_1,||u_1^0||,||v_2^0||,t)\right) \\
& \times\left(\int_0^t \left((t-s)^{n/2q}+(t-s)^{-n/2p}\right)ds\right) \\
& + \left(\rho+\frac{mb_1}{a_1-d_1}\right)\left(k_1c_3(n,d_1,||u_2^0||,||v_1^0||,t)+k_2c_4(n,d_1,||u_1^0||,||v_1^0||,t)\right) \\
& \times\left(\int_0^t \left((t-s)^{n/2q}+(t-s)^{-n/2p}\right)ds\right), \\
||v_1(x,t)|| \le{} & \frac{b_1}{a_1-d_1}||u_1^0||+||v_1^0||+\frac{mb_1}{a_1-d_1}(k_1c_1+k_2c_2)\left|\int_0^t \left((t-s)^{n/2q}+(t-s)^{-n/2p}\right)ds\right| \\
& + m\left(\rho+\frac{b_1}{a_1-d_1}\right)(k_1c_3+k_2c_4)\left|\int_0^t \left((t-s)^{n/2q}+(t-s)^{-n/2p}\right)ds\right|.
\end{aligned}
\tag{3.48}
$$

If $p>n/2$, we get

$$
\begin{aligned}
||v_1(x,t)|| \le{} & \frac{b_1}{a_1-d_1}||u_1^0||+||v_1^0||+\frac{mb_1}{a_1-d_1}(k_1c_1+k_2c_2)\left(\frac{2qt^{n/2q+1}}{n+2q}+\frac{2pt^{1-n/2p}}{2p-n}\right) \\
& + m\left(\rho+\frac{b_1}{a_1-d_1}\right)(k_1c_3+k_2c_4)\left(\frac{2qt^{n/2q+1}}{2q+n}+\frac{2pt^{1-n/2p}}{2p-n}\right), \quad t>0.
\end{aligned}
\tag{3.49}
$$

Similarly

$$
\begin{aligned}
||v_2(x,t)|| \le{} & \frac{b_2}{a_2-d_2}||u_1^0||+||v_1^0||+\frac{rb_2}{a_2-d_2}(k_1c_5+k_2c_6)\left(\frac{2qt^{n/2q+1}}{n+2q}+\frac{2pt^{1-n/2p}}{2p-n}\right) \\
& + r\left(\rho+\frac{b_2}{a_2-d_2}\right)(k_1c_7+k_2c_8)\left(\frac{2qt^{n/2q+1}}{2q+n}+\frac{2pt^{1-n/2p}}{2p-n}\right), \quad t>0.
\end{aligned}
\tag{3.50}
$$

Similarly one can show that

$$
||u_1(x,t)|| \le ||u_1^0||+m(k_1c_9+k_2c_{10})\left(\frac{2qt^{n/2q+1}}{2q+n}+\frac{2pt^{1-n/2p}}{2p-n}\right), \quad t>0, \tag{3.51}
$$

$$
||u_2(x,t)|| \le ||u_2^0||+m(k_1c_{11}+k_2c_{12})\left(\frac{2qt^{n/2q+1}}{2q+n}+\frac{2pt^{1-n/2p}}{2p-n}\right), \quad t>0. \tag{3.52}
$$

Here c_1-c_{12} are constants.

From estimates (3.49)–(3.52) and the standard parabolic regularity theory we get the existence of global classical solution $(u_1,u_2,v_1,v_2)\in(C([0,\infty),C_{\text{ub}})\cap C^1((0,\infty),C_{\text{ub}}))^4$.

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