INSCRIBING CLOSED NON-σ-LOWER POROUS SETS INTO SUSLIN NON-σ-LOWER POROUS SETS

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The main aim of this paper is to prove that every non-σ-lower porous Suslin set in a topologically complete metric space contains a closed non-σ-lower porous subset. In fact, we prove a general result of this type on "abstract porosities." This general theorem is also applied to ball small sets in Hilbert spaces and to σ-cone-supported sets in separable Banach spaces.

1. Introduction

This paper is a continuation of the work done in [9]. We are interested in the following question within the context of σ-ideals of σ-porous type.

Let X be a metric space and let $\mathcal{I}$ be a σ-ideal of subsets of X. Let $S \subset X$ be a Suslin set with $S \notin \mathcal{I}$. Does there exist a closed set $F \subset S$ which is not in $\mathcal{I}$?

The answer is positive provided that X is locally compact and $\mathcal{I}$ is a σ-ideal of σ-$\mathbf{P}$-porous sets, where $\mathbf{P}$ is a porosity-like relation satisfying some additional conditions (see the definitions below, and for the precise statement, see [9]). In the case of the σ-ideal of ordinary (i.e., upper) σ-porous sets, which satisfies the assumptions of the above-mentioned theorem in any locally compact metric space, even more is true: X can be any topologically complete metric space (see [8]). The proofs are not easy; they use either some amount of descriptive set theory (see [9]) or a quite complicated construction (see [8]).

In this paper, we deal with σ-ideals of σ-$\mathbf{P}$-porous sets again, but these σ-ideals are supposed to be generated by closed $\mathbf{P}$-porous sets, that is, every σ-$\mathbf{P}$-porous set is covered by countably many closed $\mathbf{P}$-porous sets. Note that this property does not hold for ordinary σ-porous sets but does hold for σ-lower porous sets. Although we will also work in nonseparable spaces, it turns out that the situation is much simpler than in [9]. Under a simple additional condition on the porosity-like relation $\mathbf{P}$, we prove that every such σ-ideal has the property that every non-σ-$\mathbf{P}$-porous Suslin subset of a topologically complete metric space X contains a closed non-σ-$\mathbf{P}$-porous subset. As the main tool, we use a nonseparable version of Solecki's theorem proved in [2].

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The general result will be applied to the σ-ideals of σ-lower porous sets, of σ-cone-supported sets, and of ball small sets.

2. The general result

We start with notations and definitions. Let (X,ρ) be a metric space. Then the open ball with center $x \in X$ and radius $r > 0$ is denoted by $B(x,r)$. We will use the following terminology from [7, 9]. We say that $\mathbf{R}$ is a *point-set relation on* X if it is a relation between points of X and subsets of X. Thus a point-set relation $\mathbf{R}$ is a subset of $X \times 2^X$. The symbol $\mathbf{R}(x,A)$, where $x \in X$ and $A \subset X$, means that $(x,A) \in \mathbf{R}$, that is, $\mathbf{R}$ holds for the pair (x,A).

Let $\mathbf{R}$ be a point-set relation on X. If $A \subset X$ and $B \subset X$, then $\mathbf{R}(A,B) \overset{\text{def}}{\iff} \forall a \in A : \mathbf{R}(a,B)$. The point-set relation $\neg\mathbf{R}$ on X is defined by $(\neg\mathbf{R})(x,A) \overset{\text{def}}{\iff} \neg(\mathbf{R}(x,A))$.

We consider the following properties of a point-set relation $\mathbf{R}$ on X.

(A1) If $A \subset B \subset X$, $x \in X$, and $\mathbf{R}(x,B)$, then $\mathbf{R}(x,A)$.

(A2) $\mathbf{R}(x,A)$ if and only if there is $r > 0$ such that $\mathbf{R}(x,A \cap B(x,r))$.

(A3) $\mathbf{R}(x,A)$ if and only if $\mathbf{R}(x,\overline{A})$.

We say that a point-set relation $\mathbf{P}$ on X is a *porosity-like relation* if $\mathbf{P}$ satisfies the "axioms" (A1)–(A3).

Let $\mathbf{P}$ be a porosity-like relation on X. We say that $A \subset X$ is

(i) $\mathbf{P}$-*porous at* $x \in X$ if $\mathbf{P}(x,A)$,

(ii) $\mathbf{P}$-*porous* if $\mathbf{P}(x,A)$ for every $x \in A$,

(iii) σ-$\mathbf{P}$-*porous* if A is a countable union of $\mathbf{P}$-porous sets.

If $\mathbf{P}$ is a porosity-like relation on X and $A \subset X$, then the set of all points of A, at which A is not $\mathbf{P}$-porous, is denoted by $\mathrm{N}(\mathbf{P},A)$.

The proof of our result is based on the following nonseparable version (see [2, Corollary 3.6 and Remark 3.7]) of Solecki's theorem (see [3]). We need the following definitions to formulate it.

Let $\mathcal{A}$ be a system of subsets of a metric space X. We say that $\mathcal{A}$ is *weakly locally determined* if $A \subset X$ belongs to $\mathcal{A}$ whenever for each $x \in X$ there exists a, not necessarily open, neighbourhood U of x such that $U \cap A \in \mathcal{A}$.

Let $\mathcal{F}$ be a family of closed subsets of a metric space X. We say that $\mathcal{F}$ is *hereditary* if for all sets F_1, F_2 with $F_1 \subset F_2$, $F_2 \in \mathcal{F}$, we have $F_1 \in \mathcal{F}$.

Proposition 2.1 (see [2]). *Let X be a topologically complete metric space. Let $\mathcal{F}$ be a hereditary weakly locally determined system of closed sets. Then each Suslin subset of X is either covered by countably many elements of $\mathcal{F}$ or else contains a $\mathbf{G}_\delta$ set H such that $H \cap G$ cannot be covered by countably many elements of $\mathcal{F}$, whenever G is open and $G \cap H \neq \emptyset$.*

Definition 2.2. Let X be a metric space and let $\mathbf{P}$ be a porosity-like relation on X. It is said that $\mathbf{P}$ has property $(\star)$ if the following condition is satisfied.

$(\star)$ If $H \subset X$, $x \in H'$, and H is not $\mathbf{P}$-porous at x, then there exists $J \subset H$ such that $J' = \{x\}$ and J is not $\mathbf{P}$-porous at x.

The symbol H' stands for the set of all points of accumulation of H.

Now we can formulate our abstract theorem.

THEOREM 2.3. *Let X be a topologically complete metric space and let $\mathbf{P}$ be a porosity-like relation on X such that $\mathbf{P}$ satisfies ($\star$), and each σ-$\mathbf{P}$-porous set is covered by countably many closed $\mathbf{P}$-porous sets. If $S \subset X$ is a Suslin non-σ-$\mathbf{P}$-porous set, then there exists a closed non-σ-$\mathbf{P}$-porous set $F \subset S$.*

The next lemma immediately follows by a Baire category argument.

LEMMA 2.4. *Let X and $\mathbf{P}$ be as in Theorem 2.3. Let $F \subset X$ be a closed nonempty set such that $\mathrm{N}(\mathbf{P},F)$ is dense in F. Then F is not σ-$\mathbf{P}$-porous.*

Proof of Theorem 2.3. We denote the σ-ideal of all σ-$\mathbf{P}$-porous sets by $\mathcal{I}$.

The system of all closed $\mathbf{P}$-porous sets is clearly hereditary and weakly locally determined by (A1) and (A2). According to Proposition 2.1, we may and do assume that S is a $\mathbf{G}_\delta$ set and $S \cap G \notin \mathcal{I}$ for every open $G \subset X$ intersecting S. If there is $x \in S \setminus S'$, then $\{x\} \notin \mathcal{I}$. In this case, $F := \{x\}$ can serve as the set for which we are looking. From now on, we assume that $S \subset S'$. Let $S = \bigcap_{n=1}^\infty G_n$, where $\{G_n\}_{n=1}^\infty$ is a decreasing sequence of open sets. We will construct a sequence $\{F_n\}_{n=0}^\infty$ of closed sets and a decreasing sequence $\{H_n\}_{n=1}^\infty$ of open sets such that $F_0 = \emptyset$ and for every $n \in \mathbb{N}$, we have

(a) $\emptyset \neq F_n \subset \mathrm{N}(\mathbf{P},S)$,
(b) $F_n' = F_{n-1}$,
(c) $F_n \subset H_n \subset \overline{H_n} \subset G_n$,
(d) $(\neg\mathbf{P})(F_{n-1},F_n)$.

We proceed by induction over n. Since $S \notin \mathcal{I}$, we can choose $x \in \mathrm{N}(\mathbf{P},S)$. We put $F_1 = \{x\}$. We easily find an open set H_1 such that $x \in H_1$ and $\overline{H_1} \subset G_1$. The sets F_1 and H_1 satisfy (a)–(d) for $n = 1$.

Assume that we have constructed $F_1,\dots,F_m$ and $H_1,\dots,H_m$ such that (a)–(d) hold for $n = 1,\dots,m$. We find an open set H_{m+1} with $F_m \subset H_{m+1} \subset \overline{H_{m+1}} \subset G_{m+1} \cap H_m$. The set $F_m \setminus F_m'$ is discrete in $X \setminus F_m'$, that is, for every $y \in X \setminus F_m'$, there exists $r > 0$ such that $B(y,r) \cap (F_m \setminus F_m')$ contains at most one point. It is well known and easy to prove that, for each $z \in F_m \setminus F_m'$, we can choose $r_z > 0$ such that $\mathcal{B} = (B(z,r_z))_{z \in F_m \setminus F_m'}$ is discrete in $X \setminus F_m'$, that is, for every $y \in X \setminus F_m'$, there exists $s > 0$ such that, for at most one $z \in F_m \setminus F_m'$, $B(y,s)$ intersects $B(z,r_z)$.

Since $S \cap G \notin \mathcal{I}$ for every open G intersecting S, we have that $\mathrm{N}(\mathbf{P},S)$ is dense in S. According to this, (A3), and (a), we have $F_m \subset \mathrm{N}(\mathbf{P},\mathrm{N}(\mathbf{P},S))$. Thus using the condition ($\star$) and (A2), we find for every $z \in F_m \setminus F_m'$ a set J_z such that $J_z \subset B(z,r_z) \cap H_{m+1} \cap \mathrm{N}(\mathbf{P},S)$, $(\neg\mathbf{P})(z,J_z)$, and $J_z' = \{z\}$.

We put $F_{m+1} = F_m \cup \bigcup\{J_z;\, z \in F_m \setminus F_m'\}$. Clearly, $F_{m+1} \subset \mathrm{N}(\mathbf{P},S)$ and $F_{m+1} \subset H_{m+1}$. It is easy to see that $F_{m+1}' = F_m$; in particular, F_{m+1} is closed.

Let $x \in F_m$. We distinguish two possibilities. If $x \in F_m' = F_{m-1}$, then $(\neg\mathbf{P})(x,F_m)$ by the induction hypothesis, and so $(\neg\mathbf{P})(x,F_{m+1})$ by (A1). If $x \in F_m \setminus F_m'$, then $(\neg\mathbf{P})(x,J_x)$ and we also have $(\neg\mathbf{P})(x,F_{m+1})$. We get $(\neg\mathbf{P})(F_m,F_{m+1})$. Thus the sets F_{m+1} and H_{m+1} satisfy (a)–(d) for $n = m+1$ and the construction of our sequences is finished.

The desired set F is defined by $F = \overline{\bigcup_{n=1}^\infty F_n}$. Using (c) and the monotonicity of the H_n's, we get $F \subset S$. We have $(\neg\mathbf{P})(\bigcup_{n=1}^\infty F_n, F)$ by (d). The set $\bigcup_{n=1}^\infty F_n$ is dense in F. Hence $F \notin \mathcal{I}$, by Lemma 2.4. □

3. Applications

We will apply Theorem 2.3 to the σ-ideal of σ-lower porous sets (in a topologically complete metric space) and to two of its subsystems: to the σ-ideal of σ-cone-supported sets (in a separable Banach space) and to the σ-ideal of ball small sets (in an arbitrary Hilbert space).

Note that σ-lower porous sets (called frequently simply "σ-porous sets" and sometimes "σ-very porous sets") were applied in a number of articles on exceptional sets in (sometimes also nonseparable) Banach spaces (cf. [6]). In [6], information on σ-cone-supported and ball small sets can also be found.

To verify condition ($\star$) in concrete cases, we will apply the following easy lemma.

Lemma 3.1. *Let $g : [0,\infty) \to [0,\infty)$ be a continuous increasing function with $g(0) = 0$. Let (X,ρ) be a metric space, $H \subset X$, and $a \in H'$. Then there exists $J \subset H \setminus \{a\}$, such that $J' = \{a\}$, and for each $x \in H \setminus \{a\}$, there exists $x^* \in J$ such that $g(\rho(x,x^*)) < \min(\rho(x,a), \rho(x^*,a))$.*

Proof. Let $M_1 := \{x \in X;\ 1 \le \rho(x,a)\}$ and $M_n := \{x \in X;\ 1/n \le \rho(x,a) < 1/(n-1)\}$ for $n = 2,3,\ldots$. For each natural n, choose $\varepsilon_n > 0$ such that $g(\varepsilon_n) < 1/n$ and in $H \cap M_n$, find a maximal ε_n-discrete subset D_n ($\rho(u,v) \ge \varepsilon_n$ for each $u, v \in D_n$, $u \ne v$). Put $J := \bigcup_{n=1}^{\infty} D_n$. Clearly, $J \subset H \setminus \{a\}$ and $J' = \{a\}$. Let $x \in H \setminus \{a\}$ be given. Find $n \in \mathbb{N}$ with $x \in M_n$. By maximality of D_n, we can choose $x^* \in D_n \subset J$ with $\rho(x,x^*) < \varepsilon_n$. Consequently,

$$g(\rho(x,x^*)) < g(\varepsilon_n) < \frac{1}{n} \le \min(\rho(x,a),\rho(x^*,a)). \tag{3.1}$$

□

3.1. σ-lower porous sets

Definition 3.2. Let (X,ρ) be a metric space. It is said that $A \subset X$ is *lower porous at $x \in X$* if there exist $c > 0$ and $r_0 > 0$ such that for every $r \in (0,r_0)$, there exists $y \in B(x,r)$ with $B(y,cr) \subset B(x,r) \setminus A$. The corresponding porosity-like relation is denoted by $\mathbf{P_l}$, and σ-$\mathbf{P_l}$-porous sets are called *σ-lower porous*.

It is a well known and an easy fact that the σ-ideal $\mathcal{I}_l$ of all σ-lower porous sets is generated by closed $\mathbf{P_l}$-porous sets (see, e.g., [6, Proposition 2.5]). The proof of the following lemma is also easy.

Lemma 3.3. *Let X be a metric space. Then $\mathbf{P_l}$ has property ($\star$).*

Proof. Let $x \in \mathrm{N}(\mathbf{P},H) \cap H'$. Put $g(h) := \sqrt{h}$ (then $h = o(g(h))$, $h \to 0+$) and find $J \subset H$ by Lemma 3.1. Then $J' = \{x\}$. We will prove $(\neg\mathbf{P_l})(x,J)$.

Suppose on the contrary that J is lower porous at x. Then there exist $c > 0$ and $r_0 > 0$ such that for each $0 < r < r_0$, there exists $y \in X$ with $B(y,cr) \subset B(x,r) \setminus J$. We can clearly choose $r_1 > 0$ such that $g(h) > 2h/c$ for each $0 < h < r_1$. Put $\tilde{r} := \min(r_0,r_1)$, $\tilde{c} := c/2$, and consider an arbitrary $0 < r < \tilde{r}$. Choose $y \in X$ such that $B(y,cr) \subset B(x,r) \setminus J$. To obtain a contradiction with $x \in \mathrm{N}(\mathbf{P_l},H)$, it is sufficient to show that

$$B(y,\tilde{c}r) \cap H = \emptyset. \tag{3.2}$$

Suppose that it is not the case and choose $z \in B(y,\tilde{c}r) \cap H$. By the choice of J, we can find $z^* \in J$ such that $g(\rho(z,z^*)) < \rho(z,x) < r < r_1$. Since $\tilde{c} < c$, we have $z \neq z^*$ and the definition of r_1 gives $g(\rho(z,z^*)) > 2\rho(z,z^*)/c$. Consequently, $\rho(z,z^*) < cr/2$, which implies that $z^* \in B(y,cr) \cap J$. This is a contradiction which proves (3.2). □

Theorem 2.3 thus implies the following result.

Corollary 3.4. *Let X be a topologically complete metric space and let $S \subset X$ be a Suslin set which is not σ-lower porous. Then there exists a closed $F \subset S$ which is not σ-lower porous.*

Remark 3.5. We say that $A \subset \mathbf{R}$ is *lower symmetrically porous at $x \in \mathbf{R}$* if there exist $r_0 > 0$ and $c > 0$ such that for each $0 < r < r_0$, there exist $h > 0$ and $t \geq 0$ such that $h/r > c$, $t + h \leq r$, $(x+t, x+t+h) \cap A = \emptyset$, and $(x-t-h, x-t) \cap A = \emptyset$. The notions of a *lower symmetrically porous set* and a *σ-lower symmetrically porous set* are defined in the obvious way.

Proceeding quite similarly as above, we can easily obtain that *each analytic set $S \subset \mathbf{R}$ which is not σ-lower symmetrically porous contains a closed set which is not σ-lower symmetrically porous.*

3.2. Cone-supported sets

Definition 3.6. If X is a Banach space, $v \in X$, $\|v\| = 1$, and $0 < c < 1$, then define the cone $A(v,c) := \bigcup_{\lambda>0} \lambda \cdot B(v,c)$. Define the (clearly porosity-like) point-set relation $\mathbf{P_s}$ as follows: $\mathbf{P_s}(x,M)$ if there exist $r > 0$ and a cone $A(v,c)$ such that $M \cap (x + A(v,c)) \cap B(x,r) = \emptyset$. Sets which are $\mathbf{P_s}$-porous (σ-$\mathbf{P_s}$-porous) are called *cone supported* (*σ-cone supported*).

If X is separable, it is easy to prove (see [4, Lemma 1], cf. [6]) that $M \subset X$ is σ-cone supported (i.e., σ-$\mathbf{P_s}$-porous) if and only if M can be covered by countably many Lipschitz hypersurfaces. Since each Lipschitz hypersurface is clearly a closed $\mathbf{P_s}$-porous set, every σ-$\mathbf{P_s}$-porous set is covered by countably many closed $\mathbf{P_s}$-porous sets.

Lemma 3.7. *Let X be a Banach space. Then $\mathbf{P_s}$ has property $(\star)$.*

Proof. Let $x \in \mathrm{N}(\mathbf{P_s}, H) \cap H'$. Put $g(h) := \sqrt{h}$ and find $J \subset H$ by Lemma 3.1. Then $J' = \{x\}$. We will prove $(\neg\mathbf{P_s})(x,J)$. We can and will suppose that $x = 0$.

Suppose on the contrary that $\mathbf{P_s}(0,J)$. Then there exist $v \in X$, with $\|v\| = 1$, $1 > c > 0$, and $r > 0$ such that $J \cap A(v,c) \cap B(0,r) = \emptyset$. We can suppose that $r < c/4$. To obtain a contradiction with $0 \in \mathrm{N}(\mathbf{P_s}, H)$, it is sufficient to show that

$$H \cap A\left(v, \frac{c}{2}\right) \cap B\left(0, \frac{r}{2}\right) = \emptyset. \tag{3.3}$$

Suppose that this is not the case and choose $z \in H \cap A(v,c/2) \cap B(0,r/2)$. By the choice of J, we can find $z^* \in J$ such that $\|z - z^*\| \leq \|z\|^2 < \min(r/2, c/4 \cdot \|z\|)$. Thus clearly $z^* \in B(0,r)$. Choose $\lambda > 0$ with $\|\lambda z - v\| < c/2$. Then

$$||\lambda z^* - v|| \leq \frac{c}{2} + \lambda||z - z^*|| \leq \frac{c}{2} + \|\lambda z\| \cdot \frac{c}{4} < \frac{c}{2} + \left(1 + \frac{c}{2}\right) \cdot \frac{c}{4} < c, \tag{3.4}$$

and thus $z^* \in A(v,c) \cap B(0,r)$. This is a contradiction which proves (3.3). □

Theorem 2.3 thus implies the following result.

Corollary 3.8. *Let X be a separable Banach space and let $S \subset X$ be an analytic set which cannot be covered by countably many Lipschitz hypersurfaces. Then there exists a closed set $F \subset S$ which cannot be covered by countably many Lipschitz hypersurfaces.*

3.3. Ball small sets

Definition 3.9. Let X be a Banach space and let $r > 0$. It is said that $A \subset X$ is *r-ball porous* at a point $x \in A$ if for each $\varepsilon \in (0,r)$, there exists $y \in X$ such that $\|x - y\| = r$ and $B(y, r - \varepsilon) \cap A = \emptyset$. A set $A \subset X$ is called *r-ball porous* if it is r-ball porous at each $x \in A$. It is said that $A \subset X$ is *ball small* if it can be written in the form $A = \bigcup_{n=1}^{\infty} A_n$, where each A_n is r_n-ball porous for some $r_n > 0$.

Using the obvious fact that $B(z, \|z - x\| - \varepsilon) \subset B(y, \rho - \varepsilon)$ whenever $\|y - x\| = \rho > 0$, z lies on the segment xy, and $\|z - x\| > \varepsilon > 0$, it is easy to verify the following facts.

(i) If A is r-ball porous at a and $0 < r^* < r$, then A is r^*-ball porous at a.

(ii) If A is r-ball porous, then $\overline{A}$ is $r/2$-ball porous.

For $A \subset X$ and $x \in X$, we will write $\mathbf{P_b}(x, A)$ if A is r-ball porous at x for some $r > 0$.

Using (i), it is easy to see that $\mathbf{P_b}$ is a porosity-like relation on X and that the σ-ideal $\mathcal{I}_b$ of all ball small sets coincides with the system of all σ-$\mathbf{P_b}$-porous sets.

By (ii), we easily obtain that $\mathcal{I}_b$ is generated by closed $\mathbf{P_b}$-porous sets.

The proof of the following lemma is not difficult but slightly technical.

Lemma 3.10. *Let X be a Hilbert space. Then $\mathbf{P_b}$ has property ($\star$).*

Proof (Sketch). First, observe that an elementary (two-dimensional) computation gives the following fact.

(F) If b, v, x, x^* are points of X, $\|v\| = 1$, $0 < \rho < 1/10$, $x \in B(b + \rho/2 \cdot v, \rho/2)$, and $\|x^* - x\| \le 4\|b - x\|^2$, then $x^* \in B(b + \rho v, \rho)$.

Now let $H \subset X$ and $a \in \mathrm{N}(\mathbf{P_b}, H) \cap H'$. Put $g(h) := \sqrt{h}$ and find $J \subset H$ by Lemma 3.1. Then $J' = \{a\}$. We will prove $(\neg \mathbf{P_b})(a, J)$. Suppose to the contrary that J is r-ball porous at a for some $r > 0$. By (i), we can suppose that $r < 1/10$. Then for each $0 < \varepsilon < r/4$, there exists $v \in X$ with $\|v\| = 1$ such that $B(a + rv, r - \varepsilon) \cap J = \emptyset$. It is sufficient to prove that

$$B\left(a + \frac{r}{2} \cdot v, \frac{r}{2} - 2\varepsilon\right) \cap H = \emptyset. \tag{3.5}$$

Then H is $r/2$-ball porous at a, a contradiction.

To prove (3.5), suppose on the contrary that there exists $x \in B(a + r/2 \cdot v, r/2 - 2\varepsilon) \cap H$. By the choice of J, there exists $x^* \in J$ such that $\|x - x^*\| < \|x - a\|^2$. Denote $b := a + 2\varepsilon v$ and distinguish two cases.

If $\|x - b\| < 2\varepsilon$, then $\|x - a\| < 4\varepsilon$ and therefore $\|x - x^*\| < 16\varepsilon^2 < \varepsilon$ (since $\varepsilon < r/4 < 1/40$). Consequently, $x^* \in B(a + r/2 \cdot v, r/2 - \varepsilon) \subset B(a + rv, r - \varepsilon)$, a contradiction.

If $\|x - b\| \ge 2\varepsilon$, then $\|x - a\| \le 2\varepsilon + \|x - b\| \le 2\|x - b\|$ and thus $\|x - x^*\| \le 4\|x - b\|^2$. Put $\rho := r - 4\varepsilon$. Since $x \in B(b + \rho/2 \cdot v, \rho/2) = B(a + r/2 \cdot v, r/2 - 2\varepsilon)$, fact (F) implies that

$$x^* \in B(b+\rho v,\rho) = B(a+(r-2\varepsilon)v, r-4\varepsilon) \subset B(a+rv, r-\varepsilon), \tag{3.6}$$

a contradiction. □

Corollary 3.11. *Let X be a Hilbert space and let $S \subset X$ be a Suslin set which is not ball small. Then there exists a closed set $F \subset S$ which is not ball small.*

Finally, note that Theorem 2.3 can be easily applied also to the system of σ-cone porous sets in an arbitrary Banach space (by a cone porous set, we mean a set which is α-cone porous for some $\alpha > 0$; see [5] for the definition and [1] for some properties of α-cone porous sets in Hilbert spaces). On the other hand, it seems that Theorem 2.3 can be applied neither to the (more interesting) related system of cone small sets (cf. [6]) nor to the system of σ-cone supported sets in nonseparable Banach spaces.

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References

[1] P. Holický, *Local and global σ-cone porosity*, Acta Univ. Carolin. Math. Phys. **34** (1993), no. 2, 51–57.

[2] P. Holický, L. Zajíček, and M. Zelený, *A remark on a theorem of Solecki*, Comment. Math. Univ. Carolin. **46** (2005), no. 1, 43–54.

[3] S. Solecki, *Covering analytic sets by families of closed sets*, J. Symbolic Logic **59** (1994), no. 3, 1022–1031.

[4] L. Zajíček, *On the points of multivaluedness of metric projections in separable Banach spaces*, Comment. Math. Univ. Carolin. **19** (1978), no. 3, 513–523.

[5] ———, *Smallness of sets of nondifferentiability of convex functions in non-separable Banach spaces*, Czechoslovak Math. J. **41(116)** (1991), no. 2, 288–296.

[6] ———, *On σ-porous sets in abstract spaces (a partial survey)*, submitted to Abstr. Appl. Anal., http://www.karlin.mff.cuni.cz/kma-preprints/.

[7] L. Zajíček and M. Zelený, *On the complexity of some σ-ideals of σ-P-porous sets*, Comment. Math. Univ. Carolin. **44** (2003), no. 3, 531–554.

[8] M. Zelený and J. Pelant, *The structure of the σ-ideal of σ-porous sets*, Comment. Math. Univ. Carolin. **45** (2004), no. 1, 37–72.

[9] M. Zelený and L. Zajíček, *Inscribing compact non-σ-porous sets into analytic non-σ-porous sets*, Fund. Math. **185** (2005), no. 1, 19–39.

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