AN EXISTENCE RESULT FOR A SEMIPOSITONE PROBLEM WITH A SIGN CHANGING WEIGHT

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We establish an existence result on positive solution for a class of reaction-diffusion equation with semipositone structure. In particular, our results apply to the diffusive logistic equation with a class of sign changing weight and constant yield harvesting. We establish the result via the method of subsuper solutions.

1. Introduction

In this paper we discuss the existence of positive classical solutions ($u \in C^{2,\alpha}(\overline{\Omega})$) of the boundary value problem

$$\begin{gathered} -\Delta u = \lambda\big(g(x)\big[u\big(1-u^p\big)\big] - \text{ch}(x)\big), \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.1}$$

where $p > 0$, $c > 0$, and $\lambda > 0$ are parameters and Ω is an open bounded region with boundary $\partial\Omega$ in class C^2 in $\mathbb{R}^n$ for $n \geq 1$. Here $g : \overline{\Omega} \to \mathbb{R}$ is a C^α function while $h : \Omega \to \mathbb{R}$ is a nonnegative C^α function with $\|h\|_\infty = 1$. When $p = 1$, (1.1) arises in population dynamics where $1/\lambda$ is the diffusion coefficient and $\text{ch}(x)$ represents the constant yield harvesting. In this case ($p = 1$), when $g(x)$ is a positive constant, various results have been established in [4]. Here we focus on sign changing weight functions g.

To precisely define our classes of weight functions, we first let $\lambda_1 > 0$ be the principal eigenvalue and $\phi > 0$ with $\|\phi\|_\infty = 1$ the corresponding eigenfunction of $-\Delta$ with the Dirichlet boundary conditions. It is well known that $\partial\phi/\partial\eta < 0$ on $\partial\Omega$ where η is the unit outward normal. Hence there exists $\delta > 0$, $\sigma > 0$, and $m > 0$ such that

$$|\nabla\phi|^2 - \lambda_1\phi^2 \geq m \quad \text{on } \overline{\Omega}_\delta, \tag{1.2}$$

$$\phi \geq \sigma \quad \text{on } \Omega - \overline{\Omega}_\delta, \tag{1.3}$$

where $\Omega_\delta := \{x \in \Omega \mid d(x,\partial\Omega) < \delta\}$.

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In this paper we assume that the weight g takes negative values in Ω_δ but requires g to be strictly positive in $\Omega - \Omega_\delta$. Define $\gamma := \min_{\Omega-\Omega_\delta} g(x)$, $\mu := \min_{\overline{\Omega}_\delta} g(x)$, and we assume that

$$|\mu| < \frac{m\gamma}{\lambda_1}\left(\frac{1}{p+1}\right)^{1/p}. \tag{1.4}$$

Further let $0 < x_1 < x_2 < \gamma/2\lambda_1$ be the positive roots of $q(x) = -\mu$ (see Figure 1.1), where

$$q(x) := x\left[1 - \frac{2\lambda_1}{\gamma}x\right]^{1/p}\left(\frac{p+1}{p}\right)2m. \tag{1.5}$$

Then we establish the following.

Theorem 1.1. *Suppose (1.4) holds,* $1/x_2 < \lambda < 1/x_1$ *and* $c \le c_0(\lambda)$, *where*

$$c_0(\lambda) := \min\left\{\left(\frac{1}{p+1}\right)^{1/p}\left[\frac{2m}{\lambda}\left(1 - \frac{2\lambda_1}{\lambda\gamma}\right)^{1/p} + \frac{\mu p}{(p+1)}\right], \frac{p\gamma\sigma^2}{(p+1)^{(p+1)/p}}\left[1 - \frac{2\lambda_1}{\lambda\gamma}\right]^{(p+1)/p}\right\}. \tag{1.6}$$

Then (1.1) has at least one positive solution u such that $\|u\|_\infty < 1$.

Note that when $c > 0$, (1.1) is a semipositone problem and it is well known in the literature that the study of positive solutions is mathematically challenging (see [2–4]). Here we also include the additional challenge of dealing with a sign changing weight function g.

Finally, we also deduce a result for the case when $g(x) \ge 0$ on $\overline{\Omega}_\delta$. In particular we prove the following.

Corollary 1.2. *If* $g(x) \ge 0$ *on* $\overline{\Omega}_\delta$ *and* $c = 0$, *then for any* $\lambda \ge 2\lambda_1/\gamma$ *(1.1) has a positive solution.*

We establish our results by the method of subsuper solutions. By a subsolution we mean a function $w \in C^2(\overline{\Omega})$ such that

$$\begin{aligned} -\Delta w &\le \lambda\big(g(x)[w(1-w^p)] - \mathrm{ch}(x)\big), \quad x \in \Omega, \\ w &\le 0, \quad x \in \partial\Omega, \end{aligned} \tag{1.7}$$

and by a supersolution a function $v \in C^2(\overline{\Omega})$ such that

$$\begin{aligned} -\Delta v &\ge \lambda\Big(g(x)[v(1-v^p)] - \mathrm{ch}(x)\Big), \quad x \in \Omega, \\ v &\ge 0, \quad x \in \partial\Omega. \end{aligned} \tag{1.8}$$

Then it is well known (see [1, 5]) that if there exists a subsolution w and a supersolution v such that $w < v$, then there exists a solution $u \in C^2(\overline{\Omega})$ such that $w \le u \le v$.

We will prove Theorem 1.1 in Section 2 and Corollary 1.2 in Section 3.

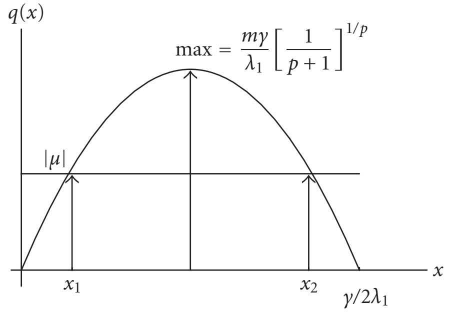

Figure 1.1

2. Proof of Theorem 1.1

Proof. Let $w = k_0\phi^2$, where

$$k_0 = \left(\frac{1}{p+1}\right)^{1/p}\left[1-\frac{2\lambda_1}{\lambda\gamma}\right]^{1/p}. \tag{2.1}$$

We will prove that w is a subsolution. Now

$$-\Delta w = -\nabla\cdot\nabla(k_0\phi^2) = -\nabla\cdot(2k_0\phi\nabla\phi) = -2k_0(\nabla\phi\cdot\nabla\phi+\phi\Delta\phi) = 2k_0(\lambda_1\phi^2-|\nabla\phi|^2). \tag{2.2}$$

First we consider the case when $x\in\overline{\Omega}_\delta$. Since the maximum of $s(1-s^p)$ is $p/(p+1)^{(p+1)/p}$, we have

$$\lambda(g(x)[w(1-w^p)]-ch(x)) \geq \lambda\left(\mu\left[\frac{p}{(p+1)^{(p+1)/p}}\right]-c\right). \tag{2.3}$$

Since

$$c<c_0\leq\left(\frac{1}{p+1}\right)^{1/p}\left[\frac{2m}{\lambda}\left(1-\frac{2\lambda_1}{\lambda\gamma}\right)^{1/p}+\frac{\mu p}{(p+1)}\right] = \frac{2k_0 m}{\lambda}+\frac{\mu p}{(p+1)^{(p+1)/p}}, \tag{2.4}$$

combining (2.3)-(2.4) and using (1.2)-(2.2), we have

$$\lambda\left(\mu\left[\frac{p}{(p+1)^{(p+1)/p}}\right]-c\right) \geq -\Delta w. \tag{2.5}$$

Hence

$$-\Delta w \leq (g(x)[w(1-w^p)]-\mathrm{ch}(x)) \quad \text{on } \overline{\Omega}_\delta. \tag{2.6}$$

Next consider the case when $x \in \Omega - \overline{\Omega}_\delta$. By the definition of γ, we have

$$
\begin{aligned}
&\lambda(g(x)[w(1-w^p)] - \mathrm{ch}(x)) \\
&\quad \geq \lambda(\gamma[k_0\phi^2(1-k_0^p\phi^{2p})] - c) \geq \lambda(\gamma[k_0\phi^2(1-k_0^p)] - c) \\
&\quad \geq \lambda\left(\gamma[k_0\phi^2(1-k_0^p)] - \frac{p\gamma}{(p+1)^{(p+1)/p}}\left[1-\frac{2\lambda_1}{\lambda\gamma}\right]^{(p+1)/p}\sigma^2\right) \quad \text{since } c \leq c_0 \\
&\quad \geq \lambda\left(\gamma[k_0\phi^2(1-k_0^p)] - \frac{p\gamma}{(p+1)}\left[1-\frac{2\lambda_1}{\lambda\gamma}\right]k_0\phi^2\right) \quad \text{using (1.3), (2.1)} \\
&\quad = \lambda\gamma k_0\phi^2\left\{1-k_0^p - \frac{p}{(p+1)}\left[1-\frac{2\lambda_1}{\lambda\gamma}\right]\right\} \\
&\quad = \lambda\gamma k_0\phi^2\{1-k_0^p - pk_0^p\} \quad \text{by (2.1)} \\
&\quad = \lambda\gamma k_0\phi^2\{1-[p+1]k_0^p\} \\
&\quad = \lambda\gamma k_0\phi^2\left\{1-\left[1-\frac{2\lambda_1}{\lambda\gamma}\right]\right\} \quad \text{by (2.1)} \\
&\quad = 2k_0\lambda_1\phi^2 \geq 2k_0[\lambda_1\phi^2 - |\nabla\phi|^2] \\
&\quad = -\Delta w \quad \text{using (2.2).}
\end{aligned}
\tag{2.7}
$$

Hence

$$-\Delta w \leq (g(x)[w(1-w^p)] - \mathrm{ch}(x)) \quad \text{on } \Omega - \overline{\Omega}_\delta. \tag{2.8}$$

From (2.6) and (2.8) we have

$$-\Delta w \leq (g(x)[w(1-w^p)] - \mathrm{ch}(x)) \quad \text{on } \Omega. \tag{2.9}$$

Thus $w = k_0\phi^2$ is a subsolution of (1.1).

Next it is easy to see that $v \equiv 1$ is a supersolution of (1.1) and $v > w$ on $\overline{\Omega}$. Thus we have a positive solution u such that $\|u\|_\infty < 1$. □

3. Proof of Corollary 1.2

Proof. Since $g(x) \geq 0$ and $c = 0$, on $\overline{\Omega}_\delta$, $\lambda(g(x)[w(1-w^p)]) \geq 0$. But $-\Delta w \leq -2k_0 m$ and is negative; hence, on $\overline{\Omega}_\delta$, we have

$$-\Delta w \leq g(x)[w(1-w^p)] \quad \text{on } \overline{\Omega}_\delta, \tag{3.1}$$

and on $\Omega - \overline{\Omega}_\delta$, we have

$$\begin{aligned}
&\lambda g(x)[w(1-w^p)] \\
&\quad \ge \lambda\gamma[k_0\phi^2(1-k_0^p\phi^{2p})] \ge \lambda\gamma[k_0\phi^2(1-k_0^p)] \\
&\quad \ge \lambda\gamma k_0\phi^2\left[1-\frac{1}{p+1}\left[1-\frac{2\lambda_1}{\lambda\gamma}\right]\right] \quad \text{by (2.1)} \\
&\quad = \frac{k_0\phi^2}{p+1}[p\lambda\gamma+2\lambda_1] \\
&\quad \ge \frac{k_0\phi^2}{p+1}[2\lambda_1(p+1)] \quad \text{since } \lambda \ge \frac{2\lambda_1}{\gamma} \\
&\quad = 2\lambda_1 k_0\phi^2 \\
&\quad \ge 2k_0[\lambda_1\phi^2 - |\nabla\phi|^2] = -\Delta w.
\end{aligned} \tag{3.2}$$

Hence we have

$$-\Delta w \le g(x)[w(1-w^p)] \quad \text{on } \Omega - \overline{\Omega}_\delta. \tag{3.3}$$

Using (3.1)–(3.3) we have that $w = k_0\phi^2$ is a subsolution. Again we note that $v \equiv 1$ is a supersolution. Hence the result holds. □

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