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Research Article

Blow-Up of Solutions with High Energies of a Coupled System of Hyperbolic Equations

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We consider an abstract coupled evolution system of second order in time. For any positive value of the initial energy, in particular for high energies, we give sufficient conditions on the initial data to conclude nonexistence of global solutions. We compare our results with those in the literature and show how we improve them.

1. Introduction

A coupled Klein-Gordon system, in electromagnetic theory, was first introduced in [1]. Posteriorly, further generalizations have been studied. In particular, the following system was analyzed in [2]

$$(\mathbf{KG}) \begin{cases} u_{tt} - \Delta u + m_1^2 u + K_1(x) u = a_1 (p+1) |v|^{q+1} |u|^{p-1} u, \\ v_{tt} - \Delta v + m_2^2 v + K_2(x) v = a_2 (q+1) |u|^{p+1} |v|^{q-1} v, \\ u(0,x) = u_0(x), \\ u_t(0,x) = u_1(x), \\ v(0,x) = v_0(x), \\ v_t(0,x) = v_1(x), \end{cases} \quad (1)$$

on $\mathbb{R} \times \mathbb{R}^N$, where $a_i > 0$, $m_i \neq 0$, $K_i(x) \geq 0$, $x \in \mathbb{R}^N$, $i = 1, 2$, and $p > 1, q > 1$. The existence and uniqueness of weak solutions of (**KG**), as well as characterizations for blow-up and globality, by means of the potential well method for values of the initial energy smaller than the mountain pass level, were proved in [2]. In the same paper, sufficient conditions were given to obtain blow-up for arbitrary positive values of the initial energy. The purpose of our work is to study an abstract hyperbolic coupled system and improve some results about nonexistence of global solutions presented in the literature for some concrete systems. In particular, we shall improve some blow-up results presented in [2], for the problem (**KG**). Precisely, we consider the following abstract problem:

$$(\mathbf{P}) \begin{cases} Pu_{tt} + Au = F(u,v), \\ Qv_{tt} + Bv = G(u,v), \\ u(0) = u_0, \\ u_t(0) = u_1, \\ v(0) = v_0, \\ v_t(0) = v_1, \end{cases} \tag{2}$$

on $\mathbb{R}$, where we assumed that the operators

$$\begin{aligned} P &: W_P \longrightarrow W_P', \\ Q &: W_Q \longrightarrow W_Q', \\ A &: V_A \longrightarrow V_A', \\ B &: V_B \longrightarrow V_B' \end{aligned} \tag{3}$$

are linear, continuous, positive, and symmetric, where

$$\begin{aligned} V_A &\subset W_P \subset H, \\ V_B &\subset W_Q \subset H \end{aligned} \tag{4}$$

are linear subspaces of the Hilbert space H with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Here, $H', W_P', W_Q', V_A', V_B'$ are the corresponding dual spaces and we identify $H = H'$. Then,

$$\begin{aligned} H &\subset W_P' \subset V_A', \\ H &\subset W_Q' \subset V_B'. \end{aligned} \tag{5}$$

By means of the operators P, Q, A, and B, we define the following bilinear forms:

$$\begin{aligned} &\mathscr{P}(u,w) \equiv (Pu,w)_{W_P \times W_P'}, \\ &\|u\|_{W_P}^2 \equiv \mathscr{P}(u,u), \\ &\qquad \forall u, w \in W_P, \\ &\mathscr{Q}(v,\widetilde{w}) \equiv (Qv,\widetilde{w})_{W_Q \times W_Q'}, \\ &\|v\|_{W_Q}^2 \equiv \mathscr{Q}(v,v), \\ &\qquad \forall v, \widetilde{w} \in W_Q, \\ &\mathscr{A}(u,w) \equiv (Au,w)_{V_A \times V_A'}, \\ &\|u\|_{V_A}^2 \equiv \mathscr{A}(u,u), \\ &\qquad \forall u, w \in V_A, \\ &\mathscr{B}(v,\widetilde{w}) \equiv (Bv,\widetilde{w})_{V_B \times V_B'}, \\ &\|v\|_{V_B}^2 \equiv \mathscr{B}(v,v), \\ &\qquad \forall v, \widetilde{w} \in V_B. \end{aligned} \tag{6}$$

We assume that there exists $c > 0$, such that

$$\begin{aligned} &(H0)\ \ \|u\|_{V_A}^2 + \|v\|_{V_B}^2 \geq c\left(\|u\|_{W_P}^2 + \|v\|_{W_Q}^2\right), \\ &\qquad \forall (u,v) \in V_A \times V_B. \end{aligned} \tag{7}$$

The nonlinear source terms $F : V_A \times V_B \longrightarrow H$ and $G : V_A \times V_B \longrightarrow H$, are such that $F(0,0) = 0 = G(0,0)$, and $(F,G) : (V_A \times V_B) \times (V_A \times V_B) \longrightarrow H \times H$ is a potential operator with potential $\mathscr{K} : V_A \times V_B \longrightarrow \mathbb{R}$, that is, $F(u,v) = \partial_u \mathscr{K}(u,v)$ and $G(u,v) = \partial_v \mathscr{K}(u,v)$, and they satisfy

$$\begin{aligned} &(H1)\ \ (F(u,v),u) + (G(u,v),v) - r\mathscr{K}(u,v) \geq 0, \\ &\qquad \forall (u,v) \in V_A \times V_B, \end{aligned} \tag{8}$$

where $r > 2$ is a constant.

2. Functional Framework

We shall analyze qualitative properties for a set of solutions of problem $(\mathbf{P})$. To this end, we define the phase space

$$\mathscr{H} \equiv (V_A \times W_P) \times (V_B \times W_Q). \tag{9}$$

We assume that the following local existence and uniqueness result is met.

Theorem 1. *For every initial data $((u_0,u_1),(v_0,v_1)) \in \mathscr{H}$, there exists $T > 0$, and a unique local solution $((u_0,u_1),(v_0,v_1)) \longmapsto ((u,\dot{u}),(v,\dot{v})) \in C([0,T);\mathscr{H})$, $\dot{u}(t) \equiv (d/dt)u(t)$, $\dot{v}(t) \equiv (d/dt)v(t)$, such that problem $(\mathbf{P})$ is satisfied in the following sense*

$$\begin{aligned} &\frac{d}{dt}\mathscr{P}(\dot{u}(t),w) + \mathscr{A}(u(t),w) = (F(u(t),v(t)),w), \\ &\frac{d}{dt}\mathscr{Q}(\dot{v}(t),\widetilde{w}) + \mathscr{B}(v(t),\widetilde{w}) = (G(u(t),v(t)),\widetilde{w}), \end{aligned} \tag{10}$$

a. e. in $(0,T)$ and for every $(w,\widetilde{w}) \in V_A \times V_B$. Furthermore, the following energy equation holds for $T > t \geq t_0 \geq 0$,

$$\begin{aligned} &E(u(t_0),\dot{u}(t_0),v(t_0),\dot{v}(t_0)) \\ &\quad = E(u(t),\dot{u}(t),v(t),\dot{v}(t)) \\ &\quad \equiv \frac{1}{2}\left\{\|\dot{u}(t)\|_{W_P}^2 + \|\dot{v}(t)\|_{W_Q}^2\right\} + J(u(t),v(t)), \\ &J(u(t),v(t)) \\ &\quad \equiv \frac{1}{2}\left\{\|u(t)\|_{V_A}^2 + \|v(t)\|_{V_B}^2\right\} - \mathscr{K}(u(t),v(t)). \end{aligned} \tag{11}$$

Remark 2. Problem $(\mathbf{P})$ is invariant if we reverse the time direction: $t \longmapsto -t$. Indeed, the solution backwards $((u(t),\dot{u}(t)),(v(t),\dot{v}(t))), t < 0$, with initial data $((u_0,u_1),(v_0,v_1))$ is the solution forwards $((u(-t),\dot{u}(-t)),(v(-t),\dot{v}(-t)))$, $-t > 0$ with initial data $((u_0,-u_1),(v_0,-v_1))$.

An important set of solutions are the equilibria, that is, solutions independent of time: $\dot{u} = 0 = \dot{v}$. In this case, (u, v) satisfies

$$\mathcal{A}(u, w) = (F(u, v), w), \quad \mathcal{B}(v, \widetilde{w}) = (G(u, v), \widetilde{w}), \tag{12}$$

for every $(w, \widetilde{w}) \in V_A \times V_B$. In particular, for $w = u, \widetilde{w} = v$,

$$\|u\|_{V_A}^2 = (F(u, v), u), \quad \|v\|_{V_B}^2 = (G(u, v), v), \tag{13}$$

and then

$$I(u, v) \equiv \|u\|_{V_A}^2 + \|v\|_{V_B}^2 - (F(u, v), u) - (G(u, v), v) = 0. \tag{14}$$

By $(H0)$, $(u, v) \equiv (0, 0)$ is an equilibrium. The set of equilibria $(u, v) \neq (0, 0)$, with minimal energy is called ground state, and the corresponding value of the energy is the mountain pass level denoted by d; see [3]. For the problem **(KG)**, the sign of $I(u_0, v_0)$ characterizes either blow-up in finite time or boundedness of solutions if $E_0 \equiv E(u_0, u_1, v_0, v_1) < d$. Indeed, blow-up and boundedness properties hold if, $I(u_0, v_0) < 0$ and $I(u_0, v_0) > 0$, respectively; see [2]. Similar analysis have been done to prove similar characterizations for coupled systems of wave equations with linear and nonlinear damping terms; see [4–8] and references therein, just to cite some works of the abundant literature in the field. The qualitative analysis of the solutions with high energies is almost unknown. There are some works that prove blow-up if $I(u_0, v_0) < 0$ and some other conditions on E_0 and the initial data are satisfied; see for instance [2]. Similar theorems have been proved in [6, 9–11], for damped systems of semilinear wave equations. The purpose of this work is to improve considerably the existing results for blow-up for systems **(P)**, with high energies. We shall generalize the technique used in a previous work for a single equation; see [12].

3. Nonexistence of Global Solutions

We consider the following orthogonal decomposition of the velocities

$$\dot{u} = \frac{\mathcal{P}(\dot{u}, u)}{\|u\|_{W_P}^2} u + h, \quad \dot{v} = \frac{\mathcal{Q}(\dot{v}, v)}{\|v\|_{W_Q}^2} v + \widetilde{h} \tag{15}$$

where $\mathcal{P}(u, h) = 0$ and $\mathcal{Q}(v, \widetilde{h}) = 0$. Then, we define the functionals

$$R(u, \dot{u}) \equiv \frac{|\mathcal{P}(\dot{u}, u)|^2}{\|u\|_{W_P}^2}, \quad S(v, \dot{v}) \equiv \frac{|\mathcal{Q}(\dot{v}, v)|^2}{\|v\|_{W_Q}^2} \tag{16}$$

Consequently,

$$\|\dot{u}\|_{W_P}^2 = \|h\|_{W_P}^2 + R(u, \dot{u}), \quad \|\dot{v}\|_{W_Q}^2 = \left\|\widetilde{h}\right\|_{W_Q}^2 + S(v, \dot{v}). \tag{17}$$

Also, we define

$$\begin{aligned} \Psi_1(u) &\equiv \|u\|_{W_P}^2, \\ \Psi_2(v) &\equiv \|v\|_{W_Q}^2, \\ \Psi(u, v) &\equiv \Psi_1(u) + \Psi_2(v) \\ \Phi(u, \dot{u}, v, \dot{v}) &\equiv c\Psi(u, v) + \frac{(\mathcal{P}(u, \dot{u}) + \mathcal{Q}(v, \dot{v}))^2}{\Psi(u, v)}. \end{aligned} \tag{18}$$

where $c > 0$ is the constant in $H(0)$. We also define the following functions

$$\begin{aligned} \eta_q(u, \dot{u}, v, \dot{v}) &\equiv \frac{1}{2}\Phi(u, \dot{u}, v, \dot{v}) - \frac{c}{r}\Psi(u, v)\left(\frac{c\Psi(u, v)}{\Phi(u, \dot{u}, v, \dot{v})}\right)^q, \quad \text{for } q \geq 0, \\ \mu_\lambda(u, \dot{u}, v, \dot{v}) &\equiv \frac{1}{2}\Phi(u, \dot{u}, v, \dot{v}) - \frac{c}{r}\Psi(u, v)\left(\frac{\lambda c\Psi(u, v)}{\Phi(u, \dot{u}, v, \dot{v})}\right)^{(r-2)/2}, \quad \text{for } \lambda \in (0, 1), \\ \sigma_\nu(u, \dot{u}, v, \dot{v}) &\equiv \frac{1}{2}\Phi(u, \dot{u}, v, \dot{v}) - \frac{c\nu}{r}\Psi(u, v), \quad \text{for } \nu > 1. \end{aligned} \tag{19}$$

If $\mathcal{P}(u, \dot{u}) + \mathcal{Q}(v, \dot{v}) > 0$, we notice that $q \longmapsto \eta_q(u, \dot{u}, v, \dot{v})$ is strictly increasing, $\lambda \longmapsto \mu_\lambda$ is strictly decreasing, and $\nu \longmapsto \sigma_\nu$ is strictly decreasing. They have the following relations

$$\lim_{\lambda \to 1} \mu_\lambda(u, \dot{u}, v, \dot{v}) = \eta_{(r-2)/2}(u, \dot{u}, v, \dot{v}), \quad \lim_{\nu \to 1} \sigma_\nu(u, \dot{u}, v, \dot{v}) = \eta_0(u, \dot{u}, v, \dot{v}), \tag{20}$$

and $\sigma_\nu(u, \dot{u}, v, \dot{v}) < \eta_0(u, \dot{u}, v, \dot{v}) < \eta_{(r-2)/2}(u, \dot{u}, v, \dot{v}) < \mu_\lambda(u, \dot{u}, v, \dot{v})$.

For the **(KG)** system, a recent work [2] proved blow-up of solutions with initial energy: $E_0 < (c(r-4)/2r)\Psi(u_0, v_0)$. We observe that $(c(r-4)/2r)\Psi(u_0, v_0) < (c(r-2)/2r)\Psi(u_0, v_0) < \eta_0(u_0, u_1, v_0, v_1)$. Here, we shall prove nonexistence of global solutions of the **(P)** system with initial energy $\sigma_{\nu^*}(u_0, u_1, v_0, v_1) < E_0 < \mu_\lambda^*(u_0, u_1, v_0, v_1)$, for some $\nu^* > 1$ and $\lambda^* \in ((r-2)/r, 1)$. Furthermore, we shall prove that for any positive value of the initial energy there are initial data implying nonexistence of global solutions.

Theorem 3. *Consider any solution of problem **(P)** in the sense of Theorem 1. Assume that hypotheses* $(H0)$ *and* $(H1)$ *are met and that*

$$\|u_0\|^2_{W_P} + \|v_0\|^2_{W_Q} > 0, \qquad \mathscr{P}(u_0, u_1) + \mathscr{Q}(v_0, v_1) > 0, \tag{21}$$

are satisfied. Then, we construct the nonempty interval

$$\mathscr{I}_0 \equiv (\alpha_0, \beta_0) \subset \left(0, \frac{1}{2}\Phi(u_0, u_1, v_0, v_1)\right), \tag{22}$$

where

$$\alpha_0 = \sigma_{\nu^*}(u_0, u_1, v_0, v_1) = \frac{r-2}{2r}\left(\frac{c\Psi(u_0, v_0)}{\nu^{*(2/(r-2))}}\right), \qquad \beta_0 = \mu_{\lambda^*}(u_0, u_1, v_0, v_1) = \frac{r-2}{2r}\left(\frac{\Phi(u_0, u_1, v_0, v_1)}{\lambda^*}\right), \tag{23}$$

for some $(r-2)/r < \lambda^* < 1$ *and* $\nu^* > 1$, *and we have the following assertions.*

(i) If the initial energy is such that $E_0 \in \mathscr{I}_0$, *then the maximal time of existence of the solution is finite.*

(ii) For fixed $\Psi(u_0, v_0)$,

$$\mathscr{P}(u_0, u_1) + \mathscr{Q}(v_0, v_1) \longmapsto |\mathscr{I}_0| = \beta_0 - \alpha_0 \tag{24}$$

is strictly increasing, and

$$\begin{aligned} \lim_{\mathscr{P}(u_0,u_1)+\mathscr{Q}(v_0,v_1)\longrightarrow\infty} \alpha_0 &= 0 \\ &= \lim_{\mathscr{P}(u_0,u_1)+\mathscr{Q}(v_0,v_1)\longrightarrow\infty} \left|\beta_0 - \frac{1}{2}\Phi(u_0, u_1, v_0, v_1)\right|, \\ \lim_{\mathscr{P}(u_0,u_1)+\mathscr{Q}(v_0,v_1)\longrightarrow\infty} \nu^* &= \infty, \\ \lim_{\mathscr{P}(u_0,u_1)+\mathscr{Q}(v_0,v_1)\longrightarrow\infty} \lambda^* &= \frac{r-2}{r}. \end{aligned} \tag{25}$$

Corollary 4. *Assume that hypotheses of Theorem 3 are met. For every number* $\mathscr{E} > 0$, *we can choose initial data with* $\mathscr{P}(u_0, u_1) + \mathscr{Q}(v_0, v_1)$ *large enough, so that* $\mathscr{E} \in \mathscr{I}_0$, *and then the corresponding solution with* $E_0 = \mathscr{E}$ *exists only up to a finite time.*

4. Proofs

Proof (of Theorem 3). First, we will assume that the solution is global and then, by means of a differential inequality in terms of Ψ, we shall get a contradiction. Then, assume that $\Psi(u(t), v(t))$ exists for any $t \geq 0$. We observe that $(d/dt)\Psi(u(t), v(t)) = 2(\mathscr{P}(u(t), \dot{u}(t)) + \mathscr{Q}(v(t), \dot{v}(t)))$. Now, we define $\mathscr{G}(t) \equiv \Psi^{-(r-2)/4}(u(t), v(t))$, and due to (21), we have that for $t \geq 0$, close to zero, the following inequality holds

$$\begin{aligned} \frac{d}{dt}\mathscr{G}(t) &= -\frac{r-2}{4} \cdot \Psi^{-(r+2)/4}(u(t), v(t)) \frac{d}{dt}\Psi(u(t), v(t)) = -\frac{r-2}{2} \\ &\quad \cdot \Psi^{-(r+2)/4}(u(t), v(t)) \\ &\quad \cdot (\mathscr{P}(u(t), \dot{u}(t)) + \mathscr{Q}(v(t), \dot{v}(t))) < 0. \end{aligned} \tag{26}$$

By energy equation and hypotheses $(H0)$ and $(H1)$, we obtain

$$\begin{aligned} \frac{d^2}{dt^2}\Psi(u(t), v(t)) &= 2\left(\|\dot{u}(t)\|^2_{W_P} + \|\dot{v}(t)\|^2_{W_Q} - I(u(t), v(t))\right) = 2\left(\|\dot{u}(t)\|^2_{W_P} + \|\dot{v}(t)\|^2_{W_Q} \right. \\ &\quad \left. - I(u(t))\right) + 2rE_0 - 2rE_0 \geq (r+2) \\ &\quad \cdot (R(u(t), \dot{u}(t)) + S(v(t), \dot{v}(t))) + (r-2) \\ &\quad \cdot \left(\|u(t)\|^2_{V_A} + \|v(t)\|^2_{V_B}\right) - 2rE_0. \\ &\geq \frac{r+2}{4}\left\{\frac{((d/dt)\Psi_1(u(t)))^2}{\Psi_1(u(t))} + \frac{((d/dt)\Psi_2(v(t)))^2}{\Psi_2(v(t))}\right\} + c(r-2)(\Psi_1(u(t)) \\ &\quad + \Psi_2(v(t))) - 2rE_0. \\ &\geq \frac{r+2}{4}\left(\frac{((d/dt)\Psi(u(t)))^2}{\Psi(u(t), v(t))}\right) + c(r-2) \\ &\quad \cdot \Psi(u(t), v(v)) - 2rE_0, \end{aligned} \tag{27}$$

where we used the following

$$\begin{aligned} &\left(\Psi_2(v(t))\frac{d}{dt}\Psi_1(u(t)) - \Psi_1(u(t))\frac{d}{dt}\Psi_2(v(t))\right)^2 \geq 0 \Longleftrightarrow \\ &\left(\frac{d}{dt}\Psi_1(u(t))\right)^2 \Psi_2^2(v(t)) + \left(\frac{d}{dt}\Psi_2(v(t))\right)^2 \Psi_1^2(u(t)) \\ &\quad \geq 2\Psi_1(u(t))\Psi_2(v(t))\frac{d}{dt}\Psi_1(u(t))\frac{d}{dt}\Psi_2(v(t)) \Longleftrightarrow \\ &\left(\frac{d}{dt}\Psi_1(u(t))\right)^2 \frac{\Psi_2(v(t))}{\Psi_1(u(t))} + \left(\frac{d}{dt}\Psi_2(v(t))\right)^2 \frac{\Psi_1(u(t))}{\Psi_2(v(t))} \end{aligned}$$

$$\begin{aligned}
&\geq 2\frac{d}{dt}\Psi_1(u(t))\frac{d}{dt}\Psi_2(v(t)) \Longleftrightarrow \\
&\left(\frac{d}{dt}\Psi_1(u(t))\right)^2\left(\frac{\Psi_2(v(t))}{\Psi_1(u(t))}+1\right) \\
&\quad+\left(\frac{d}{dt}\Psi_2(v(t))\right)^2\left(\frac{\Psi_1(u(t))}{\Psi_2(v(t))}+1\right) \\
&\geq\left(\frac{d}{dt}\Psi_1(u(t))+\frac{d}{dt}\Psi_2(v(t))\right)^2 \Longleftrightarrow \\
&\frac{((d/dt)\Psi_1(u(t)))^2}{\Psi_1(u(t))}+\frac{((d/dt)\Psi_2(v(t)))^2}{\Psi_2(v(t))} \\
&\geq\frac{((d/dt)\Psi_1(u(t))+(d/dt)\Psi_2(v(t)))^2}{\Psi_1(u(t))+\Psi_2(v(t))} \\
&=\frac{((d/dt)\Psi(u(t),v(t)))^2}{\Psi(u(t),v(t))}.
\end{aligned} \tag{28}$$

Then,

$$\begin{aligned}
\frac{d^2}{dt^2}\mathcal{G}(t)&=\left(\frac{r-2}{4}\Psi^{-(r+2)/4}(u(t),v(t))\right) \\
&\quad\times\left(\frac{r+2}{4}\frac{((d/dt)\Psi(u(t),v(t)))^2}{\Psi(u(t),v(t))}\right. \\
&\quad\left.-\frac{d^2}{dt^2}\Psi(u(t),v(t))\right)\leq -c\frac{(r-2)^2}{4}\mathcal{G}(t)+E_0 \\
&\quad\cdot\frac{r(r-2)}{2}\mathcal{G}(t)^{(r+2)/(r-2)},
\end{aligned} \tag{29}$$

and since $(d/dt)\mathcal{G}(t)<0$, we get

$$\begin{aligned}
\left(\frac{d}{dt}\mathcal{G}(t)\right)^2&\geq\frac{(r-2)^2}{4}\left(2E_0\mathcal{G}^{2r/(r-2)}(t)-c\mathcal{G}^2(t)\right) \\
&\quad+C_0,
\end{aligned} \tag{30}$$

where

$$\begin{aligned}
C_0&\equiv\left(\frac{d}{dt}\mathcal{G}(0)\right)^2 \\
&\quad-\frac{(r-2)^2}{4}\left(2E_0\mathcal{G}^{2r/(r-2)}(0)-c\mathcal{G}^2(0)\right).
\end{aligned} \tag{31}$$

We shall prove that there exists a constant $\rho_0>0$ such that

$$\left(\frac{d}{dt}\mathcal{G}(t)\right)^2\geq\rho_0^2>0,\quad\forall t\geq 0, \tag{32}$$

and then

$$\frac{d}{dt}\mathcal{G}(t)\leq-\rho_0<0,\quad\forall t\geq 0. \tag{33}$$

Hence,

$$0\leq\mathcal{G}(t)\leq-\rho_0 t+\mathcal{G}(0),\quad\forall t\geq 0 \tag{34}$$

which is impossible for any $t>\mathcal{G}(0)/\rho_0$. Then, the solution only exits up to a finite time.

Next, we prove that (32) is satisfied. To this end, we consider the right-hand side of (30) and define, for $s\geq 0$,

$$\mathcal{F}(s)\equiv\frac{(r-2)^2}{4}\left(2E_0 s^{r/(r-2)}-cs\right)+C_0, \tag{35}$$

and we notice that

$$\mathcal{F}(s)\geq\mathcal{F}(s_0),\quad\forall s\geq 0, \tag{36}$$

with $s_0\equiv(c(r-2)/2rE_0)^{(r-2)/2}>0$, and

$$\begin{aligned}
\mathcal{F}(s_0)&=\frac{(r-2)^2}{4}\left(2E_0 s_0^{r/(r-2)}-cs_0\right)+C_0, \\
&=-(r-2)E_0\left(\frac{c(r-2)}{2rE_0}\right)^{r/2}+C_0,
\end{aligned} \tag{37}$$

Also,

$$\begin{aligned}
C_0&=\frac{(r-2)^2}{4}\left(\|u_0\|_{W_P}^2+\|v_0\|_{W_Q}^2\right)^{-(r+2)/2}(\mathcal{P}(u_0,u_1) \\
&\quad+\mathcal{Q}(v_0,v_1))^2 \\
&\quad-\frac{(r-2)^2}{4}\left(2E_0\left(\|u_0\|_{W_P}^2+\|v_0\|_{W_Q}^2\right)^{-r/2}\right. \\
&\quad\left.-c\left(\|u_0\|_{W_P}^2+\|v_0\|_{W_Q}^2\right)^{-(r-2)/2}\right).
\end{aligned} \tag{38}$$

We observe that (32) is satisfied if $\rho_0^2=\mathcal{F}(s_0)>0$, which is characterized by

$$\begin{aligned}
&\frac{2c}{r}\left(\frac{c(r-2)}{2rE_0}\right)^{(r-2)/2}+2E_0\left(\|u_0\|_{W_P}^2+\|v_0\|_{W_Q}^2\right)^{-r/2} \\
&\quad<\left(\|u_0\|_{W_P}^2+\|v_0\|_{W_Q}^2\right)^{-(r+2)/2} \\
&\quad\cdot(\mathcal{P}(u_0,u_1)+\mathcal{Q}(v_0,v_1))^2 \\
&\quad+c\left(\|u_0\|_{W_P}^2+\|v_0\|_{W_Q}^2\right)^{-(r-2)/2},
\end{aligned} \tag{39}$$

and it is equivalent to

$$\begin{aligned}
&E_0+\left(\frac{c(r-2)\Psi(u_0,v_0)}{2rE_0}\right)^{(r-2)/2}\frac{c\Psi(u_0,v_0)}{r} \\
&\quad<\frac{1}{2}\Phi(u_0,u_1,v_0,v_1),
\end{aligned} \tag{40}$$

where

$$\begin{aligned}
\Phi(u_0,u_1,v_0,v_1)&\equiv c\Psi(u_0,v_0) \\
&\quad+\frac{(\mathcal{P}(u_0,u_1)+\mathcal{Q}(v_0,v_1))^2}{\Psi(u_0,v_0)}.
\end{aligned} \tag{41}$$

Now, in order to guarantee that (40) is satisfied, we define, for $s \geq 0$,

$$\mathcal{N}(s) \equiv s + \left(\frac{c(r-2)\Psi(u_0, v_0)}{2rs}\right)^{(r-2)/2} \frac{c\Psi(u_0, v_0)}{r}. \quad (42)$$

We observe that $\mathcal{N}(s) \longrightarrow \infty$ as either $s \longrightarrow 0$ or $s \longrightarrow \infty$, and

$$\mathcal{N}(s) \geq \mathcal{N}(s_1) = \frac{c}{2}\Psi(u_0, v_0), \quad \forall s \geq 0, \quad (43)$$

for $s_1 \equiv ((r-2)/2r)c\Psi(u_0, v_0)$. Moreover, by (21), there exist exactly two different roots of $\mathcal{N}(s) = (1/2)\Phi(u_0, u_1, v_0, v_1)$, denoted by α_0 and β_0, such that

$$\begin{gathered} 0 < \alpha_0 < s_1 < \beta_0 < \frac{1}{2}\Phi(u_0, u_1, v_0, v_1), \\ \frac{1}{2}\Psi(u_0, v_0) < \mathcal{N}(s) < \frac{1}{2}\Phi(u_0, u_1, v_0, v_1), \\ \forall s \in \mathcal{I}_0 \equiv (\alpha_0, \beta_0), \ s \neq s_1. \end{gathered} \quad (44)$$

And since $\mathcal{N}$ is strictly monotone for $s < s_1$ and $s > s_1$, it follows that, for fixed $\Psi(u_0, v_0)$, the interval $\mathcal{I}_0$ grows as $\mathcal{P}(u_0, u_1) + \mathcal{Q}(v_0, v_1)$ grows. Precisely,

$$\begin{aligned} \lim_{\mathcal{P}(u_0,u_1)+\mathcal{Q}(v_0,v_1)\longrightarrow\infty} \left|\frac{1}{2}\Phi(u_0, u_1, v_0, v_1) - \beta_0\right| = 0 \\ = \lim_{\mathcal{P}(u_0,u_1)+\mathcal{Q}(v_0,v_1)\longrightarrow\infty} \alpha_0. \end{aligned} \quad (45)$$

Then, (32) holds if and only if the initial energy satisfies

$$\mathcal{N}(E_0) < \frac{1}{2}\Phi(u_0, u_1, v_0, v_1), \quad (46)$$

that is, if $E_0 \in \mathcal{I}_0$. This proves that the maximum time of existence must be finite if the initial energy is within this interval.

Next, we shall find the values of α_0 and β_0. Remember that these are the roots of $\mathcal{N}(s) = (1/2)\Phi(u_0, u_1, v_0, v_1)$. To find α_0, we consider the function

$$\begin{aligned} \sigma_\nu(u_0, u_1, v_0, v_1) \equiv \frac{1}{2}\Phi(u_0, u_1, v_0, v_1) \\ - \frac{c\nu}{r}\Psi(u_0, v_0), \end{aligned} \quad (47)$$

defined for $\nu > 1$, and the equation

$$\mathcal{N}(\sigma_\nu(u_0, u_1, v_0, v_1)) = \frac{1}{2}\Phi(u_0, u_1, v_0, v_1) \quad (48)$$

which holds if and only if

$$\frac{1}{\nu^{2/(r-2)}} = \frac{2r}{c(r-2)}\left(\frac{\sigma_\nu(u_0, u_1, v_0, v_1)}{\Psi(u_0, v_0)}\right) \quad (49)$$

which is equivalent to

$$\frac{2}{r}\nu + \left(\frac{r-2}{r}\right)\frac{1}{\nu^{2/(r-2)}} = \frac{\Phi(u_0, u_1, v_0, v_1)}{c\Psi(u_0, v_0)}. \quad (50)$$

We notice that

$$f(s) \equiv \frac{2}{r}s + \left(\frac{r-2}{r}\right)\frac{1}{s^{2/(r-2)}} \longrightarrow \infty \quad (51)$$

as $s \longrightarrow 0$ and $s \longrightarrow \infty$. Also, $\min_{\{s>0\}} f(s) = f(1) = 1$. Moreover,

$$\frac{\Phi(u_0, u_1, v_0, v_1)}{c\Psi(u_0, v_0)} > 1. \quad (52)$$

Then, the equation for ν has two roots and only one is bigger than 1. Furthermore, at this root, $\nu^* > 1$,

$$\begin{aligned} \alpha_0 &= \sigma_{\nu^*}(u_0, u_1, v_0, v_1) \\ &= \left(\frac{c(r-2)\Psi(u_0, v_0)}{2r}\right)\frac{1}{\nu^{*(2/(r-2))}}, \\ &\lim_{\mathcal{P}(u_0,u_1)+\mathcal{Q}(v_0,v_1)\longrightarrow\infty} \nu^* = \infty. \end{aligned} \quad (53)$$

Next, we consider the function

$$\begin{aligned} &\mu_\lambda(u_0, u_1, v_0, v_1) \\ &\quad \equiv \frac{1}{2}\Phi(u_0, u_1, v_0, v_1) \\ &\quad - \frac{c}{r}\Psi(u_0, v_0)\left(\frac{\lambda c\Psi(u_0, v_0)}{\Phi(u_0, u_1, v_0, v_1)}\right)^{(r-2)/2}, \end{aligned} \quad (54)$$

defined for $(r-2)/r < \lambda < 1$, and the equation

$$\mathcal{N}(\mu_\lambda(u_0, u_1, v_0, v_1)) = \frac{1}{2}\Phi(u_0, u_1, v_0, v_1) \quad (55)$$

which is equivalent to

$$\begin{aligned} &\frac{c}{r}\Psi(u_0, v_0)\left(\frac{c(r-2)}{2r}\frac{\Psi(u_0, v_0)}{\mu_\lambda(u_0, u_1, v_0, v_1)}\right)^{(r-2)/2} \\ &\quad = \frac{c}{r}\Psi(u_0, v_0)\left(\lambda\frac{c\Psi(u_0, v_0)}{\Phi(u_0, u_1, v_0, v_1)}\right)^{(r-2)/2}, \end{aligned} \quad (56)$$

and it is characterized by

$$\frac{r-2}{2r} = \lambda\frac{\mu_\lambda(u_0, u_1, v_0, v_1)}{\Phi(u_0, u_1, v_0, v_1)} \quad (57)$$

which holds if and only if

$$\frac{2}{r}\left(\lambda\frac{c\Psi(u_0, v_0)}{\Phi(u_0, u_1, v_0, v_1)}\right)^{r/2} = \lambda - \frac{r-2}{r}. \quad (58)$$

Notice that

$$\begin{aligned} g(\lambda) &\equiv \frac{2}{r}\left(\lambda\frac{c\Psi(u_0, v_0)}{\Phi(u_0, u_1, v_0, v_1)}\right)^{r/2}, \\ h(\lambda) &\equiv \lambda - \frac{r-2}{r}, \end{aligned} \quad (59)$$

are strictly monotone increasing, and

$$g(0) > h(0), \quad g(1) < h(1). \tag{60}$$

Then, there exists one and only one $\lambda^* \in ((r-2)/r, 1)$ where $g(\lambda^*) = h(\lambda^*)$. Moreover,

$$\begin{aligned} \beta_0 &= \mu_{\lambda^*}(u_0, u_1, v_0, v_1) \\ &= \left(\frac{(r-2)\Phi(u_0, u_1, v_0, v_1)}{2r} \right) \frac{1}{\lambda^*}, \\ \lim_{\mathscr{P}(u_0,u_1)+\mathscr{Q}(v_0,v_1)\longrightarrow\infty} \lambda^* &= \frac{r-2}{r}. \end{aligned} \tag{61}$$

□

Proof (of Corollary 4). Since

$$\begin{aligned} &\mathscr{P} + \mathscr{Q} \longrightarrow \infty \Longrightarrow \\ &\alpha_0 \longrightarrow 0, \\ &\beta_0 \longrightarrow \infty \end{aligned} \tag{62}$$

then, for every $\mathscr{E} > 0$, there exists $M > 0$, such that

$$\begin{aligned} &\mathscr{P} + \mathscr{Q} > M \Longrightarrow \\ &\mathscr{E} \in \mathscr{I}_0 = (\alpha_0, \beta_0). \end{aligned} \tag{63}$$

Hence, the corresponding solution with initial energy $E_0 = \mathscr{E}$ satisfying (21) exists only up to a finite time. □

Remark 5. For small energies, $E_0 < d$, the potential well method characterizes the qualitative behavior of any solution in terms of the sign of $I(u_0, v_0)$; see [4–8]. In particular, blow-up is characterized if $I(u_0, v_0) < 0$. Let us examine the situation for any positive value of the initial energy.

Assume that hypotheses of Theorem 3 are met. From energy equation, $(H0)$, $(H1)$, (21), $E_0 < \beta_0$, and since $(r-2)/r < \lambda^* < 1$, we obtain

$$\begin{aligned} I(u_0, v_0) &= 2E_0 - \|u_1\|_{W_P}^2 - \|v_1\|_{W_Q}^2 + 2\mathscr{K}(u_0, v_0) \\ &\quad - (F(u_0, v_0), u_0) - (G(u_0, v_0), v_0) \\ &\le 2E_0 - \left(\frac{|\mathscr{P}(u_0, u_1)|^2}{\|u_0\|_{W_P}^2} + \frac{|\mathscr{Q}(v_0, v_1)|^2}{\|v_0\|_{W_Q}^2} \right) \\ &\quad - (r-2)\mathscr{K}(u_0, v_0) \\ &< 2\beta_0 - \frac{(\mathscr{P}(u_0, u_1) + \mathscr{Q}(v_0, v_1))^2}{\Psi(u_0, v_0)} \\ &\quad - (r-2)\mathscr{K}(u_0, v_0) \\ &= \frac{(r-2)\Phi(u_0, u_1, v_0, v_1)}{r\lambda^*} \\ &\quad - \frac{(\mathscr{P}(u_0, u_1) + \mathscr{Q}(v_0, v_1))^2}{\Psi(u_0, v_0)} \\ &\quad - (r-2)\mathscr{K}(u_0, v_0). \end{aligned} \tag{64}$$

Let us assume that

$$\mathscr{K}(u_0, v_0) > \frac{1}{r-2} c\Psi(u_0, v_0), \tag{65}$$

and then

$$I(u_0, v_0) < -\left(1 - \frac{r-2}{r\lambda^*}\right) \Phi(u_0, u_1, v_0, v_1) < 0. \tag{66}$$

Hence, if the source term is large enough at the initial data, the inequality $I(u_0, v_0) < 0$ is a necessary condition for nonexistence of global solutions. However, it seems that the condition $I(u_0, v_0) < 0$, alone, does not imply nonexistence of global solutions for high energies; see [2, 6, 9–11]. Moreover, we did not require the sign of $I(u_0, v_0)$ in the proof of Theorem 3.

From Corollary 4, global nonexistence for small positive energies is obtained if $\mathscr{P}(u_0, u_1) + \mathscr{Q}(v_0, v_1)$ is large enough and, consequently, since

$$\begin{aligned} &\left\{2(E_0 + \mathscr{K}(u_0, v_0)) - \left(\|u_0\|_{V_A}^2 + \|v_0\|_{V_B}^2\right)\right\} \Psi(u_0, v_0) \\ &\quad \ge (\mathscr{P}(u_0, u_1) + \mathscr{Q}(v_0, v_1))^2, \end{aligned} \tag{67}$$

then $\mathscr{K}(u_0, v_0)$ must be also sufficiently big. Then, by the previous discussion, $I(u_0, v_0) < 0$ is implied. Apparently, only for energies $E_0 < d$, the condition $I(u_0, v_0) < 0$ characterizes the nonexistence of global solutions of problem (**P**).

Remark 6. We shall prove the following lower bound for β_0

$$\beta_0 > \frac{1}{2} \frac{(\mathscr{P}(u_0, u_1) + \mathscr{Q}(v_0, v_1))^2}{\Psi(u_0, v_0)}. \tag{68}$$

This inequality is equivalent to

$$\begin{aligned} &\left(\frac{r\lambda^*}{r-2} - 1\right)(\mathscr{P}(u_0, u_1) + \mathscr{Q}(v_0, v_1))^2 \\ &\quad < c\Psi^2(u_0, v_0). \end{aligned} \tag{69}$$

In order to prove it, let us define, for any $s \equiv (\mathscr{P}(u_0,u_1)+\mathscr{Q}(v_0,v_1))^2>0$, the positive function

$$l(s) \equiv \left(\frac{r\lambda^*}{r-2}-1\right)s>0, \tag{70}$$

and we remember that λ^* is a function of s, defined implicitly by

$$\frac{2}{r}\left(\lambda^*\frac{c\Psi_0}{\Phi_0}\right)^{r/2}=\lambda^*-\frac{r-2}{r}, \tag{71}$$

where

$$\begin{aligned}\Phi_0 &\equiv c\Psi_0+\frac{s}{\Psi_0},\\ \Psi_0 &\equiv \Psi(u_0,v_0),\\ \Phi_0 &\equiv \Phi(u_0,u_1,v_0,v_1).\end{aligned} \tag{72}$$

On the other hand, from Theorem 3 we know that

$$\begin{aligned}\lim_{s\to\infty}\lambda^* &= \frac{r-2}{r},\\ \lim_{s\to 0}\lambda^* &= 1.\end{aligned} \tag{73}$$

Then, from the definition of λ^* and Φ_0,

$$\begin{aligned}\lim_{s\to\infty} l(s) &= \frac{2}{r-2}\lim_{s\to\infty} s\left(\lambda^*\frac{c\Psi_0}{\Phi_0}\right)^{r/2}\\ &= \frac{2}{r-2}\lim_{s\to\infty} s\left(\lambda^*\frac{c\Psi_0^2}{c\Psi_0^2+s}\right)^{r/2}=0.\end{aligned} \tag{74}$$

Also,

$$\lim_{s\to 0} l(s)=0. \tag{75}$$

Consequently, there is some $s^*\in(0,\infty)$, such that $l(s^*)=\max_{s\in(0,\infty)}l(s)$. After some calculations, we find that

$$\begin{aligned}s^* &= c\Psi_0^2\left(\frac{(r-2)(1-\lambda^*)}{r\lambda^*-(r-2)(1-\lambda^*)}\right),\\ l(s^*) &= c\Psi_0^2\left(\frac{(r\lambda^*-(r-2))(1-\lambda^*)}{r\lambda^*-(r-2)(1-\lambda^*)}\right)\end{aligned} \tag{76}$$

and consequently

$$l(s)\le l(s^*)<c\Psi_0^2,\quad \text{for any } s>0. \tag{77}$$

5. Some Examples

5.1. Nonlinear Klein-Gordon System. Klein-Gordon systems like (**KG**) were studied in [2, 4–11], where blow-up results were proved. We shall illustrate how Theorem 3 is applied for each one of these systems.

5.1.1. Wang [2]. We rewrite the system **(KG)**, introduced at the beginning of this work, as follows:

$$(\mathbf{KG})^*\begin{cases}\alpha u_{tt}-\alpha\Delta u+\alpha m_1^2u+\alpha K_1(x)u=a_2'(p+1)|v|^{q+1}|u|^{p-1}u,\\ v_{tt}-\Delta v+m_2^2v+K_2(x)v=a_2'(q+1)|u|^{p+1}|v|^{q-1}v,\\ u(0,x)=u_0(x),\\ u_t(0,x)=u_1(x),\\ v(0,x)=v_0(x),\\ v_t(0,x)=v_1(x),\end{cases} \tag{78}$$

on $\mathbb{R}\times\mathbb{R}^N$, where $\alpha\equiv a_2(p+1)/a_1(q+1)$, $a_2'\equiv a_2/(q+1)$.

Here, $P=\alpha I_d$, $Q=I_d$ = the identity operator, $H=W_P=W_Q=L_2(\mathbb{R}^N)$, and $(\cdot,\cdot)_2=\|\cdot\|_2^2$ are the inner product and the norm square in $L_2(\mathbb{R}^N)$, respectively. Also,

$$\begin{aligned}Au &= -\alpha\Delta u+\alpha m_1^2u+\alpha K_1(x)u\\ Bv &= -\Delta v+\alpha m_2^2v+\alpha K_2(x)v,\\ V_A &= \left\{u\in H^1(\mathbb{R}^N):\int_{\mathbb{R}^N}K_1(x)u(x)\,dx<\infty\right\},\end{aligned}$$

$$V_B=\left\{v\in H^1(\mathbb{R}^N):\int_{\mathbb{R}^N}K_2(x)v(x)\,dx<\infty\right\}. \tag{79}$$

Finally,

$$\begin{aligned}F(u,v) &= a_2'(p+1)|v|^{q+1}|u|^{p-1}u,\\ G(u,v) &= a_2'(q+1)|u|^{p+1}|v|^{q-1}v.\end{aligned} \tag{80}$$

Hypothesis $(H0)$ holds with $c = \min\{m_1^2, m_2^2\}$,

$$\begin{aligned}\|u\|_{V_A}^2 + \|v\|_{V_B}^2 &\geq \alpha \|\nabla u\|_2^2 + \|\nabla v\|_2^2 + c\left(\alpha \|u\|_2^2 + \|v\|_2^2\right) \\ &\geq c\left(\alpha \|u\|_2^2 + \|v\|_2^2\right) \\ &= c\left(\|u\|_{W_P}^2 + \|v\|_{W_Q}^2\right).\end{aligned} \tag{81}$$

Hypothesis $(H1)$ holds with $r \equiv p + q + 2 > 4$, since the potential operator $\mathscr{K}$ of (F, G) is

$$\mathscr{K}(u, v) = a_2'\left(|v|^{q+1}, |u|^{p+1}\right)_2, \tag{82}$$

and then,

$$(F(u, v), u)_2 + (G(u, v), v)_2 - r\mathscr{K}(u, v) = 0. \tag{83}$$

Theorem 1 is true and nonexistence of global solutions is due to blow-up; see [2] for the details, where some bounds on p, q are required. Consequently, by Theorem 3, if the initial data satisfy

$$\begin{aligned}\alpha \left\|u_0\right\|_2^2 + \left\|v_0\right\|_2^2 &> 0, \\ \alpha\left(u_0, u_1\right)_2 + \left(v_0, v_1\right)_2 &> 0,\end{aligned} \tag{84}$$

and the initial energy is such that

$$\begin{aligned}&\frac{\min\left\{m_1^2, m_2^2\right\}(p+q)}{2(p+q+2)}\left(\frac{\alpha \left\|u_0\right\|_2^2 + \left\|v_0\right\|_2^2}{\nu^{*(2/(p+q))}}\right) < E_0 \\ &< \frac{p+q}{2(p+q+2)\lambda^*}\left(\min\left\{m_1^2, m_2^2\right\}\right. \\ &\left.\cdot\left(\alpha \left\|u_0\right\|_2^2 + \left\|v_0\right\|_2^2\right) + \frac{\left(\alpha\left(u_0, u_1\right)_2 + \left(v_0, v_1\right)_2\right)^2}{\alpha \left\|u_0\right\|_2^2 + \left\|v_0\right\|_2^2}\right),\end{aligned} \tag{85}$$

then the solution blows up in finite time in the norm of $\mathscr{H}$. We notice that as $\alpha(u_0, u_1)_2 + (v_0, v_1)_2$ grows, then ν^* grows, and consequently the lower bound of E_0 is close to zero, and also the upper bound grows. That is, by Corollary 4 for every positive initial energy E_0, there exist initial data satisfying (84) such that the solution blows up. This result is new in the literature. Sufficient conditions have been given before. Indeed, blow-up is proved in [2] if (84) holds, $I(u_0, v_0) < 0$, and the initial energy is such that

$$0 < E_0 < \frac{\min\left\{m_1^2, m_2^2\right\}(p+q)}{2(p+q+2)}\left(\alpha \left\|u_0\right\|_2^2 + \left\|v_0\right\|_2^2\right). \tag{86}$$

In Theorem 3 we did not assume any sign of $I(u_0, v_0)$ and β_0 is larger than the upper bound on E_0 given in [2].

5.1.2. Liu [6]. For the corresponding linear damped problem of (**KG**) blow-up was proved under the same sufficient conditions given in [2] for high energies, and for $E_0 < d$ by means of the potential well method.

5.1.3. Korpusov [10]. A Klein-Gordon system with linear damping was analyzed with source terms of the form

$$\begin{aligned}F(u, v) &= 4(u + av)^3 + 2buv^2, \\ G(u, v) &= 4a(u + av) + 2bu^2v,\end{aligned} \tag{87}$$

where $a > 0$, $b > 0$. Hypothesis $(H1)$ holds with $r = 4$, and the potential operator

$$\mathscr{K}(u, v) = \|u + av\|_4^4 + b\|uv\|_2^2, \tag{88}$$

and then,

$$(F(u, v), u)_2 + (G(u, v), v)_2 - 4\mathscr{K}(u, v) = 0. \tag{89}$$

Blow-up of solutions was proved if (84) holds with $\alpha = 1$, $I(u_0, v_0) < 0$, and

$$E_0 < \frac{1}{2}\frac{\left(\mathscr{P}\left(u_0, u_1\right) + \mathscr{Q}\left(v_0, v_1\right) - \mu\Psi\left(u_0, v_0\right)\right)^2}{\Psi\left(u_0, v_0\right)}, \tag{90}$$

where μ is the coefficient of the linear damping term in the system studied in [13]. If $\mu = 0$,

$$E_0 < \frac{1}{2}\frac{\left(\mathscr{P}\left(u_0, u_1\right) + \mathscr{Q}\left(v_0, v_1\right)\right)^2}{\Psi\left(u_0, v_0\right)} < \beta_0, \tag{91}$$

where the second inequality is the lower bound for β_0, obtained in Remark 6. Hence, Theorem 3 improves the result presented in [10].

5.1.4. Aliev and Yusifova [9]. A system of several equations with linear damping was studied. If we consider only two, without damping, these are like those in $(\mathbf{KG})^*$ with $a_2' = 1$, and hypothesis $(H1)$ holds with $r = p + q + 2 > 2$, because $p > 0$ and $q > 0$. The blow-up of solutions is proved in [9] if (84) holds, with $\alpha = 1$, $I(u_0, v_0) < 0$, and

$$E_0 < \frac{(p+q)}{2(p+q+2)}\left(\left\|u_0\right\|_{W_P}^2 + \left\|v_0\right\|_{W_Q}^2\right) < \beta_0, \tag{92}$$

where $\|u_0\|_{W_P}^2 = (p+1)\|u_0\|_2^2$ and $\|v_0\|_{W_Q}^2 = (q+1)\|v_0\|_2^2$. Again, Theorem 3 improves the result presented in [9].

5.1.5. Wu [11]. A system like (**KG**) with a linear damping was studied with no particular source terms. However, these are assumed to satisfy $(H1)$ with $r = 2 + 4\delta > 2$, some $\delta = (r-2)/4 > 0$. Blow-up is proved for high energies, under the following conditions: $I(u_0, v_0) < 0$ and

$$\begin{aligned}&\frac{r-2}{2}\left(\left(u_0, u_1\right)_2 + \left(v_0, v_1\right)_2\right) > \left\|u_0\right\|_2^2 + \left\|v_0\right\|_2^2 \\ &\quad > \frac{r-2}{r}E_0 > 0.\end{aligned} \tag{93}$$

These conditions are more restrictive than the ones assumed in Theorem 3.

5.1.6. Ye [8], Benaissa et al. [4]. Klein-Gordon system like (**KG**) was studied with nonlinear damping and degenerated nonlinear damping terms, respectively. In both papers, blow-up was showed for energies $E_0 < d$, by the potential well method and the following source terms were considered

$$F(u,v) = a_1 (u+v)^{2(p+1)}(u+v) + a_2 |u|^p |v|^{p+2} u,$$
$$G(u,v) = a_1 (u+v)^{2(p+1)}(u+v) + a_2 |v|^p |u|^{p+2} v, \quad (94)$$

with $p > -1$, $a_1 > 0$ and $a_2 > 0$. Hypothesis $(H1)$ holds with $r = 2(p+2) > 2$, and the potential operator

$$\mathcal{K}(u,v) = \frac{1}{2(p+2)}\left(a_1 \|u+v\|_{2(p+2)}^{2(p+2)} + 2a_2 \|uv\|_{p+2}^{p+2}\right), \quad (95)$$

and then,

$$(F(u,v),u)_2 + (G(u,v),v)_2 - 2(p+2)\mathcal{K}(u,v) = 0. \quad (96)$$

Hence, Theorem 3 and Corollary 4 are applied and blow-up is proved for the undamped case in [4, 8], for high energies.

5.1.7. Wu [7], Gan and Zhang [5]. By means of the potential well method, blow-up was showed for solutions of systems like (**KG**), with $E_0 < d$, without damping, and with linear damping, respectively. In those papers, particular cases of the following source terms were considered

$$F(u,v) = \left(a_1 |u|^{2p} + a_2 |u|^{p-1} |v|^{p+1}\right) u,$$
$$G(u,v) = \left(a_1 |v|^{2p} + a_2 |v|^{p-1} |u|^{p+1}\right) v, \quad (97)$$

with $p > 0$, $a_1 > 0$, and $a_2 > 0$. Hypothesis $(H1)$ holds with $r = 2(p+1) > 2$, and the potential operator

$$\mathcal{K}(u,v) = \frac{1}{2(p+1)}\left(a_1 \|u\|_{2(p+1)}^{2(p+1)} + a_1 \|v\|_{2(p+1)}^{2(p+1)} + 2a_2 \|uv\|_{p+1}^{p+1}\right), \quad (98)$$

and then,

$$(F(u,v),u)_2 + (G(u,v),v)_2 - 2(p+1)\mathcal{K}(u,v) = 0. \quad (99)$$

Hence, Theorem 3 and Corollary 4 are applied and blow-up is proved for the undamped case in [5, 7], for high energies.

5.2. Generalized Boussinesq System. We consider the system

$$(\mathbf{GB})\begin{cases} u_{tt} - \alpha_1 \Delta u - \alpha_2 \Delta u_{tt} + \alpha_3 \Delta^2 u + m_1 u + \Delta F(u,v) = 0, \\ v_{tt} - \beta_1 \Delta v - \beta_2 \Delta v_{tt} + \beta_3 \Delta^2 v + m_2 v + \Delta G(u,v) = 0, \\ u(0,x) = u_0(x), \\ u_t(0,x) = u_1(x), \\ v(0,x) = v_0(x), \\ v_t(0,x) = v_1(x), \end{cases} \quad (100)$$

on $\mathbb{R} \times \mathbb{R}^N$, where $\alpha_i > 0$, $\beta_i > 0$, $i = 1,2,3$, $m_j > 0$, $j = 1,2$. For the physics of the problem we refer to [13, 14].

Applying $(-\Delta)^{-1}$ to the system above, we get

$$(\mathbf{GB})^*\begin{cases} \left((-\Delta)^{-1} + \alpha_2 I_d\right) u_{tt} + \left(-\alpha_3 \Delta + m_1 (-\Delta)^{-1} + \alpha_1 I_d\right) u = F(u,v), \\ \left((-\Delta)^{-1} + \beta_2 I_d\right) v_{tt} + \left(-\beta_3 \Delta + m_2 (-\Delta)^{-1} + \beta_1 I_d\right) v = G(u,v), \\ u(0,x) = u_0(x), \\ u_t(0,x) = u_1(x), \\ v(0,x) = v_0(x), \\ v_t(0,x) = v_1(x), \end{cases} \quad (101)$$

on $\mathbb{R} \times \mathbb{R}^N$. Then,

$$P\dot{u} = \left((-\Delta)^{-1} + \alpha_2 I_d\right)\dot{u},$$

$$Au = \left(-\alpha_3 \Delta + m_1 (-\Delta)^{-1} + \alpha_1 I_d\right) u,$$

$$Q\dot{v} = \left((-\Delta)^{-1} + \beta_2 I_d\right)\dot{v},$$

$$Bu = \left(-\beta_3 \Delta + m_2 (-\Delta)^{-1} + \beta_1 I_d\right) v, \tag{102}$$

and $H = L_2(\mathbb{R}^N)$,

$$\begin{aligned} W_P &= W_Q \\ &= \left\{u \in L_2\left(\mathbb{R}^N\right) : (-\Delta)^{-1/2} u \in L_2\left(\mathbb{R}^N\right)\right\}, \\ V_A &= V_B \\ &= \left\{u \in H^1\left(\mathbb{R}^N\right) : (-\Delta)^{-1/2} u \in L_2\left(\mathbb{R}^N\right)\right\}. \end{aligned} \tag{103}$$

Moreover, if

$$\|u\|_*^2 = (u, u)_* \equiv \left((-\Delta)^{-1/2} u, (-\Delta)^{-1/2} u\right)_2, \tag{104}$$

then

$$\begin{aligned} \|u\|_{W_P}^2 &= \|u\|_*^2 + \alpha_2 \|u\|_2^2, \\ \|u\|_{V_A}^2 &= \alpha_3 \|\nabla u\|_2^2 + m_1 \|u\|_*^2 + \alpha_1 \|u\|_2^2, \\ \|v\|_{W_Q}^2 &= \|v\|_*^2 + \beta_2 \|v\|_2^2, \\ \|v\|_{V_B}^2 &= \beta_3 \|\nabla v\|_2^2 + m_2 \|v\|_*^2 + \beta_1 \|v\|_2^2. \end{aligned} \tag{105}$$

Consequently, hypothesis $(H0)$ holds with $c = \min\{m_1, \alpha_1/\alpha_2, m_2, \beta_1/\beta_2\}$. We assume that the source terms (F, G) and the corresponding potential operator $\mathscr{K}$ do not have any particular form but they satisfy $(H1)$. We do not know any reference proving an existence and uniqueness result for this system. We assume that Theorem 1 is true then nonexistence of global solutions is due to blow-up. By Theorem 3 and Corollary 4 for every positive initial energy E_0, where

$$\begin{aligned} E_0 = \frac{1}{2} &\left(\left\|u_1\right\|_*^2 + \alpha_2 \left\|u_1\right\|_2^2 + \alpha_3 \left\|\nabla u_0\right\|_2^2 + m_1 \left\|u_0\right\|_*^2 \right. \\ &\left. + \alpha_1 \left\|u_0\right\|_2^2\right) + \frac{1}{2}\left(\left\|v_1\right\|_*^2 + \beta_2 \left\|v_1\right\|_2^2 + \beta_3 \left\|\nabla v_0\right\|_2^2 \right. \\ &\left. + m_2 \left\|v_0\right\|_*^2 + \beta_1 \left\|v_0\right\|_2^2\right) - \mathscr{K}\left(u_0, v_0\right), \end{aligned} \tag{106}$$

there exists initial data such that

$$\begin{aligned} \left\|u_0\right\|_*^2 + \alpha_2 \left\|u_0\right\|_2^2 + \left\|v_0\right\|_*^2 + \beta_2 \left\|v_0\right\|_2^2 > 0, \\ \left(u_0, u_1\right)_* + \alpha_2 \left(u_0, u_1\right)_2 + \left(v_0, v_1\right)_* + \beta_2 \left(v_0, v_1\right)_2 > 0 \end{aligned} \tag{107}$$

imply the nonexistence of global solutions in the norm of $\mathscr{H}$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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