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Research Article

A Convolution Theorem Related to Quaternion Linear Canonical Transform

Mawardi Bahri [1] and Ryuichi Ashino [2]

[1] *Department of Mathematics, Hasanuddin University, Makassar 90245, Indonesia*
[2] *Division of Mathematical Sciences, Osaka Kyoiku University, Osaka 582-8582, Japan*

Correspondence should be addressed to Ryuichi Ashino; ashino@cc.osaka-kyoiku.ac.jp

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We introduce the two-dimensional quaternion linear canonical transform (QLCT), which is a generalization of the classical linear canonical transform (LCT) in quaternion algebra setting. Based on the definition of quaternion convolution in the QLCT domain we derive the convolution theorem associated with the QLCT and obtain a few consequences.

1. Introduction

The linear canonical transform (LCT) plays an important role in various fields of optics and signal processing. In some papers, the LCT is also known as the affine Fourier, the ABCD, and Moshinsky-queue transforms. The LCT can be considered as a generalization of many mathematical transforms, such as Fourier, Laplace, fractional Fourier, and Fresnel transforms. Many fundamental properties of the LCT have been investigated like translation, modulation, convolution, correlation, and uncertainty principles (see, e.g., [1–8]).

The quaternion linear canonical transform (QLCT) is a generalization of the linear canonical transform (LCT) using quaternion algebra. According to definitions of the quaternion Fourier transform (QFT), there are basically two ways of obtaining the QLCT: the (right-sided) quaternion linear canonical transform and the (two-sided) quaternion linear canonical transform. The (right-sided) quaternion linear canonical transform is obtained by substituting the Fourier kernel with the right-sided QFT kernel in the LCT definition. Some important properties of the quaternion linear canonical transform such as the Parseval's theorem, reconstruction formula, and uncertainty principles are also discussed (see [9–14] and the references mentioned therein). However, there is no literature for establishing the convolution theorem associated with the QLCT as far as we know.

Therefore, it is worthwhile to study the convolution theorems associated with the QLCT, which can be useful in signal processing theory and application. Our main objective of the present paper is to establish convolution theorems for the QLCT which are generalizations of the related classical ones. We will accomplish this task by using the properties of quaternions and combining the LCT convolution and the QFT convolution definition [15]. In the beginning, we make a definition of the QLCT and obtain the relationship between the QLCT and QFT. Based on the convolution definitions of the LCT and QFT, we propose a new definition convolution for the QLCT and obtain its convolution theorem. We emphasize that the proposed convolution definition is different from the one studied in [16]. The definition uses the kernel of the classical fractional Fourier transform which is commutative with quaternion signals.

2. Basic Facts about Quaternion Algebra and Quaternion Fourier Transform

Quaternions are hypercomplex numbers, which can be written in the following form

$$\mathbb{H} = \{q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3;\ q_0, q_1, q_2, q_3 \in \mathbb{R}\}, \quad (1)$$

where $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is a basis of $\mathbb{H}$ and obeys the following multiplication rules:

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k},$$

$$\mathbf{jk} = -\mathbf{kj} = \mathbf{i},$$

$$\mathbf{ki} = -\mathbf{ik} = \mathbf{j},$$
$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1. \tag{2}$$

For a quaternion $q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 \in \mathbb{H}$, the conjugate $\overline{q}$ of the quaternion q is given by

$$\overline{q} = q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3. \tag{3}$$

and satisfies

$$\overline{qp} = \overline{p}\,\overline{q}. \tag{4}$$

From (3) we obtain the norm or modulus of $q \in \mathbb{H}$ defined as

$$|q| = \sqrt{q\overline{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}. \tag{5}$$

It is not difficult to see that

$$|qp| = |q|\,|p|, \quad \forall p, q \in \mathbb{H}. \tag{6}$$

Like in complex case, the inverse of $q \in \mathbb{H} \setminus \{0\}$ is given by

$$q^{-1} = \frac{\overline{q}}{|q|^2}. \tag{7}$$

Every quaternion-valued function $f : \mathbb{R}^2 \longrightarrow \mathbb{H}$ can be written as

$$f(\boldsymbol{x}) = f_0(\boldsymbol{x}) + \mathbf{i}f_1(\boldsymbol{x}) + \mathbf{j}f_2(\boldsymbol{x}) + \mathbf{k}f_3(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^2. \tag{8}$$

where f_0, f_1, f_2, and f_3 are real-valued functions. A quaternion module $L^p(\mathbb{R}^2; \mathbb{H})$ is then defined as

$$L^p(\mathbb{R}^2; \mathbb{H}) = \left\{ f \mid f : \mathbb{R}^2 \longrightarrow \mathbb{H}, \int_{\mathbb{R}^2} |f|^p \, d\boldsymbol{x} < \infty \right\}, \quad 1 \le p < \infty. \tag{9}$$

Definition 1. The QFT of $f \in L^1(\mathbb{R}^2; \mathbb{H})$ is the transform $\mathcal{F}_q\{f\} \in L^1(\mathbb{R}^2; \mathbb{H})$ given by the integral

$$\mathcal{F}_q\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1 x_1} f(\boldsymbol{x}) e^{-\mathbf{j}\omega_2 x_2} d\boldsymbol{x}, \quad d\boldsymbol{x} = dx_1 dx_2. \tag{10}$$

Here $\mathcal{F}_q$ is called the quaternion Fourier transform operator or the quaternion Fourier transform.

Definition 2. The inverse QFT of $g \in L^1(\mathbb{R}^2; \mathbb{H})$ is the transform $\mathcal{F}_q^{-1}\{g\} \in L^1(\mathbb{R}^2; \mathbb{H})$ given by the integral

$$\mathcal{F}_q^{-1}[g](\boldsymbol{x}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{\mathbf{i}\omega_1 x_1} g(\boldsymbol{\omega}) e^{\mathbf{j}\omega_2 x_2} d\boldsymbol{\omega}. \tag{11}$$

where $\boldsymbol{\omega} \in \mathbb{R}^2$ and $d\boldsymbol{\omega} = d\omega_1 d\omega_2$. Here $\mathcal{F}_q^{-1}$ stands for the inverse QFT operator.

3. Quaternion Linear Canonical Transform and Its Convolution Theorem

In this section we first introduce the two-dimensional quaternion linear canonical transform (QLCT). We then make a convolution definition in the QLCT domain and derive a convolution theorem related to the QLCT.

3.1. Definition of QLCT. Based on the definition of the two-sided quaternion Fourier transform (QFT) and its properties [17–22], we obtain a definition of the QLCT. We also can derive useful properties of the QLCT using fundamental relationship between the QFT and QLCT. Denote by $SL(2, \mathbb{R})$ the special linear group of degree 2 over $\mathbb{R}$, that is, the group of all real 2×2 matrices with determinant one. Let

$$A_s = (a_s, b_s, c_s, d_s) = \begin{pmatrix} a_s & b_s \\ c_s & d_s \end{pmatrix} \in SL(2, \mathbb{R}), \quad s = 1, 2. \tag{12}$$

When $b_1 b_2 \neq 0$, we define the kernel K_{A_s} of the QLCT by

$$\begin{aligned} K_{A_1}(x_1, \omega_1) &= \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}} e^{\mathbf{i}(1/2)((a_1/b_1)x_1^2 - (2/b_1)x_1\omega_1 + (d_1/b_1)\omega_1^2)} \\ K_{A_2}(x_2, \omega_2) &= \frac{1}{\sqrt{2\pi b_2 \mathbf{j}}} e^{\mathbf{j}(1/2)((a_2/b_2)x_2^2 - (2/b_2)x_2\omega_2 + (d_2/b_2)\omega_2^2)}. \end{aligned} \tag{13}$$

Observe that we can write the imaginary units above of the form

$$\sqrt{\mathbf{i}} = e^{\mathbf{i}(\pi/4)}, \quad \sqrt{\mathbf{j}} = e^{\mathbf{j}(\pi/4)}. \tag{14}$$

These facts yield

$$\frac{1}{\sqrt{2\pi b_1 \mathbf{i}}} = \frac{e^{-\mathbf{i}(\pi/4)}}{\sqrt{2\pi b_1}}, \quad \frac{1}{\sqrt{2\pi b_2 \mathbf{j}}} = \frac{e^{-\mathbf{j}(\pi/4)}}{\sqrt{2\pi b_2}}. \tag{15}$$

Definition 3 (QLCT definition). The QLCT of a quaternion signal $f \in L^1(\mathbb{R}^2; \mathbb{H})$ is defined by

$$\mathcal{F}_{A_1,A_2}^{\mathbb{H}}\{f\}(\boldsymbol{\omega}) = \begin{cases} \int_{\mathbb{R}^2} K_{A_1}(x_1, \omega_1) f(\boldsymbol{x}) K_{A_2}(x_2, \omega_2) \, d\boldsymbol{x}, & b_1 b_2 \neq 0 \\ \sqrt{d_1 d_2} e^{\mathbf{i}(c_1 d_1/2)\omega_1^2} f(d_1\omega_1, d_2\omega_2) e^{\mathbf{j}(c_2 d_2/2)\omega_2^2}, & b_1 b_2 = 0. \end{cases} \tag{16}$$

Because $e^{\mathbf{i}(c_1 d_1/2)\omega_1^2}$ and $e^{\mathbf{j}(c_2 d_2/2)\omega_2^2}$ are *chirp signals* in signal processing, then we always work for the case $b_1 b_2 \neq 0$.

The inverse transform of the QLCT above is then described by

$$f(\boldsymbol{x}) = \int_{\mathbb{R}^2} K_{A_1^{-1}}(\omega_1, x_1)\, \mathcal{F}^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})\, K_{A_2^{-1}}(\omega_2, x_2)\, d\boldsymbol{\omega}. \quad (17)$$

This form is equivalent to

$$f(\boldsymbol{x}) = \frac{1}{\sqrt{-2\pi b_1 \mathbf{i}}} \cdot \int_{\mathbb{R}^2} e^{-\mathbf{i}(1/2)((a_1/b_1)x_1^2-(2/b_1)x_1\omega_1+(d_1/b_1)\omega_1^2)} \mathcal{F}^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) \cdot \frac{1}{\sqrt{-2\pi b_2 \mathbf{j}}} \times e^{-\mathbf{j}(1/2)((a_2/b_2)x_2^2-(2/b_2)x_2\omega_2+(d_2/b_2)\omega_2^2)} d\boldsymbol{\omega}, \quad (18)$$

where $A_1^{-1} = (d_1, -b_1, -c_1, a_1)$ and $A_2^{-1} = (d_2, -b_2, -c_2, a_2)$.

It directly follows from (8) and (15) that

$$\begin{aligned}
\mathcal{F}^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) &= \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}} \cdot \int_{\mathbb{R}^2} e^{\mathbf{i}(1/2)((a_1/b_1)x_1^2-(2/b_1)x_1\omega_1+(d_1/b_1)\omega_1^2)} (f_0(\boldsymbol{x}) + \mathbf{i} f_1 + \mathbf{j} f_2(\boldsymbol{x}) + \mathbf{k} f_3(\boldsymbol{x})) \\
&\quad \times \frac{1}{\sqrt{2\pi b_2 \mathbf{j}}} e^{\mathbf{j}(1/2)((a_2/b_2)x_2^2-(2/b_2)x_2\omega_2+(d_2/b_2)\omega_2^2)} d\boldsymbol{x} \\
&= \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}} \cdot \int_{\mathbb{R}^2} e^{\mathbf{i}(1/2)((a_1/b_1)x_1^2-(2/b_1)x_1\omega_1+(d_1/b_1)\omega_1^2)} f_0(\boldsymbol{x}) \cdot \frac{1}{\sqrt{2\pi b_2 \mathbf{j}}} e^{\mathbf{j}(1/2)((a_2/b_2)x_2^2-(2/b_2)x_2\omega_2+(d_2/b_2)\omega_2^2)} d\boldsymbol{x} \\
&\quad + \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}} \cdot \int_{\mathbb{R}^2} e^{\mathbf{i}(1/2)((a_1/b_1)x_1^2-(2/b_1)x_1\omega_1+(d_1/b_1)\omega_1^2)} \mathbf{i} f_1(\boldsymbol{x}) \cdot \frac{1}{\sqrt{2\pi b_2 \mathbf{j}}} e^{\mathbf{j}(1/2)((a_2/b_2)x_2^2-(2/b_2)x_2\omega_2+(d_2/b_2)\omega_2^2)} d\boldsymbol{x} \\
&= \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}} \cdot \int_{\mathbb{R}^2} e^{\mathbf{i}(1/2)((a_1/b_1)x_1^2-(2/b_1)x_1\omega_1+(d_1/b_1)\omega_1^2)} f_2(\boldsymbol{x}) \mathbf{j} \cdot \frac{1}{\sqrt{2\pi b_2 \mathbf{j}}} e^{\mathbf{j}(1/2)((a_2/b_2)x_2^2-(2/b_2)x_2\omega_2+(d_2/b_2)\omega_2^2)} d\boldsymbol{x} \\
&= \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}} \cdot \int_{\mathbb{R}^2} e^{\mathbf{i}(1/2)((a_1/b_1)x_1^2-(2/b_1)x_1\omega_1+(d_1/b_1)\omega_1^2)} \mathbf{i} f_3(\boldsymbol{x}) \mathbf{j} \cdot \frac{1}{\sqrt{2\pi b_2 \mathbf{j}}} e^{\mathbf{j}(1/2)((a_2/b_2)x_2^2-(2/b_2)x_2\omega_2+(d_2/b_2)\omega_2^2)} d\boldsymbol{x} \\
&= \mathcal{F}^{\mathbb{H}}_{A_1,A_2}\{f_0\}(\boldsymbol{\omega}) + \mathbf{i}\mathcal{F}^{\mathbb{H}}_{A_1,A_2}\{f_1\}(\boldsymbol{\omega}) + \mathcal{F}^{\mathbb{H}}_{A_1,A_2}\{f_2\}(\boldsymbol{\omega})\mathbf{j} + \mathbf{i}\mathcal{F}^{\mathbb{H}}_{A_1,A_2}\{f_3\}(\boldsymbol{\omega})\mathbf{j}.
\end{aligned} \quad (19)$$

In the rest of the paper, we always assume that $\mathcal{F}^{\mathbb{H}}_{A_1,A_2}\{f_i\}$ for $i = 0, 1, 2, 3$ are real-valued function or $\mathcal{F}^{\mathbb{H}}_{A_1,A_2}\{f_i\} \in L^1(\mathbb{R}^2; \mathbb{R})$.

Theorem 4. *If $f(\boldsymbol{x}) \in L^1(\mathbb{R}^2; \mathbb{H})$, then $\mathcal{F}^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})$ is continuous on $\mathbb{R}^2$.*

Proof. See [9]. □

Similarly, one can obtain the following result.

Theorem 5. *If $\mathcal{F}^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) \in L^1(\mathbb{R}^2; \mathbb{H})$, then $f(\boldsymbol{x})$ is continuous on $\mathbb{R}^2$.*

3.2. Convolution Theorem for QLCT. In the following we first define the convolution for the QLCT. It is an extension of the convolution definition from the LCT (see [5, 6]) to the QLCT domain. We then investigate how the QLCT behaves under convolutions.

Definition 6. For any two quaternion functions $f, g \in L^1(\mathbb{R}^2; \mathbb{H})$, we define the convolution operator of the QLCT as

$$(f \star g)(\boldsymbol{x}) = \int_{\mathbb{R}^2} e^{\mathbf{i}(a_1/b_1)t_1(t_1-x_1)} f(\boldsymbol{t})\, g(\boldsymbol{x}-\boldsymbol{t})\, e^{\mathbf{j}(a_2/b_2)t_2(t_2-x_2)} d\boldsymbol{t}. \quad (20)$$

As a direct consequence, we get the convolution theorem associated with the QLCT, which is expressed as

Theorem 7. *Let $f, g \in L^1(\mathbb{R}^2; \mathbb{H})$ be two quaternion-valued functions. Then we have the QLCT of the convolution of f and g in the form*

$$\begin{aligned}
\mathcal{F}^{\mathbb{H}}_{A_1,A_2}\{f \star g\}(\boldsymbol{\omega}) &= \sqrt{2\pi b_1 \mathbf{i}}\, e^{-\mathbf{i}(d_1\omega_1^2/2b_1)} \Big(\Big(\mathcal{F}^{\mathbb{H}}_{A_1,A_2}\{f_0\}(\boldsymbol{\omega}) + \mathbf{i}\mathcal{F}^{\mathbb{H}}_{A_1,A_2}\{f_1\}(\boldsymbol{\omega})\Big)\Big(\mathcal{F}^{\mathbb{H}}_{A_1,A_2}\{g_0\}(\boldsymbol{\omega}) \\
&\quad + \mathbf{j}\mathcal{F}^{\mathbb{H}}_{A_1^*,A_2}\{g_2\}(\boldsymbol{\omega})\Big) + \Big(\mathcal{F}^{\mathbb{H}}_{A_1,A_2^*}\{f_0\}(\boldsymbol{\omega}) + \mathbf{i}\mathcal{F}^{\mathbb{H}}_{A_1,A_2^*}\{f_1\}(\boldsymbol{\omega})\Big)\Big(\mathbf{i}\mathcal{F}^{\mathbb{H}}_{A_1,A_2}\{g_1\}(\boldsymbol{\omega}) \\
&\quad + \mathbf{k}\mathcal{F}^{\mathbb{H}}_{A_1^*,A_2}\{g_3\}(\boldsymbol{\omega})\Big) + \Big(\mathbf{j}\mathcal{F}^{\mathbb{H}}_{A_1^*,A_2}\{f_2\}(\boldsymbol{\omega})
\end{aligned}$$

$$+\mathbf{k}\mathcal{F}^{\mathbb{H}}_{A_1^*,A_2}\{f_3\}(\boldsymbol{\omega})\Big)\Big(\mathcal{F}^{\mathbb{H}}_{A_1^*,A_2}\{g_0\}(\boldsymbol{\omega})$$
$$+\mathbf{j}\mathcal{F}^{\mathbb{H}}_{A_1,A_2}\{g_2\}(\boldsymbol{\omega})\Big)+\Big(\mathbf{j}\mathcal{F}^{\mathbb{H}}_{A_1^*,A_2^*}\{f_2\}(\boldsymbol{\omega})$$
$$+\mathbf{k}\mathcal{F}^{\mathbb{H}}_{A_1^*,A_2^*}\{f_3\}(\boldsymbol{\omega})\Big)\Big(\mathbf{i}\mathcal{F}^{\mathbb{H}}_{A_1^*,A_2}\{g_1\}(\boldsymbol{\omega})$$
$$+\mathbf{k}\mathcal{F}^{\mathbb{H}}_{A_1,A_2}\{g_3\}(\boldsymbol{\omega})\Big)\Big)e^{-\mathbf{j}(d_2\omega_2^2/2b_2)}\sqrt{2\pi b_2\mathbf{j}}, \tag{21}$$

where the matrix parameters

$$A_1^*=\begin{pmatrix} a_1 & -b_1 \\ c_1 & d_1\end{pmatrix} \quad \textit{and}\ A_2^*=\begin{pmatrix} a_2 & -b_2 \\ c_2 & d_2\end{pmatrix}. \tag{22}$$

Proof. It directly follows from (16) and (20) that

$$\begin{aligned}\mathcal{F}^{\mathbb{H}}_{A_1,A_2}\{f\star g\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_1\mathbf{i}}}e^{\mathbf{i}(1/2)((a_1/b_1)x_1^2-(2/b_1)x_1\omega_1+(d_1/b_1)\omega_1^2)}e^{\mathbf{i}(a_1/b_1)t_1(t_1-x_1)}f(\boldsymbol{t})\,g(\boldsymbol{x}-\boldsymbol{t})\\
&\times\frac{1}{\sqrt{2\pi b_2\mathbf{j}}}e^{\mathbf{j}(1/2)((a_2/b_2)x_2^2-(2/b_2)x_2\omega_2+(d_2/b_2)\omega_2^2)}e^{\mathbf{j}(a_2/b_2)t_2(t_2-x_2)}d\boldsymbol{t}d\boldsymbol{x}\\
&=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_1\mathbf{i}}}e^{\mathbf{i}(a_1/b_1)t_1(t_1-x_1)}e^{\mathbf{i}(a_1/2b_1)x_1^2}e^{-\mathbf{i}(2/2b_1)x_1\omega_1}e^{\mathbf{i}(d_1/2b_1)\omega_1^2}f(\boldsymbol{t})\,g(\boldsymbol{x}-\boldsymbol{t})\\
&\times\frac{1}{\sqrt{2\pi b_2\mathbf{j}}}e^{\mathbf{j}(a_2/b_2)t_2(t_2-x_2)}e^{\mathbf{j}(a_2/2b_2)x_2^2}e^{-\mathbf{j}(2/2b_2)x_2\omega_2}e^{\mathbf{j}(d_2/2b_2)\omega_2^2}d\boldsymbol{t}d\boldsymbol{x}.\end{aligned}\tag{23}$$

Letting $\boldsymbol{x}-\boldsymbol{t}=\boldsymbol{z}$, (23) can be rewritten as

$$\begin{aligned}\mathcal{F}^{\mathbb{H}}_{A_1,A_2}\{f\star g\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_1\mathbf{i}}}e^{-\mathbf{i}(a_1/b_1)t_1z_1}e^{\mathbf{i}(1/2)((a_1/b_1)(z_1+t_1)^2-(2/b_1)(z_1+t_1)\omega_1+(d_1/b_1)\omega_1^2)}\\
&\times\left(\{f_0(\boldsymbol{t})+\mathbf{i}f_1(\boldsymbol{t})\}+\mathbf{j}f_2(\boldsymbol{t})+\mathbf{k}f_3(\boldsymbol{t})\right)\left(\{g_0(\boldsymbol{z})+\mathbf{j}g_2(\boldsymbol{z})\}+\mathbf{i}g_1(\boldsymbol{z})+\mathbf{k}g_3(\boldsymbol{z})\right)\\
&\times\int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_2\mathbf{j}}}e^{-\mathbf{j}(a_2/b_2)t_2z_2}e^{\mathbf{j}(1/2)((a_2/b_2)(z_2+t_2)^2-(2/b_2)(z_2+t_2)\omega_2+(d_2/b_2)\omega_2^2)}d\boldsymbol{z}d\boldsymbol{t}\\
&=\int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_1\mathbf{i}}}e^{-\mathbf{i}(a_1/b_1)t_1z_1}e^{\mathbf{i}(1/2)((a_1/b_1)(z_1+t_1)^2-(2/b_1)(z_1+t_1)\omega_1+(d_1/b_1)\omega_1^2)}\times\left(f_0(\boldsymbol{t})+\mathbf{i}f_1(\boldsymbol{t})\right)\left(g_0(\boldsymbol{z})+\mathbf{j}g_2(\boldsymbol{z})\right)\\
&\times\int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_2\mathbf{j}}}e^{-\mathbf{j}(a_2/b_2)t_2z_2}e^{\mathbf{j}(1/2)((a_2/b_2)(z_2+t_2)^2-(2/b_2)(z_2+t_2)\omega_2+(d_2/b_2)\omega_2^2)}d\boldsymbol{z}d\boldsymbol{t}\\
&+\int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_1\mathbf{i}}}e^{-\mathbf{i}(a_1/b_1)t_1z_1}e^{\mathbf{i}(1/2)((a_1/b_1)(z_1+t_1)^2-(2/b_1)(z_1+t_1)\omega_1+(d_1/b_1)\omega_1^2)}\times\left(f_0(\boldsymbol{t})+\mathbf{i}f_1(\boldsymbol{t})\left(\mathbf{i}g_1(\boldsymbol{z})+\mathbf{k}g_3(\boldsymbol{z})\right)\right.\\
&\times\int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_2\mathbf{j}}}e^{-\mathbf{j}(a_2/b_2)t_2z_2}e^{\mathbf{j}(1/2)((a_2/b_2)(z_2+t_2)^2-(2/b_2)(z_2+t_2)\omega_2+(d_2/b_2)\omega_2^2)}d\boldsymbol{z}d\boldsymbol{t}\\
&+\int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_1\mathbf{i}}}e^{-\mathbf{i}(a_1/b_1)t_1z_1}e^{\mathbf{i}(1/2)((a_1/b_1)(z_1+t_1)^2-(2/b_1)(z_1+t_1)\omega_1+(d_1/b_1)\omega_1^2)}\times\left(\mathbf{j}f_2(\boldsymbol{t})+\mathbf{k}f_3(\boldsymbol{t})\right)\left(g_0(\boldsymbol{z})+\mathbf{j}g_2(\boldsymbol{z})\right)\\
&\times\int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_2\mathbf{j}}}e^{-\mathbf{j}(a_2/b_2)t_2z_2}e^{\mathbf{j}(1/2)((a_2/b_2)(z_2+t_2)^2-(2/b_2)(z_2+t_2)\omega_2+(d_2/b_2)\omega_2^2)}d\boldsymbol{z}d\boldsymbol{t}\\
&+\int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_1\mathbf{i}}}e^{-\mathbf{i}(a_1/b_1)t_1z_1}e^{\mathbf{i}(1/2)((a_1/b_1)(z_1+t_1)^2-(2/b_1)(z_1+t_1)\omega_1+(d_1/b_1)\omega_1^2)}\times\left(\mathbf{j}f_2(\boldsymbol{t})+\mathbf{k}f_3(\boldsymbol{t})\right)\left(\mathbf{i}g_1(\boldsymbol{z})+\mathbf{k}g_3(\boldsymbol{z})\right)\\
&\times\int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_2\mathbf{j}}}e^{-\mathbf{j}(a_2/b_2)t_2z_2}e^{\mathbf{j}(1/2)((a_2/b_2)(z_2+t_2)^2-(2/b_2)(z_2+t_2)\omega_2+(d_2/b_2)\omega_2^2)}d\boldsymbol{z}d\boldsymbol{t}.\end{aligned}\tag{24}$$

Simplifying this result we have

$$\begin{aligned}\mathscr{F}^{\mathbb{H}}_{A_1,A_2}\{f\star g\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_1\mathbf{i}}}e^{\mathbf{i}(a_1t_1^2/2b_1)}e^{-\mathbf{i}(t_1\omega_1/b_1)}e^{\mathbf{i}(d_1\omega_1^2/2b_1)}e^{\mathbf{i}(a_1z_1^2/2b_1)}e^{-\mathbf{i}(z_1\omega_1/b_1)}\left(f_0(\boldsymbol{t})+\mathbf{i}f_1(\boldsymbol{t})\right)\left(g_0(\boldsymbol{z})+\mathbf{j}g_2(\boldsymbol{z})\right)\\
&\times\int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_2\mathbf{j}}}e^{\mathbf{j}(a_2t_2^2/2b_2)}e^{-\mathbf{j}(t_2\omega_1/b_2)}e^{\mathbf{j}(d_2\omega_2^2/2b_2)}e^{\mathbf{j}(a_2z_2^2/2b_2)}e^{-\mathbf{j}(z_2\omega_2/b_2)}d\boldsymbol{z}d\boldsymbol{t}\\
&+\int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_1\mathbf{i}}}e^{\mathbf{i}(a_1t_1^2/2b_1)}e^{-\mathbf{i}(t_1\omega_1/b_1)}e^{\mathbf{i}(d_1\omega_1^2/2b_1)}e^{\mathbf{i}(a_1z_1^2/2b_1)}e^{-\mathbf{i}(z_1\omega_1/b_1)}(f_0(\boldsymbol{t})+\mathbf{i}f_1(\boldsymbol{t})\left(\mathbf{i}g_1(\boldsymbol{z})+\mathbf{k}g_3(\boldsymbol{z})\right)\\
&\times\int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_2\mathbf{j}}}e^{\mathbf{j}(a_2t_2^2/2b_2)}e^{-\mathbf{j}(t_2\omega_1/b_2)}e^{\mathbf{j}(d_2\omega_2^2/2b_2)}e^{\mathbf{j}(a_2z_2^2/2b_2)}e^{-\mathbf{j}(z_2\omega_2/b_2)}d\boldsymbol{z}d\boldsymbol{t}\\
&+\int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_1\mathbf{i}}}e^{\mathbf{i}(a_1t_1^2/2b_1)}e^{-\mathbf{i}(t_1\omega_1/b_1)}e^{\mathbf{i}(d_1\omega_1^2/2b_1)}e^{\mathbf{i}(a_1z_1^2/2b_1)}e^{-\mathbf{i}(z_1\omega_1/b_1)}\left(\mathbf{j}f_2(\boldsymbol{t})+\mathbf{k}f_3(\boldsymbol{t})\right)\left(g_0(\boldsymbol{z})+\mathbf{j}g_2(\boldsymbol{z})\right)\\
&\times\int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_2\mathbf{j}}}e^{\mathbf{j}(a_2t_2^2/2b_2)}e^{-\mathbf{j}(t_2\omega_1/b_2)}e^{\mathbf{j}(d_2\omega_2^2/2b_2)}e^{\mathbf{j}(a_2z_2^2/2b_2)}e^{-\mathbf{j}(z_2\omega_2/b_2)}d\boldsymbol{z}d\boldsymbol{t}\\
&+\int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_1\mathbf{i}}}e^{\mathbf{i}(a_1t_1^2/2b_1)}e^{-\mathbf{i}(t_1\omega_1/b_1)}e^{\mathbf{i}(d_1\omega_1^2/2b_1)}e^{\mathbf{i}(a_1z_1^2/2b_1)}e^{-\mathbf{i}(z_1\omega_1/b_1)}\left(\mathbf{j}f_2(\boldsymbol{t})+\mathbf{k}f_3(\boldsymbol{t})\right)\left(\mathbf{i}g_1(\boldsymbol{z})+\mathbf{k}g_3(\boldsymbol{z})\right)\\
&\times\int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_2\mathbf{j}}}e^{\mathbf{j}(a_2t_2^2/2b_2)}e^{-\mathbf{j}(t_2\omega_1/b_2)}e^{\mathbf{j}(d_2\omega_2^2/2b_2)}e^{\mathbf{j}(a_2z_2^2/2b_2)}e^{-\mathbf{j}(z_2\omega_2/b_2)}d\boldsymbol{z}d\boldsymbol{t}.\end{aligned}\tag{25}$$

Applying properties of kernel function of the QLCT we obtain

$$\begin{aligned}\mathscr{F}^{\mathbb{H}}_{A_1,A_2}\{f\star g\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^2}e^{\mathbf{i}(a_1t_1^2/2b_1)}e^{-\mathbf{i}(t_1\omega_1/b_1)}\left(f_0(\boldsymbol{t})+\mathbf{i}f_1(\boldsymbol{t})\right)\frac{1}{\sqrt{2\pi b_1\mathbf{i}}}e^{\mathbf{i}(a_1z_1^2/2b_1)}e^{-\mathbf{i}(z_1\omega_1/b_1)}e^{\mathbf{i}(d_1\omega_1^2/2b_1)}\left(g_0(\boldsymbol{z})+\mathbf{j}g_2(\boldsymbol{z})\right)\\
&\times\int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_2\mathbf{j}}}e^{\mathbf{j}(a_2z_2^2/2b_2)}e^{-\mathbf{j}(z_2\omega_2/b_2)}e^{\mathbf{j}(d_2\omega_2^2/2b_2)}e^{\mathbf{j}(a_2t_2^2/2b_2)}e^{-\mathbf{j}(t_2\omega_1/b_2)}d\boldsymbol{z}d\boldsymbol{t}\\
&+\int_{\mathbb{R}^2}e^{\mathbf{i}(a_1t_1^2/2b_1)}e^{-\mathbf{i}(t_1\omega_1/b_1)}\left(f_0(\boldsymbol{t})+\mathbf{i}f_1(\boldsymbol{t})\right)\frac{1}{\sqrt{2\pi b_1\mathbf{i}}}e^{\mathbf{i}(a_1z_1^2/2b_1)}e^{-\mathbf{i}(z_1\omega_1/b_1)}e^{\mathbf{i}(d_1\omega_1^2/2b_1)}\left(\mathbf{i}g_1(\boldsymbol{z})+\mathbf{k}g_3(\boldsymbol{z})\right)\\
&\times\int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_2\mathbf{j}}}e^{\mathbf{j}(a_2z_2^2/2b_2)}e^{-\mathbf{j}(z_2\omega_2/b_2)}e^{\mathbf{j}(d_2\omega_2^2/2b_2)}e^{\mathbf{j}(a_2t_2^2/2b_2)}e^{-\mathbf{j}(t_2\omega_1/b_2)}d\boldsymbol{z}d\boldsymbol{t}\\
&+\int_{\mathbb{R}^2}e^{\mathbf{i}(a_1t_1^2/2b_1)}e^{-\mathbf{i}(t_1\omega_1/b_1)}\left(\mathbf{j}f_2(\boldsymbol{t})+\mathbf{k}f_3(\boldsymbol{t})\right)\frac{1}{\sqrt{2\pi b_1\mathbf{i}}}e^{-\mathbf{i}(a_1z_1^2/2b_1)}e^{\mathbf{i}(z_1\omega_1/b_1)}e^{-\mathbf{i}(d_1\omega_1^2/2b_1)}\left(g_0(\boldsymbol{z})+\mathbf{j}g_2(\boldsymbol{z})\right)\\
&\times\int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_2\mathbf{j}}}e^{\mathbf{j}(a_2z_2^2/2b_2)}e^{-\mathbf{j}(z_2\omega_2/b_2)}e^{\mathbf{j}(d_2\omega_2^2/2b_2)}e^{\mathbf{j}(a_2t_2^2/2b_2)}e^{-\mathbf{j}(t_2\omega_1/b_2)}d\boldsymbol{z}d\boldsymbol{t}\\
&+\int_{\mathbb{R}^2}e^{\mathbf{i}(a_1t_1^2/2b_1)}e^{-\mathbf{i}(t_1\omega_1/b_1)}\left(\mathbf{j}f_2(\boldsymbol{t})+\mathbf{k}f_3(\boldsymbol{t})\right)\frac{1}{\sqrt{2\pi b_1\mathbf{i}}}e^{-\mathbf{i}(a_1z_1^2/2b_1)}e^{\mathbf{i}(z_1\omega_1/b_1)}e^{-\mathbf{i}(d_1\omega_1^2/2b_1)}\left(\mathbf{i}g_1(\boldsymbol{z})+\mathbf{k}g_3(\boldsymbol{z})\right)\\
&\times\int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_2\mathbf{j}}}e^{\mathbf{j}(a_2z_2^2/2b_2)}e^{-\mathbf{j}(z_2\omega_2/b_2)}e^{\mathbf{j}(d_2\omega_2^2/2b_2)}e^{\mathbf{j}(a_2t_2^2/2b_2)}e^{-\mathbf{j}(t_2\omega_1/b_2)}d\boldsymbol{z}d\boldsymbol{t}.\end{aligned}\tag{26}$$

Now multiplying both sides of the above equation by $(1/\sqrt{2\pi b_1 \mathbf{i}})e^{\mathbf{i}(d_1\omega_1^2/2b_1)}$ and $(1/\sqrt{2\pi b_2 \mathbf{j}})e^{\mathbf{j}(d_2\omega_2^2/2b_2)}$ gives

$$
\begin{aligned}
&\frac{1}{\sqrt{2\pi b_1 \mathbf{i}}} e^{\mathbf{i}(d_1\omega_1^2/2b_1)} \mathscr{F}^{\mathbb{H}}_{A_1,A_2}\{f \star g\}(\boldsymbol{\omega})\, e^{\mathbf{j}(d_2\omega_2^2/2b_2)} \\
&\quad \cdot \frac{1}{\sqrt{2\pi b_2 \mathbf{j}}} \\
&\quad = \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}} e^{\mathbf{i}(a_1 t_1^2/2b_1)} e^{-\mathbf{i}(t_1\omega_1/b_1)} e^{\mathbf{i}(d_1\omega_1^2/2b_1)} \left(f_0(\boldsymbol{t})\right. \\
&\quad \left.+ \mathbf{i} f_1(\boldsymbol{t})\right) \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}} e^{\mathbf{i}(a_1 z_1^2/2b_1)} e^{-\mathbf{i}(z_1\omega_1/b_1)} e^{\mathbf{i}(d_1\omega_1^2/2b_1)} \\
&\quad \times \int_{\mathbb{R}^2} \left(g_0(\boldsymbol{z}) + \mathbf{j} g_2(\boldsymbol{z})\right) \\
&\quad \cdot \frac{1}{\sqrt{2\pi b_2 \mathbf{j}}} e^{\mathbf{j}(a_2 z_2^2/2b_2)} e^{-\mathbf{j}(z_2\omega_2/b_2)} e^{\mathbf{j}(d_2\omega_2^2/2b_2)} \\
&\quad \cdot \frac{1}{\sqrt{2\pi b_2 \mathbf{j}}} e^{\mathbf{j}(a_2 t_2^2/2b_2)} e^{-\mathbf{j}(t_2\omega_1/b_2)} e^{\mathbf{j}(d_2\omega_2^2/2b_2)} \boldsymbol{dzdt} \\
&\quad + \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}} e^{\mathbf{i}(a_1 t_1^2/2b_1)} e^{-\mathbf{i}(t_1\omega_1/b_1)} e^{\mathbf{i}(d_1\omega_1^2/2b_1)} \left(f_0(\boldsymbol{t})\right. \\
&\quad \left.+ \mathbf{i} f_1(\boldsymbol{t})\right) \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}} e^{\mathbf{i}(a_1 z_1^2/2b_1)} e^{-\mathbf{i}(z_1\omega_1/b_1)} e^{\mathbf{i}(d_1\omega_1^2/2b_1)} \\
&\quad \times \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_2 \mathbf{j}}} \left(\mathbf{i} g_1(\boldsymbol{z}) + \mathbf{k} g_3(\boldsymbol{z})\right) \\
&\quad \cdot e^{\mathbf{j}(a_2 z_2^2/2b_2)} e^{-\mathbf{j}(z_2\omega_2/b_2)} e^{\mathbf{j}(d_2\omega_2^2/2b_2)} \\
&\quad \cdot \frac{1}{\sqrt{2\pi b_2 \mathbf{j}}} e^{\mathbf{j}(a_2 t_2^2/2b_2)} e^{-\mathbf{j}(t_2\omega_1/b_2)} e^{\mathbf{j}(d_2\omega_2^2/2b_2)} \boldsymbol{dzdt} \\
&\quad + \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}} e^{\mathbf{i}(a_1 t_1^2/2b_1)} e^{-\mathbf{i}(t_1\omega_1/b_1)} e^{\mathbf{i}(d_1\omega_1^2/2b_1)} \left(\mathbf{j} f_2(\boldsymbol{t})\right. \\
&\quad \left.+ \mathbf{k} f_3(\boldsymbol{t})\right) \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}} e^{-\mathbf{i}(a_1 z_1^2/2b_1)} e^{\mathbf{i}(z_1\omega_1/b_1)} e^{-\mathbf{i}(d_1\omega_1^2/2b_1)} \\
&\quad \times \int_{\mathbb{R}^2} \left(g_0(\boldsymbol{z}) + \mathbf{j} g_2(\boldsymbol{z})\right) \\
&\quad \cdot \frac{1}{\sqrt{2\pi b_2 \mathbf{j}}} e^{\mathbf{j}(a_2 z_2^2/2b_2)} e^{-\mathbf{j}(z_2\omega_2/b_2)} e^{\mathbf{j}(d_2\omega_2^2/2b_2)} \\
&\quad \cdot \frac{1}{\sqrt{2\pi b_2 \mathbf{j}}} e^{\mathbf{j}(a_2 t_2^2/2b_2)} e^{-\mathbf{j}(t_2\omega_1/b_2)} e^{\mathbf{j}(d_2\omega_2^2/2b_2)} \boldsymbol{dzdt} \\
&\quad + \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}} e^{\mathbf{i}(a_1 t_1^2/2b_1)} e^{-\mathbf{i}(t_1\omega_1/b_1)} e^{\mathbf{i}(d_1\omega_1^2/2b_1)} \left(\mathbf{j} f_2(\boldsymbol{t})\right. \\
&\quad \left.+ \mathbf{k} f_3(\boldsymbol{t})\right) \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}} e^{-\mathbf{i}(a_1 z_1^2/2b_1)} e^{\mathbf{i}(z_1\omega_1/b_1)} e^{-\mathbf{i}(d_1\omega_1^2/2b_1)} \\
&\quad \times \int_{\mathbb{R}^2} \left(\mathbf{i} g_1(\boldsymbol{z}) + \mathbf{k} g_3(\boldsymbol{z})\right) \\
&\quad \cdot \frac{1}{\sqrt{2\pi b_2 \mathbf{j}}} e^{\mathbf{j}(a_2 z_2^2/2b_2)} e^{-\mathbf{j}(z_2\omega_2/b_2)} e^{\mathbf{j}(d_2\omega_2^2/2b_2)} \\
&\quad \cdot \frac{1}{\sqrt{2\pi b_2 \mathbf{j}}} e^{\mathbf{j}(a_2 t_2^2/2b_2)} e^{-\mathbf{j}(t_2\omega_1/b_2)} e^{\mathbf{j}(d_2\omega_2^2/2b_2)} \boldsymbol{dzdt} \\
&\quad = \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}} e^{\mathbf{i}(a_1 t_1^2/2b_1)} e^{-\mathbf{i}(t_1\omega_1/b_1)} e^{\mathbf{i}(d_1\omega_1^2/2b_1)} \left(f_0(\boldsymbol{t})\right. \\
&\quad \left.+ \mathbf{i} f_1(\boldsymbol{t})\right) \times \left(\mathscr{F}^{\mathbb{H}}_{A_1,A_2}\{g_0\}(\boldsymbol{\omega}) + \mathbf{j}\mathscr{F}^{\mathbb{H}}_{A_1^*,A_2}\{g_2\}(\boldsymbol{\omega})\right) \\
&\quad \cdot \frac{1}{\sqrt{2\pi b_2 \mathbf{j}}} e^{\mathbf{j}(a_2 t_2^2/2b_2)} e^{-\mathbf{j}(t_2\omega_1/b_2)} e^{\mathbf{j}(d_2\omega_2^2/2b_2)} \boldsymbol{dzdt} \\
&\quad + \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}} e^{\mathbf{i}(a_1 t_1^2/2b_1)} e^{-\mathbf{i}(t_1\omega_1/b_1)} e^{\mathbf{i}(d_1\omega_1^2/2b_1)} \left(f_0(\boldsymbol{t})\right. \\
&\quad \left.+ \mathbf{i} f_1(\boldsymbol{t})\right) \times \left(\mathbf{i}\mathscr{F}^{\mathbb{H}}_{A_1,A_2}\{g_1\}(\boldsymbol{\omega}) + \mathbf{k}\mathscr{F}^{\mathbb{H}}_{A_1^*,A_2}\{g_3\}(\boldsymbol{\omega})\right) \\
&\quad \cdot \frac{1}{\sqrt{2\pi b_2 \mathbf{j}}} e^{\mathbf{j}(a_2 t_2^2/2b_2)} e^{-\mathbf{j}(t_2\omega_1/b_2)} e^{\mathbf{j}(d_2\omega_2^2/2b_2)} \boldsymbol{dzdt} \\
&\quad + \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}} e^{\mathbf{i}(a_1 t_1^2/2b_1)} e^{-\mathbf{i}(t_1\omega_1/b_1)} e^{\mathbf{i}(d_1\omega_1^2/2b_1)} \left(\mathbf{j} f_2(\boldsymbol{t})\right. \\
&\quad \left.+ \mathbf{k} f_3(\boldsymbol{t})\right) \\
&\quad \times \int_{\mathbb{R}^2} \left(\mathscr{F}^{\mathbb{H}}_{A_1^*,A_2}\{g_0\}(\boldsymbol{\omega}) + \mathbf{j}\mathscr{F}^{\mathbb{H}}_{A_1,A_2}\{g_2\}(\boldsymbol{\omega})\right) \\
&\quad \cdot \frac{1}{\sqrt{2\pi b_2 \mathbf{j}}} e^{\mathbf{j}(a_2 t_2^2/2b_2)} e^{-\mathbf{j}(t_2\omega_1/b_2)} e^{\mathbf{j}(d_2\omega_2^2/2b_2)} \boldsymbol{dzdt} \\
&\quad + \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}} e^{\mathbf{i}(a_1 t_1^2/2b_1)} e^{-\mathbf{i}(t_1\omega_1/b_1)} e^{\mathbf{i}(d_1\omega_1^2/2b_1)} \left(\mathbf{j} f_2(\boldsymbol{t})\right. \\
&\quad \left.+ \mathbf{k} f_3(\boldsymbol{t})\right) \times \left(\mathbf{i}\mathscr{F}^{\mathbb{H}}_{A_1^*,A_2}\{g_1\}(\boldsymbol{\omega}) + \mathbf{k}\mathscr{F}^{\mathbb{H}}_{A_1,A_2}\{g_3\}\right. \\
&\quad \left.\cdot(\boldsymbol{\omega})\right) \frac{1}{\sqrt{2\pi b_2 \mathbf{j}}} e^{\mathbf{j}(a_2 t_2^2/2b_2)} e^{-\mathbf{j}(t_2\omega_1/b_2)} e^{\mathbf{j}(d_2\omega_2^2/2b_2)} \boldsymbol{dzdt}.
\end{aligned} \tag{27}
$$

Based on Definition 3, the required result follows. □

Remark 8. If we use the matrix parameters $A_1 = A_2 = \left(\begin{smallmatrix} a & 1 \\ -1 & 0 \end{smallmatrix}\right)$ with $a \neq 0$. Then (21) will lead to

$$
\begin{aligned}
&\mathscr{F}^{\mathbb{H}}_{A_1,A_2}\{f \star g\}(\boldsymbol{\omega}) \\
&\quad = \sqrt{2\pi \mathbf{i}}\left(\left(\mathscr{F}^{\mathbb{H}}_{A_1,A_2}\{f_0\}(\boldsymbol{\omega}) + \mathbf{i}\mathscr{F}^{\mathbb{H}}_{A_1,A_2}\{f_1\}(\boldsymbol{\omega})\right)\right. \\
&\quad \cdot \left(\mathscr{F}^{\mathbb{H}}_{A_1,A_2}\{g_0\}(\boldsymbol{\omega}) + \mathbf{j}\mathscr{F}^{\mathbb{H}}_{A_1^*,A_2}\{g_2\}(\boldsymbol{\omega})\right) \\
&\quad + \left(\mathscr{F}^{\mathbb{H}}_{A_1,A_2^*}\{f_0\}(\boldsymbol{\omega}) + \mathbf{i}\mathscr{F}^{\mathbb{H}}_{A_1,A_2^*}\{f_1\}(\boldsymbol{\omega})\right) \\
&\quad \cdot \left(\mathbf{i}\mathscr{F}^{\mathbb{H}}_{A_1,A_2}\{g_1\}(\boldsymbol{\omega}) + \mathbf{k}\mathscr{F}^{\mathbb{H}}_{A_1^*,A_2}\{g_3\}(\boldsymbol{\omega})\right) \\
&\quad + \left(\mathbf{j}\mathscr{F}^{\mathbb{H}}_{A_1^*,A_2}\{f_2\}(\boldsymbol{\omega}) + \mathbf{k}\mathscr{F}^{\mathbb{H}}_{A_1^*,A_2}\{f_3\}(\boldsymbol{\omega})\right) \\
&\quad \cdot \left(\mathscr{F}^{\mathbb{H}}_{A_1^*,A_2}\{g_0\}(\boldsymbol{\omega}) + \mathbf{j}\mathscr{F}^{\mathbb{H}}_{A_1,A_2}\{g_2\}(\boldsymbol{\omega})\right)
\end{aligned}
$$

$$+\left(\mathbf{j}\mathscr{F}^{\mathbb{H}}_{A_1^*,A_2^*}\{f_2\}(\boldsymbol{\omega})+\mathbf{k}\mathscr{F}^{\mathbb{H}}_{A_1^*,A_2^*}\{f_3\}(\boldsymbol{\omega})\right)\cdot\left(\mathbf{i}\mathscr{F}^{\mathbb{H}}_{A_1^*,A_2}\{g_1\}(\boldsymbol{\omega})+\mathbf{k}\mathscr{F}^{\mathbb{H}}_{A_1,A_2}\{g_3\}(\boldsymbol{\omega})\right)\Big)\sqrt{2\pi\mathbf{j}}. \tag{28}$$

Observe that the above form is quite similar to the convolution theorem associated with the QFT [15]. Now we investigate some consequences of Theorem 7, which are given in the following results.

Lemma 9. *Let $f,g\in L^1(\mathbb{R}^2;\mathbb{H})$ be two quaternion-valued functions. If we assume that $\mathscr{F}^{\mathbb{H}}_{A_1,A_2}\{g\}\in L^1(\mathbb{R}^2;\mathbb{R})$, then we have*

$$\mathscr{F}^{\mathbb{H}}_{A_1,A_2}\{f\star g\}(\boldsymbol{\omega})=\sqrt{2\pi b_1\mathbf{i}}e^{-\mathbf{i}(d_1\omega_1^2/2b_1)}\Big(\left(\mathscr{F}^{\mathbb{H}}_{A_1,A_2}\{f_0\}(\boldsymbol{\omega})+\mathbf{i}\mathscr{F}^{\mathbb{H}}_{A_1,A_2}\{f_1\}(\boldsymbol{\omega})\right)\mathscr{F}^{\mathbb{H}}_{A_1,A_2}\{g\}(\boldsymbol{\omega})+\left(\mathbf{j}\mathscr{F}^{\mathbb{H}}_{A_1^*,A_2}\{f_2\}(\boldsymbol{\omega})+\mathbf{k}\mathscr{F}^{\mathbb{H}}_{A_1^*,A_2}\{f_3\}(\boldsymbol{\omega})\right)\cdot\mathscr{F}^{\mathbb{H}}_{A_1^*,A_2}\{g\}(\boldsymbol{\omega})\Big)\sqrt{2\pi b_2\mathbf{j}}e^{-\mathbf{j}(d_2\omega_2^2/2b_2)}. \tag{29}$$

Moreover, if $\mathscr{F}^{\mathbb{H}}_{A_1,A_2}\{f\}\in L^1(\mathbb{R}^2;\mathbb{R})$, then we have

$$\mathscr{F}^{\mathbb{H}}_{A_1,A_2}\{f\star g\}(\boldsymbol{\omega})=\sqrt{2\pi b_1\mathbf{i}}e^{-\mathbf{i}(d_1\omega_1^2/2b_1)}\Big(\mathscr{F}^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})\left(\mathscr{F}^{\mathbb{H}}_{A_1,A_2}\{g_0\}(\boldsymbol{\omega})+\mathbf{j}\mathscr{F}^{\mathbb{H}}_{A_1^*,A_2}\{g_2\}(\boldsymbol{\omega})\right)+\mathscr{F}^{\mathbb{H}}_{A_1^*,A_2}\{f\}(\boldsymbol{\omega})\cdot\left(\mathbf{i}\mathscr{F}^{\mathbb{H}}_{A_1,A_2}\{g_1\}(\boldsymbol{\omega})+\mathbf{k}\mathscr{F}^{\mathbb{H}}_{A_1^*,A_2}\{g_3\}(\boldsymbol{\omega})\right)\Big)\cdot\sqrt{2\pi b_2\mathbf{j}}e^{-\mathbf{j}(d_2\omega_2^2/2b_2)}. \tag{30}$$

Proof. We only verify the identity (29) and the others are quite similar. Direct computations yield

$$\begin{aligned}\mathscr{F}^{\mathbb{H}}_{A_1,A_2}\{f\star g\}(\boldsymbol{\omega})&=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_1\mathbf{i}}}e^{\mathbf{i}(1/2)((a_1/b_1)x_1^2-(2/b_1)x_1\omega_1+(d_1/b_1)\omega_1^2)}e^{\mathbf{i}(a_1/b_1)t_1(t_1-x_1)}f(\boldsymbol{t})\,g(\boldsymbol{x}-\boldsymbol{t})\\&\quad\times\frac{1}{\sqrt{2\pi b_2\mathbf{j}}}e^{\mathbf{j}(1/2)((a_2/b_2)x_2^2-(2/b_2)x_2\omega_2+(d_2/b_2)\omega_2^2)}e^{\mathbf{j}(a_2/b_2)t_2(t_2-x_2)}d\boldsymbol{t}\,d\boldsymbol{x}\\&=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_1\mathbf{i}}}e^{\mathbf{i}(a_1/b_1)t_1(t_1-x_1)}e^{\mathbf{i}(a_1/2b_1)x_1^2}e^{-\mathbf{i}(2/2b_1)x_1\omega_1}e^{\mathbf{i}(d_1/2b_1)\omega_1^2}f(\boldsymbol{t})\,g(\boldsymbol{x}-\boldsymbol{t})\\&\quad\times\frac{1}{\sqrt{2\pi b_2\mathbf{j}}}e^{\mathbf{j}(a_2/b_2)t_2(t_2-x_2)}e^{\mathbf{j}(a_2/2b_2)x_2^2}e^{-\mathbf{j}(2/2b_2)x_2\omega_2}e^{\mathbf{j}(d_2/2b_2)\omega_2^2}d\boldsymbol{t}\,d\boldsymbol{x}.\end{aligned} \tag{31}$$

By making the change of variable $\boldsymbol{x}-\boldsymbol{t}=\boldsymbol{v}$ of the above, we get

$$\begin{aligned}\mathscr{F}^{\mathbb{H}}_{A_1,A_2}\{f\star g\}(\boldsymbol{\omega})&=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_1\mathbf{i}}}e^{-\mathbf{i}(a_1/b_1)t_1v_1}e^{\mathbf{i}(1/2)(a_1/b_1)(v_1+t_1)^2}e^{-\mathbf{i}((v_1+t_1)\omega_1/b_1)}e^{\mathbf{i}(d_1\omega_1^2/2b_1)}f(\boldsymbol{t})\,g(\boldsymbol{v})\\&\quad\times\frac{1}{\sqrt{2\pi b_2\mathbf{j}}}e^{-\mathbf{j}(a_2/b_2)t_2v_2}e^{\mathbf{j}(1/2)(a_2/b_2)(v_2+t_2)^2}e^{-\mathbf{j}((v_2+t_2)\omega_2/b_2)}e^{\mathbf{j}(d_2\omega_2^2/2b_2)}d\boldsymbol{t}\,d\boldsymbol{v}\\&=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_1\mathbf{i}}}e^{\mathbf{i}(a_1t_1^2/2b_1)}e^{-\mathbf{i}(t_1\omega_1/b_1)}e^{\mathbf{i}(d_1\omega_1^2/2b_1)}e^{\mathbf{i}(a_1v_1^2/2b_1)}e^{-\mathbf{i}(v_1\omega_1/b_1)}\times\left(\{f_0(\boldsymbol{t})+\mathbf{i}f_1(\boldsymbol{t})\}+\mathbf{j}f_2(\boldsymbol{t})+\mathbf{k}f_3(\boldsymbol{t})\right)g(\boldsymbol{v})\\&\quad\times\frac{1}{\sqrt{2\pi b_j\mathbf{j}}}e^{\mathbf{j}(a_2t_2^2/2b_2)}e^{-\mathbf{j}(t_2\omega_2/b_2)}e^{\mathbf{j}(d_2\omega_2^2/2b_2)}e^{\mathbf{j}(a_2v_2^2/2b_2)}e^{-\mathbf{j}(v_2\omega_2/b_2)}d\boldsymbol{t}\,d\boldsymbol{v}\\&=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_1\mathbf{i}}}e^{\mathbf{i}(a_1t_1^2/2b_1)}e^{-\mathbf{i}(t_1\omega_1/b_1)}e^{\mathbf{i}(d_1\omega_1^2/2b_1)}e^{\mathbf{i}(a_1v_1^2/2b_1)}e^{-\mathbf{i}(v_1\omega_1/b_1)}\left(f_0(\boldsymbol{t})+\mathbf{i}f_1(\boldsymbol{t})\right)g(\boldsymbol{v})\\&\quad\times\frac{1}{\sqrt{2\pi b_2\mathbf{j}}}e^{\mathbf{j}(a_2t_2^2/2b_2)}e^{-\mathbf{j}(t_2\omega_2/b_2)}e^{\mathbf{j}(d_2\omega_2^2/2b_2)}e^{\mathbf{j}(a_2v_2^2/2b_2)}e^{-\mathbf{j}(v_2\omega_2/b_2)}d\boldsymbol{t}\,d\boldsymbol{v}\end{aligned}$$

$$+\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_1\mathbf{i}}}e^{\mathbf{i}(a_1t_1^2/2b_1)}e^{-\mathbf{i}(t_1\omega_1/b_1)}e^{\mathbf{i}(d_1\omega_1^2/2b_1)}e^{\mathbf{i}(a_1v_1^2/2b_1)}e^{-\mathbf{i}(v_1\omega_1/b_1)}\left(\mathbf{j}f_2(\boldsymbol{t})+\mathbf{k}f_3(\boldsymbol{t})\right)g(\boldsymbol{v})$$
$$\times\frac{1}{\sqrt{2\pi b_2\mathbf{j}}}e^{\mathbf{j}(a_2t_2^2/2b_2)}e^{-\mathbf{j}(t_2\omega_2/b_2)}e^{\mathbf{j}(d_2\omega_2^2/2b_2)}e^{\mathbf{j}(a_2v_2^2/2b_2)}e^{-\mathbf{j}(v_2\omega_2/b_2)}d\boldsymbol{t}d\boldsymbol{v}. \tag{32}$$

Multiplying this result by $(1/\sqrt{2\pi b_1\mathbf{i}})e^{\mathbf{i}(d_1\omega_1^2/2b_1)}$ and $(1/\sqrt{2\pi b_2\mathbf{j}})e^{\mathbf{j}(d_2\omega_2^2/2b_2)}$, we easily obtain

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi b_1\mathbf{i}}}e^{\mathbf{i}(d_1\omega_1^2/2b_1)}\mathcal{F}^{\mathbb{H}}_{A_1,A_2}\{f\star g\}(\boldsymbol{\omega})\frac{1}{\sqrt{2\pi b_2\mathbf{j}}}e^{\mathbf{j}(d_2\omega_2^2/2b_2)}\\
&=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{2\pi b_1\mathbf{i}}e^{\mathbf{i}(a_1t_1^2/2b_1)}e^{-\mathbf{i}(t_1\omega_1/b_1)}e^{\mathbf{i}(d_1\omega_1^2/2b_1)}e^{\mathbf{i}(a_1v_1^2/2b_1)}e^{-\mathbf{i}(v_1\omega_1/b_1)}e^{\mathbf{i}(d_1\omega_1^2/2b_1)}\left(f_0(\boldsymbol{t})+\mathbf{i}f_1(\boldsymbol{t})\right)g(\boldsymbol{v})\\
&\times e^{\mathbf{j}(a_2t_2^2/2b_2)}e^{-\mathbf{j}(t_2\omega_2/b_2)}e^{\mathbf{j}(d_2\omega_2^2/2b_2)}e^{\mathbf{j}(a_2v_2^2/2b_2)}e^{-\mathbf{j}(v_2\omega_2/b_2)}\frac{1}{2\pi b_2\mathbf{j}}e^{\mathbf{j}(d_2\omega_2^2/2b_2)}d\boldsymbol{t}d\boldsymbol{v}\\
&+\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{2\pi b_1\mathbf{i}}e^{\mathbf{i}(a_1t_1^2/2b_1)}e^{-\mathbf{i}(t_1\omega_1/b_1)}e^{\mathbf{i}(d_1\omega_1^2/2b_1)}e^{\mathbf{i}(a_1v_1^2/2b_1)}e^{-\mathbf{i}(v_1\omega_1/b_1)}e^{\mathbf{i}(d_1\omega_1^2/2b_1)}\\
&\times\left(\mathbf{j}f_2(\boldsymbol{t})+\mathbf{k}f_3(\boldsymbol{t})\right)g(\boldsymbol{v})\frac{1}{2\pi b_2\mathbf{j}}e^{\mathbf{j}(a_2t_2^2/2b_2)}e^{-\mathbf{j}(t_2\omega_2/b_2)}e^{\mathbf{j}(d_2\omega_2^2/2b_2)}e^{\mathbf{j}(a_2v_2^2/2b_2)}e^{-\mathbf{j}(v_2\omega_2/b_2)}e^{\mathbf{j}(d_2\omega_2^2/2b_2)}d\boldsymbol{t}d\boldsymbol{v}\\
&=\int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_1\mathbf{i}}}e^{\mathbf{i}(a_1t_1^2/2b_1)}e^{-\mathbf{i}(t_1\omega_1/b_1)}e^{\mathbf{i}(d_1\omega_1^2/2b_1)}\left(f_0(\boldsymbol{t})+\mathbf{i}f_1(\boldsymbol{t})\right)\mathcal{F}^{\mathbb{H}}_{A_1,A_2}\{g\}(\boldsymbol{\omega})\times\frac{1}{\sqrt{2\pi b_2\mathbf{j}}}e^{\mathbf{j}(a_2t_2^2/2b_2)}e^{-\mathbf{j}(t_2\omega_2/b_2)}e^{\mathbf{j}(d_2\omega_2^2/2b_2)}d\boldsymbol{t}\\
&+\int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_1\mathbf{i}}}e^{\mathbf{i}(a_1t_1^2/2b_1)}e^{-\mathbf{i}(t_1\omega_1/b_1)}e^{\mathbf{i}(d_1\omega_1^2/2b_1)}\left(\mathbf{j}f_2(\boldsymbol{t})+\mathbf{k}f_3(\boldsymbol{t})\right)\mathcal{F}^{\mathbb{H}}_{A_1^*,A_2}\{g\}(\boldsymbol{\omega})\\
&\times\frac{1}{\sqrt{2\pi b_2\mathbf{j}}}e^{\mathbf{j}(a_2t_2^2/2b_2)}e^{-\mathbf{j}(t_2\omega_2/b_2)}e^{\mathbf{j}(d_2\omega_2^2/2b_2)}d\boldsymbol{t}.
\end{aligned}\tag{33}$$

The definition of the QLCT (16) finally gives

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi b_1\mathbf{i}}}e^{\mathbf{i}(d_1\omega_1^2/2b_1)}\mathcal{F}^{\mathbb{H}}_{A_1,A_2}\{f\star g\}(\boldsymbol{\omega})\frac{1}{\sqrt{2\pi b_1\mathbf{j}}}\\
&\cdot e^{\mathbf{j}(d_2\omega_2^2/2b_2)}\\
&=\left(\mathcal{F}^{\mathbb{H}}_{A_1,A_2}\{f_0\}(\boldsymbol{\omega})+\mathbf{i}\mathcal{F}^{\mathbb{H}}_{A_1,A_2}\{f_1\}(\boldsymbol{\omega})\right)\\
&\cdot\mathcal{F}^{\mathbb{H}}_{A_1,A_2}\{g\}(\boldsymbol{\omega})\\
&+\left(\mathbf{j}\mathcal{F}^{\mathbb{H}}_{A_1^*,A_2}\{f_2\}(\boldsymbol{\omega})+\mathbf{k}\mathcal{F}^{\mathbb{H}}_{A_1^*,A_2}\{f_3\}(\boldsymbol{\omega})\right)\\
&\cdot\mathcal{F}^{\mathbb{H}}_{A_1^*,A_2}\{g\}(\boldsymbol{\omega}).
\end{aligned}\tag{34}$$

This finishes the proof of the lemma. □

Lemma 10. *Suppose that $f,g\in L^1(\mathbb{R}^2;\mathbb{H})$ has the form*

$$\begin{aligned}
f(\boldsymbol{x})&=f_0(\boldsymbol{x})+\mathbf{i}f_1(\boldsymbol{x})+\mathbf{j}f_2(\boldsymbol{x})+\mathbf{k}f_3(\boldsymbol{x}),\\
g(\boldsymbol{x})&=g_0(\boldsymbol{x})+\mathbf{j}g_2(\boldsymbol{x}).
\end{aligned}\tag{35}$$

If $\mathcal{F}^{\mathbb{H}}_{A_1^*,A_2}\{f\}\in L^1(\mathbb{R}^2;\mathbb{R})$ is satisfied, then we have

$$\begin{aligned}
\mathcal{F}^{\mathbb{H}}_{A_1,A_2}\{f\star g\}(\boldsymbol{\omega})&=\sqrt{2\pi b_1\mathbf{i}}e^{-\mathbf{i}(d_1\omega_1^2/2b_1)}\mathcal{F}^{\mathbb{H}}_{A_1,A_2}\{f\}\\
&\cdot(\boldsymbol{\omega})\mathcal{F}^{\mathbb{H}}_{A_1,A_2}\{g\}(\boldsymbol{\omega})\sqrt{2\pi b_2\mathbf{j}}e^{-\mathbf{j}(d_2\omega_2^2/2b_2)}\\
&=\sqrt{2\pi b_1\mathbf{i}}e^{-\mathbf{i}(d_1\omega_1^2/2b_1)}\mathcal{F}^{\mathbb{H}}_{A_1,A_2}\{g\}(\boldsymbol{\omega})\mathcal{F}^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})\\
&\cdot\sqrt{2\pi b_2\mathbf{j}}e^{-\mathbf{j}(d_2\omega_2^2/2b_2)}.
\end{aligned}\tag{36}$$

Remark 11. It should be noticed if $f,g\in L^1(\mathbb{R}^2;\mathbb{H})$ takes the form

$$\begin{aligned}
f(\boldsymbol{x})&=\mathbf{j}f_2(\boldsymbol{x})+\mathbf{k}f_3(\boldsymbol{x}),\\
g(\boldsymbol{x})&=g_0(\boldsymbol{x})+\mathbf{i}g_1(\boldsymbol{x})+\mathbf{j}g_2(\boldsymbol{x})+\mathbf{k}g_3(\boldsymbol{x}),
\end{aligned}\tag{37}$$

then it holds

$$\begin{aligned}
\mathcal{F}^{\mathbb{H}}_{A_1,A_2}\{f\star g\}(\boldsymbol{\omega})&=\sqrt{2\pi b_1\mathbf{i}}e^{-\mathbf{i}(d_1\omega_1^2/2b_1)}\mathcal{F}^{\mathbb{H}}_{A_1,A_2}\{f\}\\
&\cdot(\boldsymbol{\omega})\mathcal{F}^{\mathbb{H}}_{A_1^*,A_2}\{g\}(\boldsymbol{\omega})\sqrt{2\pi b_2\mathbf{j}}e^{-\mathbf{j}(d_2\omega_2^2/2b_2)}
\end{aligned}$$

$$= \sqrt{2\pi b_1 \mathbf{i}} e^{-\mathbf{i}(d_1 \omega_1^2/2b_1)} \mathscr{F}^{\mathbb{H}}_{A_1^*,A_2} \{g\} (\boldsymbol{\omega}) \mathscr{F}^{\mathbb{H}}_{A_1,A_2} \{f\} (\boldsymbol{\omega}) \cdot \sqrt{2\pi b_2 \mathbf{j}} e^{-\mathbf{j}(d_2 \omega_2^2/2b_2)}, \quad (38)$$

where $\mathscr{F}^{\mathbb{H}}_{A_1,A_2}\{f\}, \mathscr{F}^{\mathbb{H}}_{A_1,A_2}\{g\} \in L^1(\mathbb{R}^2;\mathbb{R})$.

Data Availability

All data generated or analyzed during this study are included within the article.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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