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Research Article

Results on Uniqueness of Solution of Nonhomogeneous Impulsive Retarded Equation Using the Generalized Ordinary Differential Equation

D. K. Igobi and U. Abasiekwere

Department of Mathematics and Statistics, University of Uyo, P.M.B. 1017, Nigeria

Correspondence should be addressed to D. K. Igobi; dodiigobi@gmail.com

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In this work, we consider an initial value problem of a nonhomogeneous retarded functional equation coupled with the impulsive term. The fundamental matrix theorem is employed to derive the integral equivalent of the equation which is Lebesgue integrable. The integral equivalent equation with impulses satisfying the Carathéodory and Lipschitz conditions is embedded in the space of generalized ordinary differential equations (GODEs), and the correspondence between the generalized ordinary differential equation and the nonhomogeneous retarded equation coupled with impulsive term is established by the construction of a local flow by means of a topological dynamic satisfying certain technical conditions. The uniqueness of the equation solution is proved. The results obtained follow the primitive Riemann concept of integration from a simple understanding.

1. Introduction

The dynamic of an evolving system is most often subjected to abrupt changes such as shocks, harvesting, and natural disasters. When the effects of these abrupt changes are trivial the classical differential equation is most suitable for the modeling of the system. But for short-term perturbation that acts in the form of impulses, the impulsive delay differential equation becomes handy. An impulsive retarded differential equation is a delay equation coupled with a difference equation known as the impulsive term. Among the earliest research work on impulsive differential equation was the article by Milman and Myshkis [1]. Thereafter, growing research interest in the qualitative analysis of the properties of the impulsive retarded equation increases, as seen in the works of Igobi and Ndiyo [2], Isaac and Lipcsey [3], Benchohra and Ntouyas [4], Federson and Schwabik [5], Federson and Taboas [6], Argawal and Saker [7], and Ballinger [8].

The introduction of the generalized ordinary differential equation in the Banach space function by Kurzweil [9] has become a valuable mathematical tool for the investigation of the qualitative properties of continuous and discrete systems from common sense. The topological dynamic of the Kurzweil equation considers the limit point of the translate $f_t \longrightarrow f(x, t + s)$ under the assumptions that the limiting equation satisfying the Lipschitz and Carathéodory conditions is not an ordinary differential equation, and the space of the ordinary equation is not complete. But if the ordinary differential equation is embedded in the Kurzweil equations we obtained a complete and compact space, such that the techniques of the topological translate can be applied.

A more relaxed Kurzweil condition was presented in the article by Artstein [10]. He considered the metric topology characterized by the convergence

$$f_k \longrightarrow f_0 \quad if \quad \int_0^t f_k(x,\tau)\, d\tau \longrightarrow \int_0^t f_o(x,\tau)\, d\tau, \qquad (1)$$

with the following properties:

(i) $V \subset W$ is a compact set; then there exists a locally Lebesgue integrable function $M_V(s)$ such that

$$\left| f(x,s) \right| \leq M_V(s), \quad x \in V, \qquad (2)$$

and $\int_0^{s+h} M_V(\tau)d\tau$ is uniformly continuous in s. That is, there exists $\mu_V(\varepsilon) > 0$ such that $\int_0^{s+h} M_V(\tau)d\tau < \varepsilon h \le \mu_V(\varepsilon)$

(ii) $V \subset W$ is a compact set; then there exists a locally Lebesgue integrable function $K_A(s)$ such that

$$\begin{aligned} &\left|f(x,s) - f(y,s)\right| \le K_V(s)\left|x - y\right|, \quad x, y \in V \\ &\textit{and} \int_0^s K_V(\tau)\, d\tau \le N_V \end{aligned} \tag{3}$$

for all s and N_A fixed

The metric convergence of f_k fulfilling (i) and (ii) for some N_V and μ_V guarantees the continuity of $(t, f) \longrightarrow f_t(t, x)$ and the precompactness of the function space, but not completeness. That is, by equation (1), the Cauchy sequence f_i implies $\int_0^s f(x,\tau)d\tau$ converges for all (x, s). However,

$$A(s)x = \lim \int_0^s f_i(x,\tau)\, d\tau \tag{4}$$

does not have the integral representation

$$A(s)x = \int_0^s f(x,\tau)\, d\tau. \tag{5}$$

In summary, the Kurzweil equation addresses functions whose limit exists but are nowhere differentiable, but, by using the primitive definition of Riemann integral, a correspondence is established.

Consider an ordinary equation

$$\dot{x} = f(x,t), \tag{6}$$

with integral equivalent

$$x(t) - x(t_0) = \int_{t_0}^t f(x,\tau)\, d\tau. \tag{7}$$

Suppose the integral is a Riemann integral, we can define a δ−fine partition

$t_0 = s_1 \le s_2 \le \cdots \le s_{n+1} = t$ on $[t_0, t]$ and a $\eta_i \in [s_{i-1}, s_i]$, $i = 1, 2, 3, \ldots$. The differential approximation of Equation (7) in the Riemann sense is

$$\sum_{i=1}^n \int_{s_i}^{s_{i+1}} f\left(x(\tau_i)\,\delta_i\right) d\delta. \tag{8}$$

If we defined

$$A(s)x(t) = \int_{t_0}^t f(x,\delta)\, d\delta, \tag{9}$$

So, by Equation (8), we have

$$\sum_{i=1}^t \left[A(s_{i+1}) - A(s_i)\right] x(\eta_i), = S(dA, x, P). \tag{10}$$

Equation (10) defined the Riemann-Kurzweil sum approximation of $x(t) - x(t_0)$ if and only if $x(t)$ is a solution of (7) and $[s_i, s_{i+1}]$ is a fine partition. Thus, for any $I \in R^n$,

$$I = \int_{t_0}^t dA(s)\, x(\eta) \tag{11}$$

defines the Kurzweil integral if there exists an $\varepsilon > 0$ such that

$$|I - S(dA, x, P)| \le \varepsilon \tag{12}$$

The differential equation resulting from the primitive Kurzweil integral (11) is what is known as the generalized ordinary differential equation.

The correspondence between the generalized ordinary differential equation and other types of differential system is well established in the following articles: Federson and Taboas [6], Federson and Schwabik [5], Imaz and Vorel [11], Oliva and Vorel [12], and Schwabik [13]. This was made possible by embedding the ordinary differential equation in the space of the generalized ordinary differential equation and constructing a local flow by means of a topological dynamic satisfying certain technical conditions.

In this work, we consider an initial value problem of a nonhomogeneous retarded functional equation of the form

$$\begin{aligned} \dot{x}(t) &= B_0 x(t) + B_1 x_t + u(t) \\ x_{t_0} &= \varphi, \end{aligned} \tag{13}$$

coupled with impulses

$$\Delta x(t_k) = I_k\left(x(t_k)\right), \quad k = 1, 2, \ldots m, \tag{14}$$

where $B_{0_{n\times n}}, B_{1_{n\times 1}}$ are constant matrices, $x \in G([t_0, t_0 + \delta], X)$, $u(t) : [t_0, t_0 + \delta] \longrightarrow X$, $x_t : [-r, 0] \longrightarrow X$ expresses the history of x on $[t - r, t]$by $x_t(\phi) = x(t + \phi)$, $\phi \in [-r, 0]$

$\varphi \in G([-r, 0], X)$ is the initial term, and $t_k, k = 1, 2, \ldots$ with $t_0 < t_1 < \cdots < t_k \cdots < t_m < t_0 + \delta$ are the impulses time.

We will employ the fundamental matrix theorem to derive the integral equivalent of Equation (13) and define Lebesgue integrable functions $L(\varphi(t)) : G([t_0 - r, t_0 - \delta], X) \longrightarrow X$ and $f(t) : [t_0, t_0 + \delta] \longrightarrow X$ for $\delta > 0, r \ge 0$ satisfying the conditions

$$\begin{aligned} &\text{(A)} \int_{-r}^0 K_0(t - s - r)\,\varphi(s)\, ds \le \int_{t_0}^t L(\varphi, s)\, ds \\ &\text{(B)} \int_0^t K_0(t - s - r)\, u(s)\, ds \le \int_{t_0}^t f(s)\, ds. \end{aligned} \tag{15}$$

We embed the integral equivalent equation with impulses satisfying conditions (A) and (B) in the space of generalized ordinary differential equations (GODEs), and using similar argument as presented by Federson and Taboas [6] and Federson and Schwabik [5] to show the relationship between the solutions of the generalized ordinary differential equation and the equivalent impulsive retarded differential equation, and establish the uniqueness of the equation solution.

2. Generalized Ordinary Differential Equation

Let X be a Banach space and $L(X)$ a Banach space of bounded linear operators on X, with $\|.\|_X$ and $\|.\|_{L(X)}$ defining the topological norms in X and $L(X)$, respectively. A partition is any finite set $U = \{s_0, s_1, \dots, s_{i+1}\}$ such that $a = s_0 < s_1 < \cdots < s_{i+1} = b$. Given any finite step function $A(t) : [a,b] \longrightarrow L(X)$, for $A(t)$ being a constant on (s_{i-1}, s_i), then $\mathrm{var}_a^b A(t) = \sup\{\sum_{i=1}^{n(P)} \|A(s_i) - A(s_{i-1})\|_X\}$ is the variation of $A(t)$ on $[a,b]$. The function $A(t)$ is of bounded variation on $[a,b]$ if $\mathrm{var}_a^b A(t) < \infty$.

The function $A(t) : [a,b] \longrightarrow L(X)$ is regulated on $[a,b]$ if the one-sided limits $A(t-) = \lim_{s\longrightarrow t-} A(s)$ and $A(t+) = \lim_{s\longrightarrow t+} A(s)$ exist at every point of $t \in [a,b]$. That is $A(a-) = A(a)$ and $A(b+) = A(b)$ such that $\Delta^- A(t) = A(t) - A(t-)$ and $\Delta^+ A(t) = A(t+) - A(t)$ for all $t \in [a,b]$. By $G([a,b], L(X))$ we denote the set of all regulated functions $A(t) : [a,b] \longrightarrow L(X)$, which is a Banach space when endowed with the usual supremum norm $\|A\|_\infty = \sup\{\|A(t)\|_X, t \in [a,b]\}$

A tagged division of a compact interval $U[a,b] \subset R$ is a finite collection of point-interval pairs $P = (\eta, U)$, where $U = \{s_0, s_1, \dots, s_{i+1}\}$ and $\eta_i \in [s_{i-1}, s_i]$ (that is $(\eta_i, [s_{i-1}, s_i])$). A gauge on $[a,b]$ is any positive function $\delta : [a,b] \longrightarrow (0,\infty)$. A tagged division $P(\eta_i[s_{i-1}, s_i])$ is $\delta-$*fine* if for every $i = 1, 2, \dots$. $[s_{i-1}, s_i] \subset (\eta_i - \delta(\eta_i), \eta_i + \delta(\eta_i))$.

Definition 1. Let $A : [a,b] \longrightarrow L(X)$ be a given function. A Kurzweil integral over the interval $[a,b]$ exists if there is a unique element $I \in X$ such that, for every $\varepsilon > 0$ and a gauge δ on $[a,b]$, we have

$$\|I - S(dA, x, P)\| < \varepsilon \tag{16}$$

satisfied for all δ $-$ *fine* partition P of $[a,b]$, where $S(dA, x, P) = \sum_{i=1}^{n(p)} [A(s_i) - A(s_{i-1})]x(\eta_i)$, $P = (\eta_i, [s_{i-1}, s_i])$, and $I = \int_a^b d[A(s)]x(\eta)$ is the Kurzweil integral. If the Kurzweil integral exists over $[a,b]$, then $\int_a^b d[A(s)]x(\eta) = -\int_b^a d[A(s)]x(\eta)$. In the Jaroslav Kurzweil sense $d[A(s)]x(\eta)$ is not defined; only $\int_a^b d[A(s)]x(\eta)$ might exist.

The Kurzweil integral is related to the Riemann integral when the space X is the set of real numbers such that $A : [a,b] \longrightarrow R$ and the Riemann sum is defined as

$$S(dA, x, P) = \sum_{i=1}^{n(p)} A(\eta_i)(s_i - s_{i-1}), \tag{17}$$

for all $P = [a,b]$. The properties of the Kurzweil integral such as the linearity, additivity, and convergence with respect to the nearby interval have been extensively discussed in Artstein [10], Schwabik [13], and Federson and Schwabik [5].

We state here some of the fundamental results of the Kurzweil integral on a subinterval as proved in Kurzweil [9] and Artstein [10] which are the basic concepts to be employed in this work.

Lemma 2. *Let $A : [a,b] \longrightarrow L(X)$ be continuous in s for each η. If $\int_a^b d[A(s)]x(\eta)$ exists then for each $s_1 < s_2$ the integral $\int_{s_1}^{s_2} d[A(s)]x(\eta)$ exists, and $\int_a^t d[A(s)]x(\eta)$ is continuous in $t \in [a,b]$.*

Proposition 3. *Let $A : [a,b] \longrightarrow L(X)$, and $s_1 < t \in [a,b]$ such that $|t - s_1| < \delta(s_1)$; then*

(i) $$\left\| \int_{s_1}^t d[A(s)]x(\eta) - [A(t) - A(s_1)]x(\eta) \right\| < 2\varepsilon \tag{18}$$

(ii) The continuity of $A(t)$ in t implies that $\int_{s_1}^t d[A(s)]x$ converges to zero as $s_1 \longrightarrow t$.

The consequent of Lemma 2 is the result by Schwabik [13] stated as Lemma 4.

Lemma 4. *Let $A : [a,b] \longrightarrow L(X)$ be a given function such that $A(s)$ is integrable over $[a,s]$ for $s \in [a,b)$ and let the limit*

$$\lim_{s\longrightarrow b} \left[\int_a^s d[A(s)]x(\eta) - [A(s) - A(b)]x(\eta) \right] = I \in X \text{ exist.} \tag{19}$$

Then the function $A(s)$ is integrable over $[a,b]$ and $\int_a^b dA(s)x(\eta) = I$.

Similarly, if $A(s)$ is integrable over $[s,b]$ for $s \in (a,b]$, let the limit

$$\lim_{s\longrightarrow a} \left[\int_s^b d[A(s)]x(\eta) - [A(a) - A(s)]x(\eta) \right] = I \in X \text{ exist.} \tag{20}$$

Then the function $A(s)$ is integrable over $[a,b]$ and $\int_a^b d[A(s)]x(\eta) = I$.

The result of Lemma 4 is a follow-up of Lemma (A.2.) in Artstein [7]

Lemma 5. *If $x(\eta)$ is piecewise continuous in $[a,b]$ then $G(x(\eta), t) = \int_a^b d[A(s)]x(\eta)$ exists, where $G(x(\eta), t) \in G([t_0, t_0 + \delta], X)$ is a regulated function.*

Definition 6. Let $A : [a,b] \longrightarrow L(X)$, $g : [a,b] \longrightarrow X$ and $x_0 \in X$; the linear nonhomogeneous generalized ordinary differential equation is of the form

$$\frac{dx}{dt} = d[A(t)x(t) + g(t)]$$
$$x(t_0) = x_0 \tag{21}$$

Definition 7. The linear nonhomogeneous generalized integral solution of (21) is of the form

$$x(t) = x_0 + \int_{t_0}^t d[A(t)]x + g(t) - g(t_0), \quad t \in [a,b], \tag{22}$$

if the Kurzweil integral $\int_{t_0}^{t} d[A(t)]x$ exists and $x(t_0) = x_0$ satisfies Equation (22) for each $t \in [a,b]$. The literature on Equation (21) abounds in Schwabik [13], Schwabik, Tvrdy, and Vejvoda [14], and Artstein [10].

3. Preliminary Results

In this section, we present results that are fundamental to the establishment of the main results in Section 4.

Definition 8. A matrix $K_{n\times n}(t)$ is a fundamental matrix of the system (13) if it satisfies the matrix equation

$$\frac{d}{dt}K(t) = B_0K(t) + B_1K(t-r), \quad t>0,$$
$$K(t) = 0 \quad \text{for } t<0, \; K(0) = I \tag{23}$$

Definition 9. The solution of Equation (23) with an identity initial condition $K(0) = I$ has a recurrent form

$$K_{i+1}(t) = e^{B_0(t-ir)}K_i(ir) + \int_{ir}^{t} e^{B_0(t-s)}B_1K_i(s-r)\,ds, \tag{24}$$

where $K_i(t)$ is defined in the interval $(i-1)r \le t \le ir$, $i = 0,1,\ldots$

Bastinec and Piddubna [15] used the recurrent form of (24) to define the fundamental matrix solution of Equation (23) as presented in Lemma 10 and Definition 11.

Lemma 10. *The fundamental matrix solution of Equation (23) with an identity initial condition $K(0) = I$ has the form*

$$K_0(t) = \begin{cases} 0 & -\infty \le t \le -r \\ I & -r \le t \le 0 \\ e^{B_0t} + u_1(t) & 0 \le t \le r \\ e^{B_0t} + e^{B_0(t-r)}u(r) + u_1(t) & r \le t \le 2r \\ e^{B_0(t-mr)}u_m(mr) + u_i(t) & (i-1)r \le t \le ir \end{cases} \tag{25}$$

where

$$u_p(t) = \sum_{\sum i_j=1}^{p}\prod_{i=p}^{1}\left(\sum_{k=0}^{\infty}B_0^{k_i}B_1^{ij}\right)\frac{t-(p-1)^{k(p)}}{k(p)!}$$
$$\cdot \prod_{s=p-1}^{1}\frac{\tau^{(1-i_{s+1})k(s)}}{(1-i_{s+1})k(s)!}$$
$$k(\nu) = k_\nu + i_\nu(1+k_{\nu-1}+i_{\nu-1}(1+\cdots i_2(1+k_1+i_1)\cdots)),$$
$$i_p = 1,\; i_j = \{0,1\} \tag{26}$$

Definition 11. The integral solution of system (13) satisfying the given initial condition is

$$x(t) = K_0(t)\varphi(-r) + \int_{-r}^{t_0} K_0(t-s-r)\varphi(s)\,ds + \int_{t_0}^{t} K_0(t-s-r)u(s)\,ds + \sum_{k=1}^{n} I_k(x(t_k)) \tag{27}$$

where the integral exists in the Lebesgue sense (Bastinec and Piddubna, [15]).

Definition 12. Let $L : G([t_0-r,t_0+\delta],L(X))\times[t_0,t_0+\delta] \longrightarrow X$ and $f(t) : [t_0,t_0+\delta] \longrightarrow X$ be Lebesgue integrable functions satisfying conditions (A) and (B). Also assume $t \longrightarrow L(x_t,t)$ is Kurzweil integrable function; then the integral solution (27) has the form

$$x(t) = \varphi + \int_{t_0}^{t} L(x_s,s)\,ds + \int_{t_0}^{t} f(s)\,ds + \sum_{k=1}^{n} I_k(x(t_k)). \tag{28}$$

Remark 13. One of the fundamental theories of piecewise continuous functions with respect to delay differential equation is that if $x \in PC([t_0-r,t_0+\delta],X)$ is piecewise continuous, then x_t may be discontinuous at some or all $t \in [t_0,t_0+\delta]$. This result was proved in Hale [16].

Lemma 14 (Hale, [16]). *Assume $x \in PC([t_0-r,t_0+\delta],X)$, and let $g(t) = \|x_t\|_r$ for all $t \in [t_0,t_0+\delta]$. Then $g(t) \in PC([t_0-r,t_0+\delta],R_+)$ and the only possible points of discontinuity of g are t^* or t^*+r, where t^* denotes a point of discontinuity.*

In consequence of Lemma 14 and the pioneering work of Imaz and Vorel [11] and Oliva and Vorel [12], for each $x \in G([t_0-r,t_0+\delta],X)$, and $t \longrightarrow L(x_t,t)$ being Kurzweil integrable on $[t_0,t_0+\delta]$, we define the functions $F(x,t) : G([t_0-r,t_0+\delta],X) \longrightarrow C([t_0-r,t_0+\delta],X)$ and $g(t) \in C([t_0-r,t_0+\delta],X)$ such that

(i) $F(x,t)(\nu)$

$$= \begin{cases} 0, & t_0-r \le \nu \le t_0 \\ \int_{t_0}^{\nu} L(x_s,s)\,ds & t_0 \le \nu \le t \le t_0+\delta \\ \int_{t_0}^{t} L(x_s,s)\,ds & t_0 \le t \le \nu \le t_0+\delta \end{cases} \tag{29}$$

(ii) $$g(t)(\nu) = \begin{cases} 0, & t_0-r \le \nu \le t_0 \\ \int_{t_0}^{\nu} f(s)\,ds & t_0 \le \nu \le t \le t_0+\delta \\ \int_{t_0}^{t} f(s)\,ds & t_0 \le t \le \nu \le t_0+\delta \end{cases} \tag{30}$$

Similarly, we define a unit step function concentrated at $t_k^c \in [t_0, \infty)$ as

$$\int_k^c (t) = \begin{cases} 0, & t_0 < t < t_k^c \\ 1, & t \geq t_k^c, \end{cases} \quad (31)$$

so that, given $\nu \in [t_0, t_0+\delta]$ and $x \in G([t_0 - r, t_0+\delta], X)$), the impulsive term in Equation (28) is defined as

$$\text{(iii) } J(x,t)(\nu) = \sum_{k=1}^{n} \int_k^c (t) \int_k^c (\nu) I_k(x(t_k)). \quad (32)$$

Remark 15. Let $L : G([t_0 - r, t_0+\delta], L(X)) \times [t_0, t_0+\delta] \longrightarrow X$ and $f(t) : [t_0, t_0+\delta] \longrightarrow X$ such that $t \longrightarrow L(t, x_t)$ and $t \longrightarrow f(t)$ is Kurzweil integrable. Then we make the following Carathéodory and Lipschitz assumptions on the integral of the function (unlike the usual imposition of the conditions on the functions):

(A_1) there exists a Kurzweil integrable function $M_0 : [t_0, t_0+\delta] \longrightarrow R$, such that

$$\left\| \int_{s_0}^{s_1} L(s, x_s)\, ds \right\| \leq \int_{s_0}^{s_1} M_0 ds, \qquad \text{for } s_1, s_2 \in [t_0, t_0+\delta],\ x \in G([t_0, t_0+\delta], R^n), \quad (33)$$

(A_2) there exists a Kurzweil integrable function $M_1 : [t_0, t_0+\delta] \longrightarrow R$, such that

$$\left\| \int_{s_0}^{s_1} L(x_s - y_s, s)\, ds \right\| \leq \int_{s_0}^{s_1} M_1(s) \|x_s - y_s\|\, ds, \qquad \text{for } s_1, s_2 \in [t_0, t_0+\delta],\ x, y \in G([t_0, t_0+\delta], R^n), \quad (34)$$

(A_3) there exists a real constant $M > 0$ such that $\| \int_{s_0}^{s_1} f(s) ds \| \leq \int_{s_0}^{s_1} M ds, s_1, s_2 \in [t_0, t_0+\delta]$,

(A_4) there exist positives constants K_1, K_2 such that for $k = 1, 2, \ldots, n$ and all $x, y \in R^n$

$$|I_k(x)| \leq K_1 |x| \quad \text{and} \quad |I_k(x) - I_k(y)| \leq K_2 |x - y|. \quad (35)$$

Proposition 16 (Federson and Taboas, [6]). *Equations (29), (30), and (32) satisfying assumptions (A_1 – A_4) are continuous on $C([t_0 - r, t_0+\delta], X)$*

Proposition 17 (Schwabik, [13]). *If $A : ([t_0, t_0+\delta], L(X)) \longrightarrow L(X)$ and $g \in G([t_0, t_0+\delta], X), \nu \in [t_0, t_0+\delta]$, such that $x : [t_0, t_0+\delta] \longrightarrow X$ is a solution of*

$$x = x_0 + \int_\nu^t d[A(s)]\, x(s) + g(t) - g(\nu), \qquad t \in [t_0, t_0+\delta], \quad (36)$$

then

$$\begin{aligned} x(t) &= \lim_{r \longrightarrow t^+} x(r) = [I + \Delta^+ A(t)]\, x(t) + \Delta^+ g(t) \quad \text{and} \\ x(t) &= \lim_{r \longrightarrow t^-} x(r) = [I + \Delta^- A(t)]\, x(t) + \Delta^- g(t) \end{aligned} \quad (37)$$

where I is an identity operator on X.

Proof. Let $t \in [t_0, t_0+\delta]$; then

$$\begin{aligned} x(t^-) &= \lim_{r \longrightarrow t^-} x(r) \\ &= x_0 + \lim_{r \longrightarrow t^-} \int_\nu^r d[A(s)]\, x(s) + \lim_{r \longrightarrow t^{-1}} g(r) - g(\nu) \\ &= x_0 + \int_\nu^t d[A(s)]\, x(s) - \lim_{r \longrightarrow t^-} \int_t^r d[A(s)]\, x(s) + \lim_{r \longrightarrow t^{-1}} g(r) - g(t) + g(t) - g(\nu) \\ &= x(t) - \lim_{r \longrightarrow t^-} \int_t^r d[A(s)]\, x(s) + \lim_{r \longrightarrow t^{-1}} g(r) - g(t) \\ &= x(t) - [A(t^-) - A(t)]\, x(t) + g(t^-) - g(t) \\ &= [I - \Delta^- A(t)]\, x(t) + \Delta^- g(t) \end{aligned} \quad (38)$$

□

4. Main Results

Consider for each $t \in [t_0, t_0+\delta], A(t) \in G([t_0 - r, t_0+\delta], L(X))$ given by $A(t)y = F(y,t) + J(y,t)$, such that the generalized nonhomogeneous linear ordinary differential Equation (21) holds. Then the integral equation

$$y = y_0(t_0) + \int_{t_0}^{\nu} dF(y,t) + \int_{t_0}^{\nu} dJ(y,t) + g(t) - g(t_0) \quad (39)$$

satisfying the initial condition

$$y(t_0)(\nu) = \begin{cases} \varphi(\nu - t_0), & \nu \in [t_0 - r, t_0] \\ \varphi(t_0)(t_0), & \nu \in [t_0, t_0+\delta] \end{cases} \quad (40)$$

is the solution of the generalized ordinary differential equation

$$\begin{aligned} \frac{dy}{dt} &= d[A(t) + g(t)], \\ y(t_0) &= y_0. \end{aligned} \quad (41)$$

The relationship between Equations (39) and (28) is established as in the articles Federson and Schwabik [5] and Federson and Taboas [6], though the technical manipulation of the solution in this work satisfies the Carathéodory and Lipschitz conditions in Remark 15. Sequel to this, we state a very useful assumption as stated and proved in Federson and Schwabik [5]

Lemma 18. *Assume y is a solution of Equation (39) satisfying the initial condition (40), then for all $\tau \in [t_0, t_0+\delta]$ we have*

$$y(t)(\tau) = y(\tau)(\tau), \quad \tau \in [t_0 - r, t_0 + \delta] \tag{42}$$

and

$$y(t)(\tau) = y(t)(t), \quad \tau \in [t_0 - r, t_0 + \delta] \tag{43}$$

Theorem 19. *Let $\varphi \in G([-r,0], X)$ and $L : G([-r,0], L(X)) \times [t_0-r, t_0+\delta] \longrightarrow X$ be linear functions in the first variables such that $t \longrightarrow L(x_t, t)$ is Lebesgue integrable on $[t_0 - r, t_0 + \delta]$ and the conditions (A_1), (A_2), (A_3), and (A_4) are satisfied. Assume $y(t)$ is a solution of Equation (41) on $[t_0 - r, t_0 + \delta]$ satisfying the initial condition (40); then $v \in [t_0 - r, t_0 + \delta]$ such that*

$$x(v) = \begin{cases} y(t_0)(v), & v \in [t_0 - r, t_0] \\ y(v)(v), & v \in [t_0, t_0 + \delta], \end{cases} \tag{44}$$

is a solution of Equation (28) on $[t_0 - r, t_0 + \delta]$ if, for any $\varepsilon > 0$,

$$\left\| S(dA, x, P) - \int d[A]\, x \right\| < \varepsilon \tag{45}$$

holds.

Proof. Using the result of Lemma 18 and Equation (22), we have

$$\begin{aligned} x(v) - x(t_0) &= [y(v) - y(t_0)](v) \\ &= \int_{t_0}^{v} d[A]\, y + g(v) - g(t_0) \\ &\le \left[\int_{t_0}^{v} d[A]\, y - S(dA, y, P) \right] \\ &\quad + S(dA, y, P) + g(v) - g(t_0) \end{aligned} \tag{46}$$

We make the choice of the gauge function

$$\begin{aligned} &\delta(\eta) \\ &\quad < \min\left\{ \frac{t_k - t_{k-1}}{2} : \ k = 1, \ldots, n, \ \eta \in [t_0, t_0 + \delta] \right\} \\ &0 < \delta(\eta) < \min\{|\eta - t_k|, |\eta - t_{k-1}|\}, \\ &\qquad\qquad\qquad \text{for } \eta \in (t_k, t_{k-1}), \ k = 1 \ldots n \end{aligned} \tag{47}$$

These ensure that each subinterval of δ-fine partition contains at most one of the points t_k, $k = 1, \ldots, n$, corresponding to a tag of the interval. Hence, by Equation (46), we have

$$\begin{aligned} &x(s_i) - x(s_{i-1}) \\ &\quad \le \sum_{i=1}^{n} (A(s_i) - A(s_{i-1}))\, y(\eta_i)(v) + g(v) - g(t_0) \\ &\quad = (F(y(\eta_i)s_i) - F(y(\eta_{i-1})s_{i-1}))(v) \\ &\qquad + \sum \left(\int_{t_k}(s_i) - \int_{t_k}(s_{i-1}) \right)(y(\eta_i)) \\ &\qquad + (g(s_i) - g(s_{i-1}))(v) \\ &\quad = \begin{cases} 0, & t_0 - r \le v \le t_0 \\ \int_{s_{i-1}}^{v} L(y_s, s)\, ds, & s_{i-1} \le v \le s_i \le t_0 + \delta \\ \int_{s_{i-1}}^{s_i} L(y_s, s)\, ds, & s_{i-1} \le s_i \le v \le t_0 + \delta \end{cases} \\ &\qquad + \begin{cases} 0, & t_0 - r \le v \le t_0 \\ \int_{s_{i-1}}^{v} f(s)\, ds, & s_{i-1} \le v \le s_i \le t_0 + \delta \\ \int_{s_{i-1}}^{s_i} f(s)\, ds, & s_{i-1} \le s_i \le v \le t_0 + \delta \end{cases} \\ &\qquad + \sum \left(\int_{t_k}(s_i) - \int_{t_k}(s_{i-1}) \right) \int_{t_k}(v)\, I_t(y(\eta_i)) \end{aligned} \tag{48}$$

This implies that

$$\begin{aligned} &\|x(s_i) - x(s_{i+1})\| \\ &\le \sup_{v \in [t_0 - r, t_0 + \delta]} \begin{cases} \left\| \int_{s_{i-1}}^{v} L(y_s(\eta_i), s)\, ds \right\|, & s_{i-1} \le v \le s_i \le t_0 + \delta \\ \left\| \int_{s_{i-1}}^{s_i} L(y_s(\eta_i), s)\, ds \right\|, & s_{i-1} \le s_i \le v \le t_0 + \delta \end{cases} \\ &\quad + \sup_{v \in [t_0 - r, t_0 + \delta]} \begin{cases} \left\| \int_{s_{i-1}}^{v} f(s)\, ds \right\|, & s_{i-1} \le v \le s_i \le t_0 + \delta \\ \left\| \int_{s_{i-1}}^{s_i} f(s)\, ds \right\|, & s_{i-1} \le s_i \le v \le t_0 + \delta \end{cases} \\ &\quad + I_k(y(t_k)) \int_{t_k}(v) \\ &= \sup_{v \in [s_{i-1}, s_{i-1}]} \left\| \int_{s_i}^{v} L(y_s(\eta_i), s)\, ds \right\| + \sup_{v \in [s_{i-1}, s_{i-1}]} \left\| \int_{s_i}^{v} f(s)\, ds \right\| \\ &\quad + I_k(y(t_k)) \int_{t_k}(v) \end{aligned} \tag{49}$$

Also by Lemma 14 and Equation (44), for $t \in [t_0, v]$, we have

$$\begin{aligned} \|y_t(t_0, \varphi)\| &= \|y_t\| = \sup_{-r \le \theta \le 0} |y(t+\theta)| \le \sup_{t_0 - r \le s \le v} |y(s)| \\ &= \sup_{t_0 - r \le s \le v} |x(v)(s)| = \sup_{t_0 - r \le s \le +\infty} |x(v)(s)| \\ &= \|x(v)\| < \varepsilon \end{aligned} \tag{50}$$

Hence, by Remark 15, we have

$$\begin{aligned}\|x(s_i)-x(s_{i-1})\| &\le \sup_{v\in[s_{i-1},s_{i-1}]}\left\|\int_{s_i}^{v} L(x_s(\eta_i),s)\,ds\right\| \\ &\quad+\sup_{v\in[s_{i-1},s_{i-1}]}\left\|\int_{s_i}^{v} f(s)\,ds\right\| \\ &\quad+I_k(x(t_k))\int_{t_k}(v)\operatorname{var}_{t_0}^{t}(x) \\ &\le \int_{t_0}^{t} M_0(s)\,ds\,\|x\|_X+M|t-t_0| \\ &\quad+K_0\|x\|_X \\ &= M|t-t_0| \\ &\quad+\left(K_0+\int_{t_0}^{t}M_0(s)\,ds\right)\|x\|_X \end{aligned}\tag{51}$$

which implies that the function $x\in G([t_0,t_0+\delta],X)\longrightarrow X$ is of bounded variation, and hence the existence of solution of Equation (28). □

Theorem 20. *Let $x(t)\in G([t_0,t_0+\delta],X)$ satisfy equation (28) on $[t_0-r,t_0+\delta]$ and let $t\longrightarrow L(x_t,t)$ be Lebesgue integrable. Then, there exists a $y(t)$ satisfying the initial condition (44) such that for any $v\in[t_0-r,t_0+\delta]$*

$$\text{(i) } y(t)(v)=\begin{cases}x(v), & v\in[t_0-r,t_0]\\ x(t), & v\in[t_0,t_0+\delta]\end{cases}\tag{52}$$

(ii) for $t_c\in[t_0,t_0+\delta]$, there exists a $k=k(t_c)<t_0+\delta,\ 0<k<1$ and $\Delta t_c>0$ such that

$$\operatorname*{var}_{[t_c-\Delta t_c,t_c]\cap[t_0,t_0+\delta]} A(t)<k.\tag{53}$$

Then Equation (39) has a unique solution.

Proof. By Equations (28) and (52)

$$\begin{aligned}y(t)(v)&=x(t)(v)\\ &=x(t_0)(v)+\int_{t_0}^{v}L(x_s,s)\,ds+\int_{t_0}^{v}f(s)\,ds\\ &\quad+\sum_{k=1}^{n}I_k(x(t_k))\\ &=x(t_0)(v)+F(x,t)+J(x,t)+\int_{t_0}^{v}f(s)\,ds\\ &=x(t_0)(v)+A(t)x+\int_{t_0}^{v}f(s)\,ds\end{aligned}\tag{54}$$

Let $t\longrightarrow L(x_t,t)$ be a Lebesgue integrable, $A:G([t_0,t_0+\delta],L(X)\longrightarrow L(X)$ a regulated function on $[t_0-r,t_0+\delta]$, and $x:[t_0,t_0+\delta]\longrightarrow X$; then for a $\delta-$ *fine* partition $[s_{i-1},s_i]\subset[\eta_i-\delta(\eta_i),\eta_i+\delta(\eta_i)],\ i=1,2,\ldots$, and using the relation $x(s)=x(\eta_i)=y(s),\ s\in[\eta_i,\eta_{i-1}]$, we have

$$\begin{aligned}y(t)(v)&=x(t_0)(v)+\sum_{i=1}^{n}(A(s_i)-A(s_{i-1}))x(\eta_i)(v)\\ &\quad+\int_{t_0}^{v}f(s)\,ds\\ &=y(t_0)(v)+\int_{t_0}^{v}d[A(s)]\,y(\eta)+g(v)\\ &\quad-g(t_0)\end{aligned}\tag{55}$$

Using Proposition 17 and for $y(t)$ being a solution of equation (39) we obtained

$$\begin{aligned}[I-\Delta^{-}A(t)]\,y(t)&=y(t_0)+\int_{t_0}^{v}d[A]\,y+g(v)\\ &\quad-g(t_0)\end{aligned}\tag{56}$$

Also, the results in Schwabik [1] show that for $A\in G([t_0,t_0+\delta],X)$ a regulated function there exists $\varepsilon>0$ such that the set $\{t\in[t_0,t_0+\delta],\|\Delta^{-}A(t)\|\ge\varepsilon\}$ is finite. This implies that the set of discontinuity points of A is at most countable, and there is a finite set $\{t_1,t_2,\ldots,t_n\}$ such that for $t\in[t_0,t_0+\delta],\ t=t_k,\ k=1,2,\ldots$. the operator $I-\Delta^{-}A(t)\in L(X)$ is invertible and $[I-\Delta^{-}A(t)]^{-1}$ exists.

Therefore,

$$\begin{aligned}y(t)&=[I-\Delta^{-}A(t)]^{-1}\\ &\quad\cdot\left[y(t_0)+g(v)-g(t_0)+\int_{t_0}^{v}d[A]\,y(s)\right]\end{aligned}\tag{57}$$

and

$$\begin{aligned}\|y(t)\|&\le\left\|[I-\Delta^{-}A(t)]^{-1}\right\|\\ &\quad\cdot\left[\|y(t_0)\|+\|g(v)\|+g(t_0)+\int_{t_0}^{v}d[A]\,\|y(s)\|\right]\\ &\le\left\|[I-\Delta^{-}A(t)]^{-1}\right\|N_0\exp[I-\Delta^{-}A(t)]^{-1}\operatorname{var}_{t_0}^{v}A,\\ &\qquad v\in[t_0-r,t_0+\delta]\end{aligned}\tag{58}$$

where $A(t)$ is nonnegative, nondecreasing, and left continuous function, $y(t)$ is nonnegative and bounded function.

Using preposition 2.1 (Schwabik, [1]), we defined a bounded linear operator $T:G([t_0,t_0+\delta],X)\longrightarrow G([t_0,t_0+\delta],X)$ on $G([t_0,t_0+\delta],X)$ so that by Equation (53)

$$(Ty)(t)=y(t_0)+\int_{t_0}^{v}d[A]\,y(s)+g(v)-g(t_0).\tag{59}$$

If $y,z \in G([t_c-\Delta t_c,t_c]\cap[t_0,t_0+\delta],X) \longrightarrow X$ for $y(t_c)=z(t_c)=y(t_0)$ then

$$\begin{aligned}
&\left\|((Ty)-(Tz))(t)\right\| \\
&\le \lim_{\varepsilon\longrightarrow 0^-}\left\|\int_{t_c-\varepsilon}^{t} d[A(s)](y(s)-z(s))\right\| \\
&\le \lim_{\varepsilon\longrightarrow 0^-}\operatorname*{var}_{[t,t_c-\varepsilon]} A\left\|y-z\right\|_{[t_c-\Delta t_c,t_c]\cap[t_0,t_0+\delta]} \\
&\le \operatorname*{var}_{[t_c-\Delta t_c,t_c]\cap[t_0,t_0+\delta]} A\left\|y-z\right\|_{[t_c-\Delta t_c,t_c]\cap[t_0,t_0+\delta]} \\
&\le k\left\|y-z\right\|
\end{aligned} \tag{60}$$

where k satisfied the hypothesis of the theorem as stated and the operator $T \in G([t_c-\Delta t_c,t],X)$ is a contraction, and, by Banach contraction principle, it has a unique fixed point. Hence the theorem is proved. □

Example 21. We consider the model of a circulating fuel reactor originally studied in [17] and modified in [18] by the inclusion of constants impulsive terms. We further modify the model equation by including an input function (a forcing term) $w(t):[t_0,t_0+\delta]\longrightarrow X$ which is Lebesgue integrable. The system equation is of the form

$$\begin{aligned}
&\dot{x}(t)=\int_{t-r}^{t} p(t-s)\,g(x(s))\,ds+w(t), \\
&\qquad\qquad t\neq\tau_k(x(t_k)) \\
&\Delta x(t)=I_k(x(t_k)),\quad t\neq\tau_k(x(t_k)),\ k=1,2,\ldots \\
&x(t_0)=\varphi
\end{aligned} \tag{61}$$

where $r>0,\varphi\in G([-r,0],X), p:R\longrightarrow R_+$ and $w(t):[t_0,t_0+\delta]\longrightarrow X$ are locally Lebesgue integrable functions such that $|w(t)|\le M, M>0$ and $p(u)\le B$ for all $u\in R, g: R\longrightarrow R$ is increasing function such that $g(0)=0, |g(x)|\le K|x-y|, x,y\in R$ with $K\ge 0$. Let there exists a function $m:R\longrightarrow R$ locally Lebesgue integrable such that

$$\left|\int_{s_1}^{s_2} g(x(s))\,ds\right|\le\int_{s_1}^{s_2} m(s)\,ds, \tag{62}$$

for all $s_1,s_2\in R$, and for $k=1,2\ldots$, maps R to $(0,\infty)$.

Defining $L(x_t,t)=\int_{t-r}^{t} p(t-s)g(x(s))ds$, Afonso [18] proved that conditions (A_1), and (A_2) of Remark 15 are satisfied.

Also, by the hypothesis of the problem, we have that

$$\begin{aligned}
\left\|\int_{u_1}^{u_2} f(s)\,ds\right\| &\le \left\|\int_{u_1}^{u_2} w(s)\,ds\right\| \le \int_{u_1}^{u_2}\left\|w(s)\right\|ds \\
&\le \int_{u_1}^{u_2} M ds,
\end{aligned} \tag{63}$$

Hence condition (A_3) is satisfied.

Considering the impulsive term $\Delta x(t)=I_k(x(t_k))$, we have

$$\begin{aligned}
&\left\|I_k(x(s))-I_k(y(s))\right\| \\
&\quad\le \sup\sum_{k=1}^{m}\left[I_k(x(s))-I_k(y(s))\right]\le K_2\left\|x-y\right\|,
\end{aligned} \tag{64}$$

$$\left\|I_k(x(s_2))\right\|\le K_1\left\|x\right\|,\quad \text{for } K_1,K_2>0,$$

and condition (A_4) is satisfied. Hence, in consequence of Remark 15 the integral solution

$$\begin{aligned}
x(t)=\varphi&+\int_{u_1}^{u_2}\left(-\int_{s-r}^{s} p(s-u)\,g(x(u))\,du\right)ds \\
&+\int_{u_1}^{u_2} w(s)\,ds+\sum_{k=1}^{m} I_k(x(t_k))
\end{aligned} \tag{65}$$

exists and is unique.

5. Conclusion

An initial value problem of a nonhomogeneous retarded functional equation coupled with the impulsive term was considered. The integral equivalent of the equation which is Lebesgue integrable was obtained using the fundamental matrix theorem. The integral equivalent equation with impulses satisfying the Carathéodory and Lipschitz conditions was embedded in the space of generalized ordinary differential equations (GODEs) and, using similar argument as presented in Federson and Taboas [6] and Federson and Schwabik [5], we showed the relationship between the generalized ordinary differential equation and the nonhomogeneous retarded functional equation coupled with impulsive term by the construction of a local flow using topological dynamic satisfying certain technical conditions. The uniqueness of the equation solution was proved.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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