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Research Article

Stability Analysis of Additive Runge-Kutta Methods for Delay-Integro-Differential Equations

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This paper is concerned with stability analysis of additive Runge-Kutta methods for delay-integro-differential equations. We show that if the additive Runge-Kutta methods are algebraically stable, the perturbations of the numerical solutions are controlled by the initial perturbations from the system and the methods.

1. Introduction

Spatial discretization of many nonlinear parabolic problems usually gives a class of ordinary differential equations, which have the stiff part and the nonstiff part; see, e.g., [1–5]. In such cases, the most widely used time-discretizations are the special organized numerical methods, such as the implicit-explicit numerical methods [6, 7], the additive Runge-Kutta methods [8–12], and the linearized methods [13, 14]. When applying the split numerical methods to numerically solve the equations, it is important to investigate the stability of the numerical methods.

In this paper, it is assumed that the spatial discretization of time-dependent partial differential equations yields the following nonlinear delay-integro-differential equations:

$$\begin{aligned} y'(t) &= f^{[1]}(t, y(t)) \\ &\quad + f^{[2]}\left(t, y(t), y(t-\tau), \int_{t-\tau}^{t} g(t, s, y(s))\, ds\right), \quad t > 0, \\ y(t) &= \psi(t), \quad -\tau \le t \le 0. \end{aligned} \tag{1}$$

Here τ is a positive delay term, $\psi(t)$ is continuous, $f^{[1]}$: $[t_0, +\infty) \times X \to X$, and $f^{[2]}$: $[t_0, +\infty) \times X \times X \times X \to X$, such that problem (1) owns a unique solution, where X is a real or complex Hilbert space. Particularly, when $g \equiv 0$, problem (1) is reduced to the nonlinear delay differential equations. When the delay term $\tau = 0$, problem (1) is reduced to the ordinary differential equations.

The investigation on stability analysis of different numerical methods for problem (1) has fascinated generations of researchers. For example, Torelli [15] considered stability of Euler methods for the nonautonomous nonlinear delay differential equations. Hout [16] studied the stability of Runge-Kutta methods for systems of delay differential equations. Baker and Ford [17] discussed stability of continuous Runge-Kutta methods for integrodifferential systems with unbounded delays. Zhang and Vandewalle [18] discussed the stability of the general linear methods for integrodifferential equations with memory. Li and Zhang obtained the stability and convergence of the discontinuous Galerkin methods for nonlinear delay differential equations [19, 20]. More references for this topic can be found in [21–30]. However, few works have been found on the stability of splitting methods for the proposed methods.

In the present work, we present the additive Runge-Kutta methods with some appropriate quadrature rules to numerically solve the nonlinear delay-integrodifferential equations (1). It is shown that if the additive Runge-Kutta methods are algebraically stable, the obtained numerical solutions are globally and asymptotically stable under the given assumptions, respectively. The rest of the paper is organized as follows. In Section 2, we present the numerical methods for problems (1). In Section 3, we consider stability analysis of the numerical schemes. Finally, we present some extensions in Section 4.

2. The Numerical Methods

In this section, we present the additive Runge-Kutta methods with the appropriate quadrature rules to numerically solve problem (1).

The coefficients of the additive Runge-Kutta methods can be organized in Buther tableau as follows (cf. [31]):

$$\begin{array}{c|c|c} c & A^{[1]} & A^{[2]} \\ \hline & \left(b^{[1]}\right)^T & \left(b^{[2]}\right)^T \end{array}, \tag{2}$$

where $c = [c_1, \cdots, c_s]^T$, $b^{[k]} = [b_1^{[k]}, \cdots, b_s^{[k]}]^T$, and $A^{[k]} = (a_{ij}^{[k]})_{i,j=1}^{s}$ for $k = 1, 2$.

Then, the presented ARKMs for problem (1) can be written by

$$\begin{aligned} y_{n+1} &= y_n + h\sum_{j=1}^{s} b_j^{[1]} f^{[1]}\left(t_n + c_j h, y_j^{(n)}\right) \\ &\quad + h\sum_{j=1}^{s} b_j^{[2]} f^{[2]}\left(t_n + c_j h, y_j^{(n)}, \widetilde{y}_j^{(n)}\right), \\ y_i^{(n)} &= y_n + h\sum_{j=1}^{s} a_{ij}^{[1]} f^{[1]}\left(t_n + c_j h, y_j^{(n)}\right) \\ &\quad + h\sum_{j=1}^{s} a_{ij}^{[2]} f^{[2]}\left(t_n + c_j h, y_j^{(n)}, y_j^{(n-m)}, \widetilde{y}_j^{(n)}\right), \\ &\qquad i = 1, 2, \cdots, s, \end{aligned} \tag{3}$$

where y_n and $y_i^{(n)}$ are approximations to the analytic solution $y(t_n)$ and $y(t_n + c_i h)$, respectively, $y_n = \psi(t_n)$ for $n \le 0$, $y_i^{(n)} = \psi(t_n + c_i h)$ for $t_n + c_i h \le 0$, and $\widetilde{y}_i^{(n)}$ denotes the approximation to $\int_{t_n+c_ih-\tau}^{t_n+c_ih} g(t_n + c_i h, \xi, y(\xi))d\xi$, which can be computed by some appropriate quadrature rules

$$\begin{aligned} \widetilde{y}_i^{(n)} &= h\sum_{k=0}^{m} p_k g\left(t_n + c_i h, t_{n-k} + c_i h, y_i^{(n-k)}\right), \\ &\qquad i = 1, 2, \cdots, s. \end{aligned} \tag{4}$$

For example, we usually adopt the repeated Simpson's rule or Newton-Cotes rule, etc., according to the requirement of the convergence of the method (cf. [18]).

3. Stability Analysis

In this section, we consider the numerical stability of the proposed methods. First, we introduce a perturbed problem, whose solution satisfies

$$\begin{aligned} z'(t) &= f^{[1]}(t, z(t)) \\ &\quad + f^{[2]}\left(t, z(t), z(t-\tau), \int_{t-\tau}^{t} g(t, s, z(s))\, ds\right), \\ &\qquad t > 0, \\ y(t) &= \phi(t), \quad -\tau \le t \le 0. \end{aligned} \tag{5}$$

It is assumed that there exist some inner product $< \cdot, \cdot >$ and the induced norm $\|\cdot\|$ such that

$$\begin{aligned} &\operatorname{Re}\left\langle y - z, f^{[1]}(t, y) - f^{[1]}(t, z)\right\rangle \le \alpha \|y - z\|^2, \\ &\operatorname{Re}\left\langle y - z, f^{[2]}(t, y, u_1, v_1) - f^{[2]}(t, z, u_2, v_2)\right\rangle \\ &\quad \le \beta_1 \|y - z\|^2 + \beta_2 \|u_1 - u_2\|^2 + \gamma \|v_1 - v_2\|^2, \\ &\|g(t, v, s_1) - g(t, v, s_2)\| \le \theta \|s_1 - s_2\|, \end{aligned} \tag{6}$$

where $\alpha < 0$, $\beta_1 < 0$, $\beta_2 > 0$, $\gamma > 0$, and $\theta > 0$ are constants. It is remarkable that the conditions can be equivalent to the assumptions in [32, 33] (see. [34] *Remark* 2.1).

Definition 1 (cf. [9]). An additive Runge-Kutta method is called algebraically stable if the matrices

$$\begin{aligned} B_\nu &:= \operatorname{diag}\left(b_1^{[\nu]}, \cdots, b_s^{[\nu]}\right), \quad \nu = 1, 2, \\ M_{\nu\mu} &:= B_\nu A^{[\mu]} + A^{[\nu]T} B_\mu - b^{[\nu]} b^{[\mu]T} \end{aligned} \tag{7}$$

are nonnegative.

Theorem 2. *Assume an additive Runge-Kutta method is algebraically stable and $\beta_1 + \beta_2 + 4\gamma\tau^2\eta^2\theta^2 < 0$, where $\eta = \max\{p_1, p_2, \cdots, p_k\}$. Then, it holds that*

$$\begin{aligned} \|y_n - z_n\| &\le \sqrt{\left(1 + 2\sum_{i=1}^{s} \tau b_i^{[2]} \beta_2 + 4\gamma\tau^2\eta^2\theta^2\right)} \\ &\quad \cdot \max_{-\tau \le t \le 0} \|\psi(t) - \phi(t)\|, \end{aligned} \tag{8}$$

where y_n and z_n are numerical approximations to problems (1) and (5), respectively.

Proof. Let $\{y_n, y_i^{(n)}, \widetilde{y}_i^{(n)})\}$ and $\{z_n, z_i^{(n)}, \widetilde{z}_i^{(n)})\}$ be two sequences of approximations to problems (1) and (5), respectively, by ARKMs with the same stepsize h and write

$$\begin{aligned} U_i^{(n)} &= y_i^{(n)} - z_i^{(n)}, \\ \widetilde{U}_i^{(n)} &= \widetilde{y}_i^{(n)} - \widetilde{z}_i^{(n)}, \\ U_0^{(n)} &= y_n - z_n, \\ W_i^{[1]} &= h\left[f^{[1]}\left(t_n + c_i^{[1]}h, y_i^{(n)}\right)\right. \\ &\quad \left. - f^{[1]}\left(t_n + c_i^{[1]}h, z_i^{(n)}\right)\right], \\ W_i^{[2]} &= h\left[f^{[2]}\left(t_n + c_i^{[2]}h, y_i^{(n)}, y_i^{(n-m)}, \widetilde{y}_i^{(n)}\right)\right. \\ &\quad \left. - f^{[2]}\left(t_n + c_i^{[2]}h, z_i^{(n)}, z_i^{(n-m)}, \widetilde{z}_i^{(n)}\right)\right]. \end{aligned} \tag{9}$$

With the notation, the ARKMs for (1) and (5) yield

$$\begin{aligned} U_0^{(n+1)} &= U_0^n + \sum_{\mu=1}^{2}\sum_{j=1}^{s} b_j^{[\mu]} W_j^{[\mu]}, \\ U_i^{(n)} &= U_0^{(n)} + \sum_{\mu=1}^{2}\sum_{j=1}^{s} a_{ij}^{[\mu]} W_j^{[\mu]}, \quad i = 1, 2, \cdots, s. \end{aligned} \tag{10}$$

Thus, we have

$$\begin{aligned} \left\|U_0^{(n+1)}\right\|^2 &= \left\langle U_0^{(n)} + \sum_{\mu=1}^{2}\sum_{j=1}^{s} b_j^{[\mu]} W_j^{[\mu]}, U_0^{(n)}\right. \\ &\quad \left. + \sum_{\nu=1}^{2}\sum_{i=1}^{s} b_i^{[\nu]} W_i^{[\nu]}\right\rangle = \left\|U_0^{(n)}\right\|^2 + 2\sum_{\mu=1}^{2}\sum_{i=1}^{s} b_i^{[\mu]} \\ &\quad \cdot \operatorname{Re}\left\langle U_0^{(n)}, W_i^{[\mu]}\right\rangle + \sum_{\mu,\nu=1}^{2}\sum_{i,j=1}^{s} b_i^{[\mu]} b_j^{[\nu]} \left\langle W_i^{[\mu]}, W_j^{[\nu]}\right\rangle \\ &= \left\|U_0^{(n)}\right\|^2 + 2\sum_{\mu=1}^{2}\sum_{i=1}^{s} b_i^{[\mu]} \\ &\quad \cdot \operatorname{Re}\left\langle U_i^{(n)} - \sum_{\nu=1}^{2}\sum_{j=1}^{s} a_{ij}^{[\nu]} W_j^{[\nu]}, W_i^{[\mu]}\right\rangle \\ &\quad + \sum_{\mu,\nu=1}^{2}\sum_{i,j=1}^{s} b_i^{[\mu]} b_j^{[\nu]} \left\langle W_i^{[\mu]}, W_j^{[\nu]}\right\rangle = \left\|U_0^{(n)}\right\|^2 \\ &\quad + 2\sum_{\mu=1}^{2}\sum_{i=1}^{s} b_i^{[\mu]} \operatorname{Re}\left\langle U_i^{(n)}, W_i^{[\mu]}\right\rangle \\ &\quad - \sum_{\mu,\nu=1}^{2}\sum_{i,j=1}^{s} \left(b_i^{[\mu]} a_{ij}^{[\nu]} + a_{ji}^{[\mu]} b_j^{[\nu]} - b_i^{[\mu]} b_j^{[\nu]}\right) \\ &\quad \cdot \left\langle W_i^{[\mu]}, W_j^{[\nu]}\right\rangle. \end{aligned} \tag{11}$$

Since that the matrix $\mathscr{M}$ is a nonnegative matrix, we obtain

$$\begin{aligned} &- \sum_{\mu,\nu=1}^{2}\sum_{i,j=1}^{s} \left(b_i^{[\mu]} a_{ij}^{[\nu]} + a_{ji}^{[\mu]} b_j^{[\nu]} - b_i^{[\mu]} b_j^{[\nu]}\right) \left\langle W_i^{[\mu]}, W_j^{[\nu]}\right\rangle \\ &\leq 0. \end{aligned} \tag{12}$$

Furthermore, by conditions (6), we find

$$\operatorname{Re}\left\langle U_i^{(n)}, W_i^{[1]}\right\rangle \leq \alpha h \left\|U_i^{(n)}\right\|^2, \tag{13}$$

and

$$\begin{aligned} \operatorname{Re}\left\langle U_i^{(n)}, W_i^{[2]}\right\rangle &\leq \beta_1 h \left\|U_i^{(n)}\right\|^2 + \beta_2 h \left\|U_i^{(n-m)}\right\|^2 \\ &\quad + \gamma h \left\|\widetilde{U}_i^{(n)}\right\|^2. \end{aligned} \tag{14}$$

Together with (11), (12), (13), and (14), we get

$$\begin{aligned} \left\|U_0^{(n+1)}\right\|^2 &\leq \left\|U_0^{(n)}\right\|^2 + 2\sum_{i=1}^{s} h b_i^{[1]} \alpha \left\|U_i^{(n)}\right\|^2 \\ &\quad + 2\sum_{i=1}^{s} h b_i^{[2]} \left(\beta_1 \left\|U_i^{(n)}\right\|^2 + \beta_2 \left\|U_i^{(n-m)}\right\|^2\right. \\ &\quad \left. + \gamma \left\|\widetilde{U}_i^{(n)}\right\|^2\right) \leq \left\|U_0^{(n)}\right\|^2 + 2\sum_{i=1}^{s} h b_i^{[2]} \left(\beta_1 \left\|U_i^{(n)}\right\|^2\right. \\ &\quad \left. + \beta_2 \left\|U_i^{(n-m)}\right\|^2 + \gamma \left\|\widetilde{U}_i^{(n)}\right\|^2\right). \end{aligned} \tag{15}$$

Note that

$$\begin{aligned} \left\|\widetilde{U}_i^{(n)}\right\|^2 &= \left\| h\sum_{k=0}^{m} p_k \left[g\left(t_n + c_i h, t_{n-k} + c_i h, y_i^{n-k}\right)\right.\right. \\ &\quad \left.\left. - g\left(t_n + c_i h, t_{n-k} + c_i h, z_i^{n-k}\right)\right]\right\|^2 \leq (m+1) \\ &\quad \cdot \eta^2 \theta^2 h^2 \sum_{k=0}^{m} \left\|U_i^{(n-k)}\right\|^2. \end{aligned} \tag{16}$$

Then, we obtain

$$\begin{aligned} \left\|U_0^{(n+1)}\right\|^2 &\leq \left\|U_0^{(n)}\right\|^2 + 2\sum_{i=1}^{s} h b_i^{[2]} \left(\beta_1 \left\|U_i^{(n)}\right\|^2\right. \\ &\quad \left. + \beta_2 \left\|U_i^{(n-m)}\right\|^2 + \gamma (m+1) \eta^2 \theta^2 h^2 \sum_{k=0}^{m} \left\|U_i^{(n-k)}\right\|^2\right) \\ &\leq \left\|U_0^{(0)}\right\|^2 + 2\sum_{j=0}^{n}\sum_{i=1}^{s} h b_i^{[2]} \left(\beta_1 \left\|U_i^{(j)}\right\|^2\right. \\ &\quad \left. + \beta_2 \left\|U_i^{(j-m)}\right\|^2 + \gamma (m+1) \eta^2 \theta^2 h^2 \sum_{k=0}^{m} \left\|U_i^{(j-k)}\right\|^2\right) \end{aligned}$$

$$\leq \left\|U_0^{(0)}\right\|^2 + 2\sum_{j=0}^{n}\sum_{i=1}^{s} hb_i^{[2]}\left(\beta_1\left\|U_i^{(j)}\right\|^2 + \beta_2\left\|U_i^{(j)}\right\|^2 + \gamma(m+1)^2 h^2\eta^2\theta^2\left\|U_i^{(j)}\right\|^2\right) + 2\sum_{j=-m}^{-1}\sum_{i=1}^{s} hb_i^{[2]}\left(\beta_2\left\|U_i^{(j)}\right\|^2 + \gamma(m+1)^2 h^2\eta^2\theta^2\left\|U_i^{(j)}\right\|^2\right) \leq \left\|U_0^{(0)}\right\|^2 + 2\sum_{j=0}^{n}\sum_{i=1}^{s} hb_i^{[2]}\left(\beta_1+\beta_2+4\gamma\tau^2\eta^2\theta^2\right)\left\|U_i^{(j)}\right\|^2 + 2\sum_{j=-m}^{-1}\sum_{i=1}^{s} hb_i^{[2]}\left(\beta_2+4\gamma\tau^2\eta^2\theta^2\right)\left\|U_i^{(j)}\right\|^2 \leq \left\|U_0^{(0)}\right\|^2 + 2\sum_{j=-m}^{-1}\sum_{i=1}^{s} hb_i^{[2]}\left(\beta_2+4\gamma\tau^2\eta^2\theta^2\right)\left\|U_i^{(j)}\right\|^2 \leq \left\|U_0^{(0)}\right\|^2 + 2\sum_{i=1}^{s} mhb_i^{[2]}\left(\beta_2+4\gamma\tau^2\eta^2\theta^2\right)\max_{-m\leq j\leq -1}\left\|U_i^{(j)}\right\|^2. \tag{17}$$

Hence,

$$\left\|U_0^{(n+1)}\right\|^2 \leq C\max_{-\tau\leq t\leq 0}\left\|\psi(t)-\phi(t)\right\|^2, \tag{18}$$

where $C = [(1 + 2\sum_{i=1}^{s}\tau b_i^{[2]}\beta_2 + 4\gamma\tau^2\eta^2\theta^2)]$. This completes the proof. □

Theorem 3. *Assume an additive Runge-Kutta method is algebraically stable and $\beta_1+\beta_2+4\gamma\tau^2\eta^2\theta^2 < 0$. Then, it holds that*

$$\lim_{n\to\infty}\left\|U_0^{(n)}\right\| = 0. \tag{19}$$

Proof. Similar to the proof of Theorem 2, it holds that

$$\left\|U_0^{(n+1)}\right\|^2 \leq \left\|U_0^{(0)}\right\|^2 + 2\sum_{j=0}^{n}\sum_{i=1}^{s} hb_i^{[2]}\left(\beta_1+\beta_2+4\gamma\tau^2\eta^2\theta^2\right)\left\|U_i^{(j)}\right\|^2 + 2\sum_{j=-m}^{-1}\sum_{i=1}^{s} hb_i^{[2]}\left(\beta_2+4\gamma\tau^2\eta^2\theta^2\right)\left\|U_i^{(j)}\right\|^2. \tag{20}$$

Note that $\beta_1+\beta_2+4\gamma\tau^2\eta^2\theta^2 < 0$ and $b_i^{[2]} > 0$; we have

$$\lim_{n\to\infty}\sum_{i=1}^{s} b_i^{[2]}\left\|U_i^{(n)}\right\| = 0. \tag{21}$$

On the other hand,

$$\left\|W_i^{[1]}\right\| = \left\|h\left[f^{[1]}\left(t_n + c_i^{[1]}h, y_i^{(n)}\right) - f^{[1]}\left(t_n + c_i^{[1]}h, z_i^{(n)}\right)\right]\right\| \leq L_1\left\|U_i^{(n)}\right\| \tag{22}$$

$$\left\|W_i^{[2]}\right\| = \left\|h\left[f^{[2]}\left(t_n + c_i^{[2]}h, y_i^{(n)}, y_i^{(n-m)}, \tilde{y}_i^{(n)}\right) - f^{[2]}\left(t_n + c_i^{[2]}h, z_i^{(n)}, z_i^{(n-m)}, \tilde{z}_i^{(n)}\right)\right]\right\| \leq L_2\left(\left\|U_i^{(n)}\right\| + \left\|U_i^{(n-m)}\right\| + \left\|\tilde{y}_i^{(n)} - \tilde{z}_i^{(n)}\right\|\right). \tag{23}$$

Now, in view of (10), (21), (22), and (23), we obtain

$$\lim_{n\to\infty}\left\|U_0^{(n)}\right\| = 0. \tag{24}$$

This completes the proof. □

Remark 4. In [35], Yuan et al. also discussed nonlinear stability of additive Runge-Kutta methods for multidelay-integro-differential equations. However, the main results are different. The main reason is that the results in [35] imply that the perturbations of the numerical solutions tend to infinity when the time increase, while the stability results in present paper indicate that the perturbations of the numerical solutions are independent of the time. Besides, the asymptotical stability of the methods is also discussed in the present paper.

4. Conclusion

The additive Runge-Kutta methods with some appropriate quadrature rules are applied to solve the delay-integro-differential equations. It is shown that if the additive Runge-Kutta methods are algebraically stable, the obtained numerical solutions can be globally and asymptotically stable, respectively. In the future works, we will apply the methods to solve more real-world problems.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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