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*Research Article*

# Existence of Solutions for Unbounded Elliptic Equations with Critical Natural Growth

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We investigate existence and regularity of solutions to unbounded elliptic problem whose simplest model is $\{-\operatorname{div}[(1+|u|^q)\nabla u]+u=\gamma(|\nabla u|^2/(1+|u|)^{1-q})+f$ in $\Omega$, $u=0$ on $\partial\Omega,\}$, where $0<q<1$, $\gamma>0$ and $f$ belongs to some appropriate Lebesgue space. We give assumptions on $f$ with respect to $q$ and $\gamma$ to show the existence and regularity results for the solutions of previous equation.

## 1. Introduction

In this paper, we consider the Dirichlet problem for some nonlinear elliptic problems such as

$$-\operatorname{div}\left(\left[a(x)+|u|^q\right]\nabla u\right)+u=H(x,u,\nabla u)+f, \quad x\in\Omega,\ u\in H_0^1(\Omega), \tag{1}$$

under the following assumptions: $\Omega$ is a bounded open subset of $\mathbb{R}^N$, where $N\geq 3$, $0<q<1$, and $f\in L^m$ with $m\geq 2$ and $a:\Omega\longrightarrow\mathbb{R}$ is a measurable function satisfying the following conditions:

$$\alpha\leq a(x)\leq\beta, \tag{2}$$

for almost every $x\in\Omega$, where $\alpha$ and $\beta$ are positive real constants. $H(x,s,\xi)$ is a Carathéodory-type function satisfying to:

$$\left|H(x,s,\xi)\right|\leq\gamma\frac{|\xi|^2}{(1+|s|)^{1-q}} \tag{3}$$

for some $\gamma>0$.

In [1], Arcoya, Boccardo, and Leonor obtained the existence and regularity results for the following elliptic problem with degenerate coercivity:

$$-\operatorname{div}\left(\frac{\alpha\nabla u}{(1+|u|)^2}\right)+u=\gamma\frac{|\nabla u|^2}{(1+|u|)^3}+f, \quad x\in\Omega,\ u\in H_0^1(\Omega), \tag{4}$$

where $\alpha,\gamma>0$, $f\in L^m(\Omega)$ with $m\geq 2$, and $\Omega$ is a bounded subset of $\mathbb{R}^N$, $N\geq 3$.

The purpose of the present paper is to study the same kind of lower order terms as in problem (4) in the case of an elliptic operator with unbounded coefficients such as (1).

There are several papers concerned with existence and regularity of the solution for the following problem:

$$\begin{aligned}-\operatorname{div}(M(x,u)\nabla u)+g(x,u,\nabla u)&=f(x) \quad x\in\Omega,\\ u(x)&=0 \quad x\in\partial\Omega.\end{aligned} \tag{5}$$

We refer the intersting articles: Boccardo, Murat and Puel [2], Bensoussan, Boccardo and Murat [3], and Boccardo, Gallout [4]. In all these works $g$ is a nonlinear lower term having natural growth with respect to $\nabla u$, data $f$ in suitable Lebesgue spaces, and $M(x,u)$ is a Carathéodory-type bounded function subject to certain structural inequalities.

Another motivation for studying these problem arises from the calculus of variations in the case where $0 \leq f \in L^m(\Omega)$ with $m \geq N/2$ and

$$g(x,u,\nabla u) = \frac{|\nabla u|^2}{u^\theta}, \tag{6}$$

where $\theta \in (0,1)$, which is considered by Puel in [5].

We point out that in [6] the authors considered $M(x,u)$ as a bounded function and

$$g(x,u,\nabla u) = \frac{Q(x,u)|\nabla u|^2}{u^\theta}, \tag{7}$$

where $\theta \in (0,1]$. The function $Q(x,s) : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}^{N^2}$ is symmetric, measurable with respect to $x$ and continuous with respect to $s$ with the following uniform ellipticity condition: for $x \in \Omega$, and $s \in \mathbb{R}$,

$$\mu|\xi|^2 \leq Q(x,s)\xi\xi \leq \nu|\xi|^2, \quad 0 < \mu \leq \nu. \tag{8}$$

We shall prove the following main results on existence and regularity of solutions for problem (1).

**Theorem 1.** *Let $\tilde{\alpha} = \min\{1,\alpha\}$. Assuming that the functions $a$ and $H$ satisfy (2) and (3) then, if $f$ belong to $L^m(\Omega)$, with*

$$m > 2\left(\frac{\gamma}{\tilde{\alpha}} + 1\right) + q, \tag{9}$$

*there exists a distributional solution $u \in W_0^{1,1}(\Omega)$ of problem (1) such that*

$$\begin{gathered} H(x,u,\nabla u) \in L^1(\Omega), \quad [a(x) + |u|^q]|\nabla u| \in L^1(\Omega), \\ \int_\Omega [a(x) + |u|^q]\nabla u \nabla \psi + \int_\Omega u\psi \\ = \int_\Omega H(x,u,\nabla u)\psi + \int_\Omega f\psi, \quad \forall \psi \in C_0^\infty(\Omega). \end{gathered} \tag{10}$$

*Furthermore, any solution of the problem (1) belongs to $H_0^1(\Omega)$.*

In the next result, we consider the case where $f$ has a high summability.

**Theorem 2.** *Let $\tilde{\alpha} = \min\{1,\alpha\}$, and assume that (2) and (3) hold true. If $u$ the solution given by Theorem 1 and $f$ belongs to $L^m(\Omega)$, with*

$$m > \max\left\{2\left(\frac{\gamma}{\tilde{\alpha}} + 1\right) + q, \frac{N}{2}\left(\frac{\gamma}{\tilde{\alpha}} + 1\right)\right\}, \tag{11}$$

*then $u$ belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$.*

The rest of the paper is organized as follows: Section 2 is devoted to give some a priori estimates for the approximated problem associated with problem (1); while in Section 3, we give the detailed proofs of Theorems 1 and 2.

## 2. The Approximated Problem

In this section, we use the hypotheses (2) and (3) and we suppose that

$$\tilde{\alpha}(m-1) - \gamma > 0, \tag{12}$$

where $\tilde{\alpha} = \min\{1,\alpha\}$ holds true. To prove Theorem 1 and Theorem 2, we will use the following approximating problems associated with problem (1):

$$\begin{gathered} -\operatorname{div}\left(\left[a(x) + |u_n|^q\right]\nabla u_n\right) + u_n \\ = H_n(x,u_n,\nabla u_n) + f_n, \quad x \in \Omega, \end{gathered} \tag{13}$$

where

$$f_n(x) = \frac{f(x)}{1 + (1/n)|f(x)|}, \tag{14}$$

and

$$H_n(x,s,\xi) = \frac{H(x,s,\xi)}{1 + (1/n)|\xi|^2}. \tag{15}$$

By the results of [2, 4] there exists a weak solution $u_n$ in $H_0^1(\Omega) \cap L^\infty(\Omega)$ of problem (13) in the sense that

$$\begin{gathered} \int_\Omega \left[a(x) + |u_n|^q\right]\nabla u_n \nabla \varphi + \int_\Omega u_n \varphi \\ = \int_\Omega H_n(x,u_n,\nabla u_n)\varphi + \int_\Omega f_n \varphi \end{gathered} \tag{16}$$

for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

The following lemma will be very useful, as it gives us an a priori estimate on the summability of the solutions to problems (13).

**Lemma 3.** *If $u_n$ is a solution to problem (13), then for every $k \geq 0$,*

$$\int_\Omega |G_k(u_n)|^m \leq \int_{\{|u_n| \geq k\}} |f|^m. \tag{17}$$

*Moreover, there exist $R > 0$ depending on $\|f\|_{L^m(\Omega)}$, $\alpha$, $q$, and $\gamma$, such that*

$$\|u_n\|_{H_0^1(\Omega)} \leq R. \tag{18}$$

*Remark 4.* (i) Let $\{u_n\}$ be a sequence of solutions $u_n$ of (13). As a consequence of Lemma 3, there exists $u \in H_0^1(\Omega)$ such that, up to a subsequence, $u_n$ converges weakly to $u$ in $H_0^1(\Omega)$ and a.e. in $\Omega$.

(ii) By the previous lemma we deduce from (3) that

$$\|H_n(x,u_n,\nabla u_n)\|_{L^1(\Omega)} \leq \gamma \int_\Omega \frac{|\nabla u_n|^2}{(1 + |u_n|)^{1-q}} \leq \gamma R^2. \tag{19}$$

*Proof of Lemma 3.* In order to prove (17), we claim that by assumption (2) and $q < 1$, there exist positive constant $c_0$ such that

$$\tilde{\alpha}(1 + |t|)^q \leq a(x) + |t|^q \leq c_0(1 + |t|)^q, \quad \forall t \in \mathbb{R}. \tag{20}$$

Choosing $\varphi = |G_k(u_n)|^{m-1}\mathrm{sgn}(u_n)$ in (16) and using (20), we obtain

$$\begin{aligned}\tilde{\alpha}(m-1)\int_\Omega (1+|u_n|)^q |\nabla u_n|^2 |G_k(u_n)|^{m-2} \\ + \int_\Omega |u_n| |G_k(u_n)|^{m-1} \\ \le \gamma \int_\Omega \frac{|\nabla u_n|^2}{(1+|u_n|)^{1-q}} |G_k(u_n)|^{m-1} \\ + \int_\Omega |f_n| |G_k(u_n)|^{m-1}.\end{aligned} \tag{21}$$

Thus, joining the terms involving the gradient, we get

$$\begin{aligned}\int_\Omega \left[\tilde{\alpha}(m-1) - \gamma \frac{|G_k(u_n)|}{1+|u_n|}\right](1+|u_n|)^q |\nabla u_n|^2 \\ \cdot |G_k(u_n)|^{m-2} + \int_\Omega |G_k(u_n)|^m \le \int_\Omega |f_n| \\ \cdot |G_k(u_n)|^{m-1}.\end{aligned} \tag{22}$$

Using (12) we deduce that

$$\int_\Omega |G_k(u_n)|^m \le \int_\Omega |f| |G_k(u_n)|^{m-1}, \tag{23}$$

and the Hölder inequality on the right hand side yields

$$\begin{aligned}\int_\Omega |G_k(u_n)|^m \\ \le \left(\int_{\{|u_n|\ge k\}} |f|^m\right)^{1/m} \left(\int_\Omega |G_k(u_n)|^m\right)^{1-1/m},\end{aligned} \tag{24}$$

which implies (17).

Let us choose now $\varphi = [(1+|u_n|)^{m-1} - 1]\mathrm{sgn}(u_n)$ as a test function in (16), and we obtain

$$\begin{aligned}(\tilde{\alpha}(m-1) - \gamma)\int_\Omega (1+|u_n|)^{m-2+q} |\nabla u_n|^2 \\ \le \|f\|_{L^m(\Omega)} \left(\int_\Omega (1+|u_n|)^m\right)^{1-1/m}.\end{aligned} \tag{25}$$

Since $m \ge 2$, the previous calculations imply

$$\int_\Omega |\nabla u_n|^2 \le c\left(\int_\Omega (1+|u_n|)^m\right)^{1-1/m}. \tag{26}$$

Using (17) with $k = 0$, (18) follows. □

**Lemma 5.** *Let $u_n$ be the sequence of solutions to problems (13) and let the function $u$ given by Remark 4. Then $u_n$ strongly converges to $u$ in $L^m(\Omega)$. Moreover $\nabla u_n$ strongly converges to $\nabla u$ in $L^1(\Omega)^N$.*

*Remark 6.* Note that (25) implies that there exists $\delta > 0$ independent of $n$ such that

$$\int_\Omega (1+|u_n|)^{m-2+q} |\nabla u_n|^2 \le \delta. \tag{27}$$

By using the previous lemma, we deduce that

$$\begin{aligned}(1+|u_n|)^{(m-2+q)/2} |\nabla u_n| \longrightarrow (1+|u|)^{(m-2+q)/2} |\nabla u| \\ \text{weakly in } L^2(\Omega)^N.\end{aligned} \tag{28}$$

*Proof of Lemma 5.* We use (17) written for $k = 0$:

$$\int_\Omega |u_n|^m \le \int_\Omega |f|^m \le c. \tag{29}$$

Since $u_n$ almost everywhere converges to $u$, we have from Fatou's lemma that

$$\int_\Omega |u|^m \le c. \tag{30}$$

Hence $u$ belongs to $L^m(\Omega)$. Using assumption (17), for any $k > 0$ we have

$$\begin{aligned}\int_E |u_n|^m \le \int_{E\cap\{|u_n|\le k\}} |u_n|^m + \int_{E\cap\{|u_n|\ge k\}} |u_n|^m \\ \le k^m meas(E) + \int_{\{|u_n|\ge k\}} |f|^m.\end{aligned} \tag{31}$$

As before, we first choose $k$ such that the second integral is small, uniformly with respect to $n$, and then the measure of $E$ small enough such that the first term is small. The almost everywhere convergence of $u_n$ to $u$ and Vitali's theorem imply that $u_n$ strongly converges to $u$ in $L^m(\Omega)$.

For the second convergence, we will follow the same technique as in [1] (see also [7]). Let $h, k > 0$. In the sequel $C$ will denote a constant independent of $n, h, k$. Let us consider $T_h[u_n - T_k(u)]$ as a test function in problems (16). Then,

$$\begin{aligned}\int_\Omega [a(x) + |u_n|^2] \nabla u_n \nabla T_h[u_n - T_k(u)] \\ + \int_\Omega u_n T_h[u_n - T_k(u)] \\ \le \left(\|f\|_{L^1(\Omega)} + \|H_n(x, u_n, \nabla u_n)\|_{L^1(\Omega)}\right) h.\end{aligned} \tag{32}$$

Moreover, thanks to the $L^m(\Omega)$ convergence of $u_n$, the second integral in (32) converges (as n diverges) to a positive number. Thus, it yields to

$$\begin{aligned}\alpha \int_\Omega |\nabla T_h[u_n - T_k(u)]|^2 \\ \le \left(\|f\|_{L^1(\Omega)} + \gamma R^2\right) h \\ - \int_\Omega [a(x) + |u_n|^q] \nabla T_k(u) \nabla T_h[u_n - T_k(u)].\end{aligned} \tag{33}$$

Let $\mathscr{K} = h+k$, observing that $\nabla T_h[u_n - T_k(u)] = 0$ if $|u_n| > \mathscr{K}$, then

$$\begin{aligned}\int_\Omega [a(x) + |u_n|^q] \nabla T_k(u) \nabla T_h[u_n - T_k(u)] \\ = \int_\Omega [a(x) + |T_{\mathscr{K}}(u_n)|^q] \nabla T_k(u) \nabla T_h[u_n - T_k(u)].\end{aligned} \tag{34}$$

Since $T_h[u_n - T_k(u)]$ converges to $T_h[u - T_k(u)]$ weakly in $(L^2(\Omega))^N$ and $[a(x) + |T_{\mathcal{K}}(u_n)|^q]\nabla T_k(u)$ strongly converges to $[a(x) + |T_{\mathcal{K}}(u)|^q]\nabla T_k(u)$ in $(L^2(\Omega))^N$, we have

$$\lim_{n\longrightarrow+\infty} \int_\Omega \left[a(x) + |u_n|^q\right] \nabla T_k(u) \nabla T_h\left[u_n - T_k(u)\right] = 0, \tag{35}$$

thus, yielding

$$\int_\Omega \left|\nabla T_h\left[u_n - T_k(u)\right]\right|^2 \le Ch + \varepsilon(n), \tag{36}$$

where $\varepsilon(n)$ denote any quantity that vanishes as $n$ diverges. Hence, by Hölder's inequality, we deduce that

$$\int_{\{|u_n-u|\le h, |u|\le k\}} |\nabla(u_n - u)| = \int_\Omega \left|\nabla T_h\left[u_n - T_k(u)\right]\right| \le |\Omega|^{1/2}\sqrt{Ch + \varepsilon(n)}. \tag{37}$$

Fix, now, $\epsilon > 0$ there exist $h_0$ such that, for $h < h_0$, we have

$$|\Omega|^{1/2}\sqrt{Ch} < \epsilon. \tag{38}$$

Thanks to the weak convergence of $u_n$ in $H_0^1(\Omega)$ and the absolute continuity of the integral, there exists $k_0$ independent from $n$ such that, for $k > k_0$, we have

$$\int_{\{|u|>k\}} |\nabla u_n| + \int_{\{|u|>k\}} |\nabla u| \le \epsilon. \tag{39}$$

In addition, by Dunford Pettis Theorem, we deduce that there exists $n(h,\epsilon)$ such that, for $n > n(h,\epsilon)$, we have

$$\int_{\{|u_n-u|>h\}} |\nabla(u_n - u)| \le \epsilon. \tag{40}$$

We can write

$$\begin{aligned}\int_\Omega |\nabla(u_n - u)| = & \int_{\{|u_n-u|\le h, |u|\le k\}} |\nabla(u_n - u)| \\ & + \int_{\{|u_n-u|\le h, |u|>k\}} |\nabla(u_n - u)| \\ & + \int_{\{|u_n-u|>h\}} |\nabla(u_n - u)|.\end{aligned} \tag{41}$$

Using (37), (39), and (40), for $h < h_0$ and $n > n(h,\epsilon)$, we have

$$\int_\Omega |\nabla(u_n - u)| \le 3\epsilon + \varepsilon(n). \tag{42}$$

This proves the strong convergence of $\nabla u_n$ to $\nabla u$ in $L^1(\Omega)^N$. ☐

The following lemma yields some a priori estimate on $\{u_n\}$.

**Lemma 7.** *Let $u$ be the function given by Remark 4. Then $|u|^q|\nabla u|$ belongs to $L^r(\Omega)$, for every $r < N/(N-1)$.*

*Proof.* For every $\lambda > 1$, we take $[1 - 1/(1 + |u_n|)^{\lambda-1}]\mathrm{sign}(u_n)$ as a test function in (16). Droping positive terms yields

$$\tilde{\alpha}(\lambda - 1)\int_\Omega \frac{\left(1 + |u_n|^q\right)|\nabla u_n|^2}{\left(1 + |u_n|\right)^\lambda} \le \|f\|_{L^1(\Omega)} + \left\|H_n(x, u_n, \nabla u_n)\right\|_{L^1(\Omega)}. \tag{43}$$

Hence, using $q < 1$, it follows that

$$\int_\Omega \frac{|\nabla u_n|^2}{\left(1 + |u_n|\right)^{\lambda-q}} \le \frac{\|f\|_{L^1(\Omega)} + \gamma R^2}{\tilde{\alpha}(\lambda - 1)}. \tag{44}$$

On the other hand, for every $\lambda > 1$; we have

$$\begin{aligned}\int_\Omega |u_n|^{qr}|\nabla u_n|^r & \le \int_\Omega \frac{|\nabla u_n|^r}{\left(1 + |u_n|\right)^{r(\lambda-q)/2}}\left(1 + |u_n|\right)^{r(\lambda+q)/2}, \\ & \le \left(\frac{\|f\|_{L^1(\Omega)} + \gamma R^2}{\tilde{\alpha}(\lambda - 1)}\right)^{r/2} \\ & \quad \cdot \left(\int_\Omega \left(1 + |u_n|\right)^{r(\lambda+q)/(2-r)}\right)^{(2-r)/2}.\end{aligned} \tag{45}$$

Then, we obtain

$$\left(\int_\Omega |u_n|^{(q+1)r^*}\right)^{r/r^*} \le c\left(\int_\Omega |u_n|^{r(\lambda+q)/(2-r)}\right)^{(2-r)/2}. \tag{46}$$

Let us choose $r$ such that $(q+1)r^* = r(\lambda+q)/(2-r)$, that is

$$r = \frac{N(2 + q - \lambda)}{N(q+1) - (\lambda + q)}. \tag{47}$$

Since $\lambda > 1$, we then have an estimate on $|u_n|^q|\nabla u_n|$ in $L^r(\Omega)$, for every $r < N/(N-1)$. ☐

The next result will be used in the proof of Theorem 2.

**Lemma 8.** *Suppose that (2), (3), and (11) hold true. Let $f \in L^m(\Omega)$ and $\{u_n\}$ be a solution of (13) with $f_n = f$ for every $n \in \mathbb{N}$. Then the norms of $\{u_n\}$ in $L^\infty(\Omega)$ and in $H_0^1(\Omega)$ are bounded by a constant which depends on $q, m, N, \alpha, \gamma$, meas$(\Omega)$ and on the norm of $f$ in $L^m(\Omega)$.*

*Proof.* Since $m > (N/2)(\gamma/\tilde{\alpha} + 1)$, we have $(1/2)(\gamma/\tilde{\alpha} + 1) < m/N$. Let us choose $\sigma > 0$ such that

$$\frac{1}{2}\left(\frac{\gamma}{\tilde{\alpha}} + q + 1\right) < \sigma < \frac{m}{N} + \frac{q}{2}. \tag{48}$$

The use of

$$\left[\left(1 + |u_n|\right)^{2\sigma-q-1} - (1+k)^{2\sigma-q-1}\right]^+ \mathrm{sign}(u_n), \tag{49}$$

as test function in (16), (3), and (20), implies that

$$
\begin{aligned}
&(2\sigma-q-1)\tilde{\alpha}\int_{A_k}|\nabla u_n|^2(1+|u_n|)^{2\sigma-2}\\
&\quad+\int_{A_k}|u_n|\left[(1+|u_n|)^{2\sigma-q-1}-(1+k)^{2\sigma-q-1}\right]\\
&\le \gamma\int_{A_k}\frac{|\nabla u_n|^2}{(1+|u_n|)^{1-q}}(1+|u_n|)^{2\sigma-q-1}\\
&\quad+\int_{A_k}|f_n|(1+|u_n|)^{2\sigma-q-1},
\end{aligned}
\tag{50}
$$

where

$$A_k=\{x\in\Omega:|u_n|>k\}. \tag{51}$$

By Young and Hölder's inequalities, we find

$$
\begin{aligned}
&[(2\sigma-q-1)\tilde{\alpha}-\gamma]\int_{A_k}|\nabla u_n|^2(1+|u_n|)^{2\sigma-2}\\
&\le C_1\int_{A_k}(1+|u_n|)^{2\sigma-q}+C_2\int_{A_k}|f|^{2\sigma-q}\\
&\le C_m(\operatorname{meas}A_k)^{1-(2\sigma-q)/m}.
\end{aligned}
\tag{52}
$$

Then, using Sobolev's inequality gives

$$
\begin{aligned}
&[(2\sigma-q-1)\tilde{\alpha}-\gamma]\\
&\cdot\frac{\mathcal{S}^2}{\sigma^2}\left(\int_{A_k}\left[(1+|u_n|)^{\sigma}-(1+k)^{\sigma}\right]^{2^*}\right)^{2/2^*}\\
&\le C_m(\operatorname{meas}A_k)^{1-(2\sigma-q)/m},
\end{aligned}
\tag{53}
$$

where $\mathcal{S}$ denotes the best constant in Sobolev inequality. Now, we set

$$(1+|u_n|)^{\sigma}=v_n \tag{54}$$

and

$$(1+k)^{\sigma}=h. \tag{55}$$

and the fact that $A_k=\{x\in\Omega:v_n>h\}$., the last inequality gives

$$
\begin{aligned}
&[(2\sigma-q-1)\tilde{\alpha}-\gamma]\frac{\mathcal{S}^2}{\sigma^2}\left(\int_{A_k}(v_n-h)^{2^*}\right)^{2/2^*}\\
&\le C_m(\operatorname{meas}A_k)^{1-(2\sigma-q)/m}.
\end{aligned}
\tag{56}
$$

Note that $\sigma < m/N+q/2$ implies that $[1-(2\sigma-q)/m](2^*/2)>1$. Then Stampacchia's technique implies the following relation for some positive constant $C_3$,

$$\|v_n\|_{L^\infty(\Omega)}=\left\|(1+|u_n|)^{\sigma}\right\|_{L^\infty(\Omega)}\le C_3, \tag{57}$$

that is, $\|u_n\|_{L^\infty(\Omega)}$ is bounded. □

## 3. Proof of the Main Results

We are now ready to prove the main result of this paper. We first observe that condition (9) implies (12). Hence the results of the previous section hold true. In order to prove the result, we have to pass to the limit in (16). To this aim, let $g$ be a function in $C^1(\mathbb{R})$ such that

$$
g(s)=\begin{cases}\dfrac{1+s}{\tilde{\alpha}\rho-\gamma} & \text{if } s\ge 0\\[2mm] \dfrac{1}{(1-s)(\tilde{\alpha}\rho-\gamma)} & \text{if } s<0,\end{cases}
\tag{58}
$$

where

$$\rho=\frac{m-q-2}{2}. \tag{59}$$

Observe that, by (9), $g$ is positive, increasing, and it verifies

$$\tilde{\alpha}\rho g'(s)-\gamma\frac{g(s)}{1+|s|}>0,\quad \forall s\in\mathbb{R}. \tag{60}$$

We will use, for $k>0$ and $s\in\mathbb{R}$,

$$R_k(s)=1-T_1(G_k(s)), \tag{61}$$

to define a test function. Remark that $R_k\ge 0$, $-k-1\le R_k(s)\le k+1$ and

$$
R_k'(s)=\begin{cases}1 & \text{if } -k-1\le s\le -k\\ -1 & \text{if } k\le s\le k+1\\ 0 & \text{otherwise.}\end{cases}
\tag{62}
$$

First of all, note that the a.e. convergence of $\nabla u_n$ (see Lemma 5), Remark 6, and (20) imply both that

$$
\begin{aligned}
&\left[a(x)+|u_n|^q\right]g^{\rho}(u_n)\nabla u_n\longrightarrow\\
&\left[a(x)+|u|^q\right]g^{\rho}(u)\nabla u\quad\text{weakly in } L^2(\Omega)^N
\end{aligned}
\tag{63}
$$

and

$$
\begin{aligned}
&\left[a(x)+|u_n|^q\right]\frac{1}{g^{\rho}(u_n)}\nabla u_n\longrightarrow\\
&\left[a(x)+|u|^q\right]\frac{1}{g^{\rho}(u)}\nabla u\quad\text{weakly in } L^2(\Omega)^N,
\end{aligned}
\tag{64}
$$

where $\rho$ is defined in (59).

The proof of the result will be achieved in two steps.

*Step 1* (The first inequality). We fix $\psi\in H_0^1(\Omega)\cap L^\infty(\Omega)$, with $\psi\ge 0$, and take

$$\phi=\frac{g^{\rho}(u_n)}{g^{\rho}(u)}R_k(u)\psi \tag{65}$$

As test function in (16), we have that

$$\begin{aligned}
&\int_\Omega \left[a(x)+|u_n|^q\right]\nabla u_n \nabla\psi \frac{g^\rho(u_n)}{g^\rho(u)} R_k(u) \\
&- \rho\int_\Omega \left[a(x)+|u_n|^q\right]\nabla u_n \nabla u \frac{g^\rho(u_n)}{g^{\rho+1}(u)} g'(u) R_k(u) \\
&\cdot \psi + \int_\Omega \left[a(x)+|u_n|^q\right]\nabla u_n \nabla u \frac{g^\rho(u_n)}{g^\rho(u)} R_k'(u)\psi \\
&+ \rho\int_\Omega \left[a(x)+|u_n|^q\right]\nabla u_n \nabla u_n \frac{g^{\rho-1}(u_n)}{g^\rho(u)} g'(u_n) \\
&\cdot R_k(u)\psi - \int_\Omega H_n(x,u_n,\nabla u_n)\frac{g^\rho(u_n)}{g^\rho(u)} R_k(u)\psi \\
&+ \int_\Omega u_n \frac{g^\rho(u_n)}{g^\rho(u)} R_k(u)\psi = \int_\Omega f_n \frac{g^\rho(u_n)}{g^\rho(u)} R_k(u)\psi.
\end{aligned} \tag{66}$$

Remark now that, by the assumptions on $a$, $H$, relation (60) and the fact that $\psi \geq 0$, then we have

$$\begin{aligned}
&\rho\left[a(x)+|u_n|^q\right]\nabla u_n \nabla u_n \frac{g^{\rho-1}(u_n)}{g^\rho(u)} g'(u_n) R_k(u)\psi \\
&- H_n(x,u_n,\nabla u_n)\frac{g^\rho(u_n)}{g^\rho(u)} R_k(u)\psi \geq (1+|u_n|)^q \\
&\cdot |\nabla u_n|^2 \frac{g^{\rho-1}(u_n)}{g^\rho(u)} R_k(u) \\
&\cdot \psi\left[\tilde{\alpha}\rho g'(u_n) - \gamma\frac{g(u_n)}{1+|u_n|}\right] \geq 0.
\end{aligned} \tag{67}$$

Therefore, using the almost everywhere convergence of both $\nabla u_n$ and $u_n$, and applying Fatou's lemma, we get

$$\begin{aligned}
&\liminf_{n\to\infty} \rho\int_\Omega \left[a(x)+|u_n|^q\right]\nabla u_n \nabla u_n \frac{g^{\rho-1}(u_n)}{g^\rho(u)} g'(u_n) \\
&\cdot R_k(u)\psi - \int_\Omega H_n(x,u_n,\nabla u_n)\frac{g^\rho(u_n)}{g^\rho(u)} R_k(u)\psi \\
&\geq \rho\int_\Omega \left[a(x)+|u|^q\right]\nabla u \nabla u \frac{g'(u)}{g(u)} R_k(u)\psi \\
&- \int_\Omega H(x,u,\nabla u) R_k(u)\psi.
\end{aligned} \tag{68}$$

Furthermore, by using Lebesgue's theorem and (63), we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \int_\Omega \left[a(x)+|u_n|^q\right]\nabla u_n \nabla\psi \frac{g^\rho(u_n)}{g^\rho(u)} R_k(u) \\
&= \int_\Omega \left[a(x)+|u|^q\right]\nabla u \nabla\psi R_k(u),
\end{aligned} \tag{69}$$

and

$$\begin{aligned}
&\lim_{n\to\infty} \rho\int_\Omega \left[a(x)+|u_n|^q\right]\nabla u_n \nabla u \frac{g^\rho(u_n)}{g^{\rho+1}(u)} g'(u) R_k(u) \\
&\cdot \psi = \rho\int_\Omega \left[a(x)+|u|^q\right]\nabla u \nabla u \frac{g'(u)}{g(u)} R_k(u)\psi.
\end{aligned} \tag{70}$$

Similarly, using the convergence $(u_n - f_n) \longrightarrow (u - f)$ in $L^m(\Omega)$, we have

$$\begin{aligned}
&\lim_{n\to\infty} \int_\Omega (u_n - f_n)\frac{g^\rho(u_n)}{g^\rho(u)} R_k(u)\psi \\
&= \int_\Omega (u-f) R_k(u)\psi.
\end{aligned} \tag{71}$$

Now, from (62), we get

$$\begin{aligned}
&\lim_{n\to\infty} \int_\Omega \left[a(x)+|u_n|^q\right]\nabla u_n \nabla u \frac{g^\rho(u_n)}{g^\rho(u)} R_k'(u)\psi \\
&= \int_\Omega \left[a(x)+|u|^q\right]\nabla u \nabla u R_k'(u) \\
&= \int_{\{k\leq|u|\leq k+1\}} \left[a(x)+|u|^q\right]\nabla u \nabla u.
\end{aligned} \tag{72}$$

Passing to the limit in (66) when $n$ tends to infinity and gathering together (68)-(72), weobtain

$$\begin{aligned}
&\int_\Omega \left[a(x)+|u|^q\right]\nabla u \nabla\psi R_k(u) + \int_\Omega u R_k(u)\psi \\
&+ \int_{\{k\leq|u|\leq k+1\}} \left[a(x)+|u|^q\right]\nabla u \nabla u \\
&- \int_\Omega H(x,u,\nabla u) R_k(u)\psi \leq \int_\Omega f R_k(u)\psi.
\end{aligned} \tag{73}$$

Choosing $T_1(G_k(u_n))$ in (16), we get

$$\begin{aligned}
&\int_{\{k\leq|u_n|\leq k+1\}} \left[a(x)+|u_n|^q\right]\nabla u_n \nabla u_n \\
&\leq \int_{\{k\leq|u_n|\}} |f| + \int_{\{k\leq|u_n|\}} |H(x,u_n,\nabla u_n)| \\
&\leq \int_{\{k\leq|u_n|\}} |f| + \frac{\gamma R^2}{(1+k)^{1-q}}.
\end{aligned} \tag{74}$$

By Fatou's lemma, we have

$$\lim_{k\to\infty} \int_{\{k\leq|u|\leq k+1\}} \left[a(x)+|u|^q\right]\nabla u \nabla u = 0. \tag{75}$$

In order to pass to the limit as $k$ tends to infinity in the inequality (73), we recall that $H(x,u,\nabla u) \in L^1(\Omega)$ and $[a(x)+|u|^q]\nabla u \in L^1(\Omega)$,. We obtain

$$\begin{aligned}
&\int_\Omega \left[a(x)+|u|^q\right]\nabla u \nabla\psi + \int_\Omega u\psi \\
&\leq \int_\Omega H(x,u,\nabla u)\psi + \int_\Omega f\psi,
\end{aligned} \tag{76}$$

for every $\psi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, with $\psi \geq 0$; that is, $u$ is a subsolution of problem (1).

*Step 2* (The second inequality). Let $\psi$ be in $H_0^1(\Omega) \cap L^\infty(\Omega)$, with $\psi \leq 0$, and $g$ be given by (58), and choose

$$\phi = \frac{g^\rho(u)}{g^\rho(u_n)} R_k(u)\psi \tag{77}$$

asa test function in (16). We obtain

$$\begin{aligned}
&\int_\Omega \left[a(x) + |u_n|^q\right] \nabla u_n \nabla \psi \frac{g^\rho(u)}{g^\rho(u_n)} R_k(u) \\
&+ \rho \int_\Omega \left[a(x) + |u_n|^q\right] \nabla u_n \nabla u \frac{g^{\rho-1}(u)}{g^\rho(u_n)} g'(u) R_k(u) \\
&\cdot \psi + \int_\Omega \left[a(x) + |u_n|^q\right] \nabla u_n \nabla u \frac{g^\rho(u)}{g^\rho(u_n)} R_k'(u)\psi \\
&- \rho \int_\Omega \left[a(x) + |u_n|^q\right] \nabla u_n \nabla u_n \frac{g^\rho(u)}{g^{\rho+1}(u_n)} g'(u_n) \\
&\cdot R_k(u)\psi - \int_\Omega H_n(x, u_n, \nabla u_n) \frac{g^\rho(u)}{g^\rho(u_n)} R_k(u)\psi \\
&+ \int_\Omega u_n \frac{g^\rho(u)}{g^\rho(u_n)} R_k(u)\psi = \int_\Omega f_n \frac{g^\rho(u)}{g^\rho(u_n)} R_k(u)\psi.
\end{aligned} \tag{78}$$

We observe that, by (60) and the fact that $\psi \leq 0$, we have

$$\begin{aligned}
&- \rho \left[a(x) + |u_n|^q\right] \nabla u_n \nabla u_n \frac{g^\rho(u)}{g^{\rho+1}(u_n)} g'(u_n) R_k(u)\psi \\
&- H_n(x, u_n, \nabla u_n) \frac{g^\rho(u)}{g^\rho(u_n)} R_k(u)\psi \geq -(1 + |u_n|)^q \\
&\cdot |\nabla u_n|^2 \frac{g^\rho(u)}{g^{\rho+1}(u_n)} R_k(u) \\
&\cdot \psi \left[\tilde{\alpha}\rho g'(u_n) - \gamma \frac{g(u_n)}{1 + |u_n|}\right] \geq 0.
\end{aligned} \tag{79}$$

Applying the same argument of Step 1 and using (64) instead of (63), we deduce that

$$\begin{aligned}
&\int_\Omega [a(x) + |u|^q] \nabla u \nabla \psi + \int_\Omega u\psi \\
&\quad \leq \int_\Omega H(x, u, \nabla u)\psi + \int_\Omega f\psi,
\end{aligned} \tag{80}$$

for every $\psi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, with $\psi \leq 0$.

Consequently, summarizing Steps 1 and 2, we have

$$\begin{aligned}
&\int_\Omega [a(x) + |u|^q] \nabla u \nabla \psi + \int_\Omega u\psi \\
&\quad \leq \int_\Omega H(x, u, \nabla u)\psi + \int_\Omega f\psi,
\end{aligned} \tag{81}$$

for every $\psi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

Finally, interchanging $\psi$ and $-\psi$ we conclude that

$$\begin{aligned}
&\int_\Omega [a(x) + |u|^q] \nabla u \nabla \psi + \int_\Omega u\psi \\
&\quad = \int_\Omega H(x, u, \nabla u)\psi + \int_\Omega f\psi,
\end{aligned} \tag{82}$$

for every $\psi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

## Data Availability

The authors do not have data available.

## Disclosure

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## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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