Hindawi
International Journal of Differential Equations
Volume 2018, Article ID 6402576, 13 pages
http://dx.doi.org/10.1155/2018/6402576

Research Article

Linear θ-Method and Compact θ-Method for Generalised Reaction-Diffusion Equation with Delay

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Received 7 March 2018; Accepted 15 May 2018; Published 11 June 2018

Academic Editor: Jiwei Zhang

This paper is concerned with the analysis of the linear θ-method and compact θ-method for solving delay reaction-diffusion equation. Solvability, consistence, stability, and convergence of the two methods are studied. When $\theta \in [0, 1/2)$, sufficient and necessary conditions are given to show that the two methods are asymptotically stable. When $\theta \in [1/2, 1]$, the two methods are proven to be unconditionally asymptotically stable. Finally, several examples are carried out to confirm the theoretical results.

1. Introduction

Partial functional differential equations (PFDEs) are widely used to model many natural phenomena in various scientific fields [1–9]. In order to gain a better understanding of the complicated dynamics, numerous researchers have investigated PFDEs. For instance, Garrido-Atienza and Real discussed the existence and uniqueness of solutions for delay evolution equations [10]. Mei et al. analysed the stability of travelling waves for nonlocal time-delayed reaction-diffusion equations [11]. Polyanin and Zhurov constructed exact solutions for delay reaction-diffusion equations and more complex nonlinear equations by the functional constraints method [12].

However, the exact solutions are difficult to be obtained [1]. Most researchers have to seek efficient and effective numerical methods to numerically solve PFDEs. Jackiewicz and Zubik-Kowal utilised spectral collocation and waveform relaxation methods to study nonlinear delay partial differential equations [13]. Chen and Wang utilised the variational iteration method to solve a neutral functional-differential equation with proportional delays [14]. Li et al. used the discontinuous Galerkin methods to solve the delay differential equations [15–17]. Bhrawy et al. applied an accurate Chebyshev pseudospectral scheme to study the multidimensional parabolic problems with time delays [18]. Aziz and Amin employed the Haar wavelet to study the numerical solution of a class of delay differential and delay partial differential equations [19].

When it comes to solving PFDEs numerically, here comes one question, that is, whether the numerical solution approximates the exact solution in a stable manner, especially for a long time. In this study, we use the following model as the test equation for analysing stability of the numerical method, which is an extension to the previous work [20–22].

$$\begin{aligned} \frac{\partial}{\partial t}u(x,t) &= r_1\frac{\partial^2}{\partial x^2}u(x,t) + r_2\frac{\partial^2}{\partial x^2}u(x,t-\tau) \\ &\quad + r_3 u(x,t) + r_4 u(x,t-\tau), \\ &\qquad t>0,\ 0<x<\pi, \\ u(x,t) &= u_0(x,t), \quad -\tau \le t \le 0,\ 0 \le x \le \pi, \\ u(0,t) &= u(\pi,t) = 0, \quad t \ge -\tau. \end{aligned} \tag{1}$$

Here and hereafter parameters $r_1 > 0$ and $r_2 > 0$ denote the diffusion coefficients, $r_3 \in \mathbb{R}$, $r_4 \in \mathbb{R}$, and $\tau > 0$ is the delay term. In particular, when $r_3 = r_4 = 0$, the above model (1) is reduced to the original test equation in [20–22].

For the case where $r_3 = r_4 = 0$, model (1) has been studied by many researchers [2, 20–27]. In this work, we will examine the case where $r_1 > 0$, $r_2 > 0$, $r_3 \in \mathbb{R}$, and $r_4 \in \mathbb{R}$, which is a generalisation of above mentioned work, and analyse the stability condition of the numerical method. The standard second-order central difference method and compact finite difference method are utilised to discrete the diffusion operator, respectively, and the linear θ-method is utilised to discrete the temporal direction. For convenience, we name the standard second-order central difference version as linear θ-method and the compact finite difference method version as compact θ-method. With the spectral radius condition, we consider the stability of the linear θ-method and compact θ-method, respectively.

The rest of this paper is organized as follows. In Section 2, we give a sufficient delay-independent condition for Problem (1) to be asymptotically stable. In Section 3, we propose the linear θ-method for solving Problem (1); solvability, stability, and convergence of the method are discussed. In Section 4, we extend the compact θ-method to solve Problem (1). In Section 5, several numerical tests are performed to validate the theoretical results.

2. Stability of PFDE (1)

In this section, based on Tian's work [20], we give a sufficient condition for the trivial solution of Problem (1) to be asymptotically stable.

Definition 1. The trivial solution $u(x,t) \equiv 0$ of PFDE (1) is called asymptotically stable if its solution $u(x,t)$ corresponding to a sufficiently differentiable function $u_0(x,t)$ with $u_0(0,t) = u_0(\pi,t) = 0$ satisfies

$$\lim_{t\to\infty} u(x,t) = 0. \tag{2}$$

Lemma 2 (cf. [3]). *All the roots of $pe^z + q - ze^z = 0$, where p and q are real, have negative real parts if and only if*

(1) $p < 1$,

(2) $p < -q < \sqrt{a_1^2 + p^2}$,

where a_1 is the root of $g = p\tan(g)$ such that $0 < g < \pi$. If $p = 0$, we take $a_1 = \pi/2$.

Theorem 3. *Assume that the solution of Problem (1) is $u(x,t) = e^{\lambda t}e^{inx}$, where $\lambda \in \mathbb{C}$, $n \in \mathbb{R}$, $x \in [0,\pi]$, and $t \geq 0$. Then the sufficient condition for the trivial solution of Problem (1) to be asymptotically stable is that*

(1) $\tau r_3 < 1 + \tau r_1 n^2$,

(2) $\tau(r_3 - r_1 n^2) < \tau(r_2 n^2 - r_4) < \sqrt{a_1^2 + \tau^2(r_3 - r_1 n^2)^2}$,

where a_1 is the root of $g = \tau(r_3 - r_1 n^2)\tan(g)$ such that $0 < g < \pi$. If $r_3 = r_1 n^2$, we take $a_1 = \pi/2$.

Proof. Let $X = B[0,\pi]$ denote the Banach space equipped with the maximum norm, and $D(\mathscr{A}) = \{y \in X : y'' \in X, y(0) = y(\pi) = 0\}$, and $\mathscr{A}y = y''$ for $y \in D(\mathscr{A})$.

Let $r_1 n^2 (n = 1, 2, \cdots)$ be the eigenvalues of $-\mathscr{A}$. According to [1, 28], if all zeros of the characteristic equations

$$f(\lambda) = \lambda + r_1 n^2 - r_3 + \left(r_2 n^2 - r_4\right) e^{-\lambda\tau} \tag{3}$$

have negative real part, then the trivial solution is asymptotically stable. Meanwhile, if at least one zero has positive real part, then it is unstable.

Let $f(\lambda) = 0$; that is, $\lambda = r_3 - r_1 n^2 + (r_4 - r_2 n^2)e^{-\lambda\tau}$. Multiplying by $e^{\lambda\tau}$, we have

$$\lambda e^{\lambda\tau} = \left(r_3 - r_1 n^2\right) e^{\lambda\tau} + \left(r_4 - r_2 n^2\right). \tag{4}$$

Setting $\lambda\tau = z$, we get

$$ze^z = \tau\left(r_3 - r_1 n^2\right) e^z + \tau\left(r_4 - r_2 n^2\right). \tag{5}$$

Denote $p = \tau(r_3 - r_1 n^2)$, and $q = \tau(r_4 - r_2 n^2)$, and rewrite the above equation as

$$ze^z = pe^z + q. \tag{6}$$

Applying Lemma 2, if

(1) $\tau r_3 < 1 + \tau r_1 n^2$,

(2) $\tau(r_3 - r_1 n^2) < \tau(r_2 n^2 - r_4) < \sqrt{a_1^2 + \tau^2(r_3 - r_1 n^2)^2}$,

then the real parts of all zeros of the characteristic equations are negative. Therefore, the trivial solution of Problem (1) is asymptotically stable. Otherwise, there exists a zero λ_0 whose real part is positive such that $f(\lambda_0) = 0$. Hence, the trivial solution is unstable. It completes the proof. □

3. Linear θ-Method

In this section, the linear θ-method is presented to solve Problem (1).

Denote $\Omega_{\Delta t} = \{t_k \mid k = -m, -m+1, \cdots\}$ as a uniform partition on the time interval $[-\tau, \infty)$, where $\Delta t = \tau/m$ is the time step size and $t_k = k\Delta t$. Denote $\Omega_{\Delta x} = \{x_j \mid j = 0, 1, \cdots, N\}$ as a uniform mesh on the space interval $\Omega = [0,\pi]$, where $\Delta x = \pi/N$ is the space step size and $x_j = j\Delta x$. Here m and N are two positive integers. Let u_j^k be the numerical approximation of $u(x_j, t_k)$ and let $\mathscr{V} = \{u_j^k \mid 0 \leq j \leq N, k \geq -m\}$ be the grid function defined on $\Omega_{\Delta x} \times \Omega_{\Delta t}$. For any grid function $u \in \mathscr{V}$, we use the following notations:

$$\begin{aligned} \delta_t u_j^k &= \frac{u_j^{k+1} - u_j^k}{\Delta t}, \\ \delta_x^2 u_j^k &= \frac{u_{j+1}^k - 2u_j^k + u_{j-1}^k}{\Delta x^2}, \\ u_j^{k+1/2} &= \frac{1}{2}\left(u_j^k + u_j^{k+1}\right). \end{aligned} \tag{7}$$

Now, applying the standard second-order central difference method to discrete the diffusion operator, we obtain the following linear θ-method:

$$\begin{aligned}\delta_t u_j^k &= r_1\left[(1-\theta)\,\delta_x^2 u_j^k + \theta\delta_x^2 u_j^{k+1}\right]\\ &\quad + r_2\left[(1-\theta)\,\delta_x^2 u_j^{k-m} + \theta\delta_x^2 u_j^{k-m+1}\right]\\ &\quad + r_3\left[(1-\theta)\,u_j^k + \theta u_j^{k+1}\right]\\ &\quad + r_4\left[(1-\theta)\,u_j^{k-m} + \theta u_j^{k-m+1}\right],\\ &\qquad\qquad j = 1,2,\cdots,N-1,\ \ k = 0,1,\cdots,\\ u_j^k &= u_0\left(x_j, t_k\right),\\ &\qquad j = 1,2,\cdots,N-1,\ \ k = -m,-m+1,\cdots,0,\\ u_0^k &= u_N^k = 0, \quad k = -m,-m+1,\cdots.\end{aligned} \tag{8}$$

We can rewrite the linear θ-method (8) as the following matrix form:

$$\begin{aligned}\phi_0(S)\,U^{k+1} &= \phi_1(S)\,U^k - \phi_m(S)\,U^{k+1-m}\\ &\quad - \phi_{m+1}(S)\,U^{k-m},\end{aligned} \tag{9}$$

where

$$\begin{aligned}U^k &= \left(u_1^k, u_2^k, \cdots, u_{N-1}^k\right)^T,\\ a &= \frac{r_1\Delta t}{\Delta x^2},\\ b &= \frac{r_2\Delta t}{\Delta x^2},\\ c &= r_3\Delta t,\\ d &= r_4\Delta t,\\ \phi_0(\eta) &= 1 + 2a\theta - c\theta - a\theta\eta,\\ \phi_1(\eta) &= 1 - 2a(1-\theta) + c(1-\theta) + a(1-\theta)\,\eta,\\ \phi_m(\eta) &= 2b\theta - d\theta - b\theta\eta,\\ \phi_{m+1}(\eta) &= 2b(1-\theta) - d(1-\theta) - b(1-\theta)\,\eta,\\ S &= \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0\\ 1 & 0 & 1 & \dots & 0 & 0\\ & \ddots & \ddots & \ddots & \ddots & \\ 0 & 0 & \dots & 1 & 0 & 1\\ 0 & 0 & \dots & 0 & 1 & 0\end{pmatrix}_{(N-1)\times(N-1)},\end{aligned} \tag{10}$$

and the eigenvalues of matrix S are $\lambda_j = 2\cos(j\Delta x)$, $j = 1,2,\cdots,N-1$.

3.1. Solvability of Linear θ-Method

Theorem 4. *The linear θ-method (8) is solvable and has a unique solution.*

Proof. The mathematical induction is utilised to prove it. We can obtain the solution of U^1 according to the initial condition. Now, assume that the solution of U^l has been determined. Then we can derive the solution of U^{l+1} with (9). It follows from (9) that the coefficient matrix of the linear system is

$$\phi_0(S) = (1 + 2a\theta - c\theta)\,I - a\theta S. \tag{11}$$

It is easy to verify that the matrix $\phi_0(S)$ is symmetric positive definite. Therefore, the solution of U^{l+1} is determined uniquely. By mathematical induction, the existence and uniqueness of the solution of difference system (8) are obtained immediately. □

3.2. Asymptotic Stability of Linear θ-Method. Section 2 gives the sufficient condition for the trivial solution of Problem (1) to be asymptotically stable. Next, we will analyse the numerical stability of the linear θ-method (8) under this condition.

Definition 5. A numerical method applied to Problem (1) is called asymptotically stable about the trivial solution if its approximate solution u_j^k corresponding to a sufficiently differentiable function $u_0(x,t)$ with $u_0(0,t) = u_0(\pi,t) = 0$ satisfies

$$\lim_{k\to\infty}\max_{1\le j\le N}\left|u_j^k\right| = 0. \tag{12}$$

In order to prove that a polynomial is a Schur polynomial, the following lemma is needed.

Lemma 6 (cf. [29]). *Let $\gamma_m(z) = \alpha(z)z^m - \beta(z)$ be a polynomial, where $\alpha(z)$ and $\beta(z)$ are polynomials of constant degree. Then, the polynomial $\gamma_m(z)$ is a Schur polynomial for any $m \ge 1$ if and only if the following conditions hold:*

(i) $\alpha(z) = 0 \Rightarrow |z| < 1$,

(ii) $|\alpha(z)| \ge |\beta(z)|$, *for all* $z \in \mathbb{C}, |z| = 1$,

(iii) $\gamma_m(z) \ne 0$, *for all* $z \in \mathbb{C}, |z| = 1$.

Taking the analytical technique in [20–22], we know that the linear θ-method (8) is asymptotically stable about the trivial solution if and only if

$$\begin{aligned}P_{m,j}^{\theta}(z) \equiv \phi_0\left(\lambda_j\right)z^{m+1} - \phi_1\left(\lambda_j\right)z^m + \phi_m\left(\lambda_j\right)z\\ + \phi_{m+1}\left(\lambda_j\right)\end{aligned} \tag{13}$$

is a Schur polynomial for any $m \ge 1$.

Basic calculations give

$$P_{m,j}^{\theta}(z) = \mu_j(z)\,z^m - \nu_j(z), \tag{14}$$

where

$$\begin{aligned}\mu_j(z) &= \left\{1 + \theta\left[a\left(2-\lambda_j\right) - c\right]\right\}z - 1\\ &\quad + (1-\theta)\left[a\left(2-\lambda_j\right) - c\right],\\ \nu_j(z) &= \left[d - b\left(2-\lambda_j\right)\right]\left[\theta z + (1-\theta)\right].\end{aligned} \tag{15}$$

With the help of Lemma 6, we obtain the following theorem when $\theta \in [0,1/2)$, which offers a sufficient and necessary condition of asymptotic stability for the linear θ-method.

Theorem 7. *Suppose that $a > c/2(1-\cos(\Delta x))$, $b > d/2(1-\cos(\Delta x))$, and $2a(1-\cos(\Delta x)) - c > 2b(1+\cos(\Delta x)) - d$.*

Then the linear θ-method (8) is asymptotically stable about the trivial solution for $\theta \in [0, 1/2)$ *if and only if*

$$(1-2\theta)\left[a+b-\frac{c+d}{2(1+\cos(\Delta x))}\right] < \frac{1}{1+\cos(\Delta x)}, \quad (16)$$

where $a = r_1\Delta t/\Delta x^2$, $b = r_2\Delta t/\Delta x^2$, $c = r_3\Delta t$, *and* $d = r_4\Delta t$.

Proof. (⇒) First, let us verify item (*i*) of Lemma 6. According to $\mu_j(z) = 0$, we derive that

$$|z| = \left|1 - \frac{a(2-\lambda_j)-c}{1+\theta\left[a(2-\lambda_j)-c\right]}\right|. \quad (17)$$

By the same technique used in [22], one can check that $|z| < 1$.

Next, to verify the rest of items of Lemma 6, that is, $\mu_j(z) > \nu_j(z)$, $j = 1, 2, \cdots, N-1$, for all $z \in \mathbb{C}$, $|z| = 1$, we define the following complex variable function:

$$\begin{aligned} w &= \frac{\mu_j(z)}{\nu_j(z)} \\ &= \frac{\left\{1+\theta\left[a(2-\lambda_j)-c\right]\right\}z - 1 + (1-\theta)\left[a(2-\lambda_j)-c\right]}{\left[d-b(2-\lambda_j)\right]\left[\theta z + (1-\theta)\right]}. \end{aligned} \quad (18)$$

Letting $w = x + yi$ and $|z| = 1$, after some basic calculations (see [20–22]), we find

$$\begin{aligned} \min_{|z|=1, z\in\mathbb{C}} |w| &= \min_{|z|=1, z\in\mathbb{C}} \left|\frac{\mu_j(z)}{\nu_j(z)}\right| \\ &= \min\left\{\left|\frac{a(2-\lambda_j)-c}{b(2-\lambda_j)-d}\right|, \right. \\ &\quad \left. \left|\frac{2-\left[a(2-\lambda_j)-c\right](1-2\theta)}{\left[b(2-\lambda_j)-d\right](1-2\theta)}\right|\right\}. \end{aligned} \quad (19)$$

The value of $\min_{|z|=1, z\in\mathbb{C}}|w|$ will be discussed in the following two different cases.

Case a ($\min_{|z|=1,z\in\mathbb{C}}|\mu_j(z)/\nu_j(z)| = |(a(2-\lambda_j)-c)/(b(2-\lambda_j)-d)|$). By conditions $a > c/2(1-\cos(\Delta x))$, $b > d/2(1-\cos(\Delta x))$, and $2a(1-\cos(\Delta x)) - c > 2b(1+\cos(\Delta x)) - d$, and noting that $2(1-\cos(\Delta x)) \le (2-\lambda_j) \le 2(1-\cos((N-1)\Delta x)) = 2(1+\cos(\Delta x))$, we have $a(2-\lambda_j) - c > b(2-\lambda_j) - d > 0$. Therefore, for all $z \in \mathbb{C}$, $|z| = 1$, we find

$$\left|\frac{\mu_j(z)}{\nu_j(z)}\right| \ge \min_{|z|=1,z\in\mathbb{C}} \left|\frac{\mu_j(z)}{\nu_j(z)}\right| = \frac{a(2-\lambda_j)-c}{b(2-\lambda_j)-d} > 1. \quad (20)$$

Case b ($\min_{|z|=1,z\in\mathbb{C}}|\mu_j(z)/\nu_j(z)| = |(2-[c+a(2-\lambda_j)](1-2\theta))/[d+b(2-\lambda_j)](1-2\theta)|$). It follows from condition (16) that

$$2 - \left[a(2-\lambda_j)-c\right](1-2\theta) > 0. \quad (21)$$

Noticing the fact that $2-\lambda_j \le 2(1+\cos(\Delta x))$ and condition (16), we have

$$\begin{aligned} \min_{|z|=1,z\in\mathbb{C}} \left|\frac{\mu_j(z)}{\nu_j(z)}\right| &= \frac{2-\left[a(2-\lambda_j)-c\right](1-2\theta)}{\left[b(2-\lambda_j)-d\right](1-2\theta)} \\ &\ge \frac{2-\left[2a(1+\cos(\Delta x))-c\right](1-2\theta)}{\left[2b(1+\cos(\Delta x))-d\right](1-2\theta)} > 1. \end{aligned} \quad (22)$$

In brief, combining **Case a** and **Case b**, we conclude that, for all $z \in \mathbb{C}$, $|z| = 1$, $\mu_j(z) > \nu_j(z)$, $j = 1, 2, \cdots, N-1$ holds, which implies that items (*ii*) and (*iii*) of Lemma 6 hold.

Now, with the help of Lemma 6, we derive that the linear θ-method (8) is asymptotically stable about the trivial solution.

(⇐) Next, we prove the necessary part from two points by contradiction:

(i) Suppose that $(1-2\theta)[a+b-(c+d)/2(1+\cos(\Delta x))] = 1/(1+\cos(\Delta x))$. Let m be even, $j = N-1$, and $z = -1$, and then, for $|z| = 1$, we get that $P^{\theta}_{m,N-1}(-1) = 0$, which indicates that condition (*iii*) of Lemma 6 does not hold. Thus, the linear θ-method (8) is not asymptotically stable.

(ii) Suppose that $(1-2\theta)[a+b-(c+d)/2(1+\cos(\Delta x))] > 1/(1+\cos(\Delta x))$. Let $j = N-1$ and $z = -1$, and then, after some basic calculations, we arrive at $|\nu_{N-1}(-1)| > |\mu_{N-1}(-1)|$. This signifies that condition (*ii*) of Lemma 6 does not hold. Therefore, the linear θ-method (8) is not asymptotically stable.

Then, we know that (16) is a necessary condition for asymptotic stability. This completes the proof. □

Remark 8. When $r_3 = r_4 = 0$, the sufficient and necessary condition (16) in Theorem 7 is simplified to

$$(1-2\theta)(a+b) < \frac{1}{1+\cos(\Delta x)}, \quad (23)$$

which is consistent with the previous work [20].

Next, when $\theta \in [1/2, 1]$, we will prove that the linear θ-method (8) is unconditionally asymptotically stable with respect to the trivial solution.

Theorem 9. *Suppose that* $a > c/2(1-\cos(\Delta x))$, $b > d/2(1-\cos(\Delta x))$, *and* $2a(1-\cos(\Delta x)) - c > 2b(1+\cos(\Delta x)) - d$. *Then the linear θ-method (8) is unconditionally asymptotically stable about the trivial solution for* $\theta \in [1/2, 1]$.

Proof. We will prove the theorem with Lemma 6. First, it follows from $\mu_j(z) = 0$ that

$$|z| = \left|1 - \frac{a(2-\lambda_j)-c}{1+\theta\left[a(2-\lambda_j)-c\right]}\right|. \quad (24)$$

Similar to the proof of Theorem 7, we get $|z| < 1$.

Then, we check items (ii) and (iii) of Lemma 6. To do that, we introduce the following complex variable function:

$$\begin{aligned} w &= \frac{\mu_j(z)}{\nu_j(z)} \\ &= \frac{\{1+\theta[a(2-\lambda_j)-c]\}z - 1 + (1-\theta)[a(2-\lambda_j)-c]}{[d-b(2-\lambda_j)][\theta z + (1-\theta)]}. \end{aligned} \tag{25}$$

(i) $\theta = 1/2$. Set $w = x + yi$ and $|z| = 1$. Basic calculations give

$$\min_{|z|=1, z\in\mathbb{C}} \left|\frac{\mu_j(z)}{\nu_j(z)}\right| = \left|\frac{a(2-\lambda_j)-c}{b(2-\lambda_j)-d}\right|. \tag{26}$$

It follows from assumptions $a > c/2(1-\cos(\Delta x))$, $b > d/2(1-\cos(\Delta x))$, and $2a(1-\cos(\Delta x))-c > 2b(1+\cos(\Delta x)) - d$ that $a(2-\lambda_j) - c > b(2-\lambda_j) - d > 0$. Then, for all $z \in \mathbb{C}$, $|z| = 1$, we find that

$$\left|\frac{\mu_j(z)}{\nu_j(z)}\right| \geq \min_{|z|=1, z\in\mathbb{C}} \left|\frac{\mu_j(z)}{\nu_j(z)}\right| = \frac{a(2-\lambda_j)-c}{b(2-\lambda_j)-d} > 1, \tag{27}$$

which indicates that items (*ii*) and (*iii*) of Lemma 6 hold.

(ii) $\theta \in (1/2, 1]$. Similarly, we can get

$$\min_{|z|=1, z\in\mathbb{C}} \left|\frac{\mu_j(z)}{\nu_j(z)}\right| = \left|\frac{a(2-\lambda_j)-c}{b(2-\lambda_j)-d}\right|. \tag{28}$$

In this case, for all $z \in \mathbb{C}$, $|z| = 1$, we also obtain that

$$\left|\frac{\mu_j(z)}{\nu_j(z)}\right| \geq \min_{|z|=1, z\in\mathbb{C}} \left|\frac{\mu_j(z)}{\nu_j(z)}\right| = \frac{a(2-\lambda_j)-c}{b(2-\lambda_j)-d} > 1. \tag{29}$$

According to Lemma 6, we conclude that the linear θ-method (8) is asymptotically stable about the trivial solution. This completes the proof of the theorem. □

3.3. Convergence of Linear θ-Method. Here and below, when we discuss the convergence of numerical methods, we will always assume that the solution $u(x,t)$ of Problem (1) is smooth enough and satisfies

$$\left|\frac{\partial^{j+k}}{\partial x^j \partial t^k} u(x,t)\right| \leq C, \quad 0 \leq j \leq 6, \ 0 \leq k \leq 3, \ C > 0, \tag{30}$$

where C is a constant.

Let $U_j^k = u(x_j, t_k)$, $j = 0, 1, \cdots, k = -m, -m+1, \cdots$. Then, we get

$$\begin{aligned} \delta_t U_j^k = r_1 &\left[(1-\theta)\delta_x^2 U_j^k + \theta \delta_x^2 U_j^{k+1}\right] \\ &+ r_2\left[(1-\theta)\delta_x^2 U_j^{k-m} + \theta\delta_x^2 U_j^{k-m+1}\right] \\ &+ r_3\left[(1-\theta)U_j^k + \theta U_j^{k+1}\right] \\ &+ r_4\left[(1-\theta)U_j^{k-m} + \theta U_j^{k-m+1}\right] + R_j^k, \end{aligned} \tag{31}$$

where R_j^k is the local truncation error. Taylor expansion yields that there exists a constant $\overline{C}$ such that, for $j = 1, 2, \cdots, N-1$, $k = 0, 1, \cdots$,

$$\left|R_j^k\right| \leq \begin{cases} \overline{C}\left(\Delta t^2 + \Delta x^2\right), & \theta = \dfrac{1}{2}, \\ \overline{C}\left(\Delta t + \Delta x^2\right), & 0 \leq \theta < \dfrac{1}{2} \text{ or } \dfrac{1}{2} < \theta \leq 1. \end{cases} \tag{32}$$

Thus, the consistence of linear θ-method (8) is obtained. Now, the convergence result is presented in the following theorem.

Theorem 10. *Assume that the assumptions in Theorems 7 and 9 hold. Then, for $k = 1, 2, \cdots$, we have the following convergent result:*

$$\left\|e^k\right\| \leq \begin{cases} \widehat{C}\left(\Delta t^2 + \Delta x^2\right), & \theta = \dfrac{1}{2}, \\ \widehat{C}\left(\Delta t + \Delta x^2\right), & 0 \leq \theta < \dfrac{1}{2} \text{ or } \dfrac{1}{2} < \theta \leq 1, \end{cases} \tag{33}$$

where $e^k = [u_1^k - U_1^k, u_2^k - U_2^k, \cdots, u_{N-1}^k - U_{N-1}^k]^T$ and $\widehat{C}$ is a constant that is independent of $\Delta t, \Delta x$.

Proof. It follows from Theorem 4 that difference system (8) is solvable and has a unique solution. Moreover, the assumptions in Theorems 7 and 9 hold, signifying that the method is stable. Together with the consistence of the method, we derive that (33) holds by the Lax equivalence theorem [30, 31]. □

4. Extension to Compact θ-Method

In this section, we would like to use the compact θ-method with a higher convergence order in space to extend our work. We introduce the compact difference operator,

$$\mathcal{A}_h u_j^k = \begin{cases} \dfrac{u_{j-1}^k + 10u_j^k + u_{j+1}^k}{12}, & j = 1, 2, \cdots, N-1, \\ u_j^k & j = 0, N, \end{cases} \tag{34}$$

and an important lemma below, which will be needed to construct and prove our main results.

Lemma 11 (cf. [32]). *Assume that $v(x) \in C^6[0,\pi]$. Then*

$$\begin{aligned} &\frac{v''(x_{j-1}) + 10v''(x_j) + v''(x_{j+1})}{12} \\ &\quad - \frac{v(x_{j-1}) - 2v(x_j) + v(x_{j+1})}{\Delta x^2} \\ &= \frac{\Delta x^4}{240} v^{(6)}(\omega_j), \end{aligned} \tag{35}$$

where $\omega_j \in (x_{j-1}, x_{j+1})$.

Now, applying the compact difference operator (34) to discrete the diffusion operator, we have the compact θ-method:

Table 1: Stability and convergence order of different methods.

	$\theta \in \left[0, \frac{1}{2}\right)$	$\theta \in \left[\frac{1}{2}, 1\right]$	Order
Linear θ-method for problem of [20]	$(1-2\theta)(a+b) < \frac{1}{1+\cos(\Delta x)}$	Unconditionally stable	2
Compact θ-method for problem of [20]	$\frac{1}{6} + (1-2\theta)(a+b) < \frac{1}{1+\cos(\Delta x)}$	Unconditionally stable	4
Linear θ-method for problem (1)	$(1-2\theta)\left[a+b-\frac{c+d}{2(1+\cos(\Delta x))}\right] < \frac{1}{1+\cos(\Delta x)}$	Unconditionally stable	2
Compact θ-method for problem (1)	$\frac{1}{6} + (1-2\theta)\left[a+b-\frac{c+d}{2(1+\cos(\Delta x))}\right] < \frac{1}{1+\cos(\Delta x)}$	Unconditionally stable	4

$$\begin{aligned}
\mathscr{A}_h \delta_t u_j^k &= r_1 \left[(1-\theta)\,\delta_x^2 u_j^k + \theta \delta_x^2 u_j^{k+1}\right] \\
&\quad + r_2 \left[(1-\theta)\,\delta_x^2 u_j^{k-m} + \theta \delta_x^2 u_j^{k-m+1}\right] \\
&\quad + r_3 \left[(1-\theta)\,u_j^k + \theta u_j^{k+1}\right] \\
&\quad + r_4 \left[(1-\theta)\,u_j^{k-m} + \theta u_j^{k-m+1}\right], \\
&\qquad j = 1, 2, \cdots, N-1, \;\; k = 0, 1, \cdots, \\
u_j^k &= u_0\left(x_j, t_k\right), \\
&\qquad j = 1, 2, \cdots, N-1, \;\; k = -m, -m+1, \cdots, 0, \\
u_0^k &= u_N^k = 0, \quad k = -m, -m+1, \cdots .
\end{aligned} \tag{36}$$

The compact θ-method (36) can be rewritten in the following matrix form:

$$\begin{aligned}
\psi_0(S)\,U^{k+1} &= \psi_1(S)\,U^k - \psi_m(S)\,U^{k+1-m} \\
&\quad - \psi_{m+1}(S)\,U^{k-m},
\end{aligned} \tag{37}$$

where

$$\begin{aligned}
\psi_0(\eta) &= \frac{5}{6} + 2a\theta - c\theta + \left(\frac{1}{12} - a\theta\right)\eta, \\
\psi_1(\eta) &= \frac{5}{6} - 2a(1-\theta) + c(1-\theta) \\
&\quad + \left(\frac{1}{12} + a(1-\theta)\right)\eta, \\
\psi_m(\eta) &= 2b\theta - d\theta - b\theta\eta, \\
\psi_{m+1}(\eta) &= 2b(1-\theta) - d(1-\theta) - b(1-\theta)\eta.
\end{aligned} \tag{38}$$

Similarly, the solvability, asymptotic stability, and convergence of the compact θ-method (36) can also be obtained. For conciseness, we merely list our main results and omit the details.

Theorem 12. *Suppose that $a > c/2(1-\cos(\Delta x))$, $b > d/2(1-\cos(\Delta x))$, and $2a(1-\cos(\Delta x)) - c > 2b(1+\cos(\Delta x)) - d$. Then the compact θ-method (36) is asymptotically stable about the trivial solution for $\theta \in [0, 1/2)$ if and only if*

$$\begin{aligned}
&\frac{1}{6} + (1-2\theta)\left[a + b - \frac{c+d}{2(1+\cos(\Delta x))}\right] \\
&\quad < \frac{1}{1+\cos(\Delta x)},
\end{aligned} \tag{39}$$

where $a = r_1 \Delta t/\Delta x^2$, $b = r_2 \Delta t/\Delta x^2$, $c = r_3 \Delta t$, and $d = r_4 \Delta t$.

Remark 13. When $r_3 = r_4 = 0$, the sufficient and necessary condition (39) in Theorem 12 is reduced to

$$\frac{1}{6} + (1-2\theta)(a+b) < \frac{1}{1+\cos(\Delta x)}, \tag{40}$$

which is consistent with the previous work [22].

Theorem 14. *Suppose that $a > c/2(1-\cos(\Delta x))$, $b > d/2(1-\cos(\Delta x))$, and $2a(1-\cos(\Delta x)) - c > 2b(1+\cos(\Delta x)) - d$. Then the compact θ-method (36) is unconditionally asymptotically stable about the trivial solution for $\theta \in [1/2, 1]$.*

Theorem 15. *Assume that the assumptions in Theorems 12 and 14 hold. Then, for $k = 1, 2, \cdots$, we have the following convergent result:*

$$\left\|e^k\right\| \le \begin{cases} \widetilde{C}\left(\Delta t^2 + \Delta x^4\right), & \theta = \dfrac{1}{2}, \\ \widetilde{C}\left(\Delta t + \Delta x^4\right), & 0 \le \theta < \dfrac{1}{2} \text{ or } \dfrac{1}{2} < \theta \le 1, \end{cases} \tag{41}$$

where $\widetilde{C}$ is a constant that is independent of temporal and spatial stepsizes.

Remark 16. The comparison of linear θ-method and compact θ-method applied to problem in [20] and Problem (1) is presented in Table 1.

5. Numerical Tests

In this section, several numerical experiments are carried out to illustrate the theoretical results.

5.1. Stability Tests of Linear θ-Method and Compact θ-Method. We use the following equation to test stability of the proposed method:

$$\begin{aligned}
\frac{\partial}{\partial t} u(x,t) &= \frac{\partial^2}{\partial x^2} u(x,t) + 0.5 \frac{\partial^2}{\partial x^2} u(x, t-\tau) \\
&\quad - u(x,t) - 0.5 u(x, t-\tau), \\
&\qquad 0 < t \le T, \;\; 0 < x < \pi, \\
u(x,t) &= \sin(x), \quad -\tau \le t \le 0, \;\; 0 \le x \le \pi, \\
u(0,t) &= u(\pi,t) = 0, \quad t \ge -\tau.
\end{aligned} \tag{42}$$

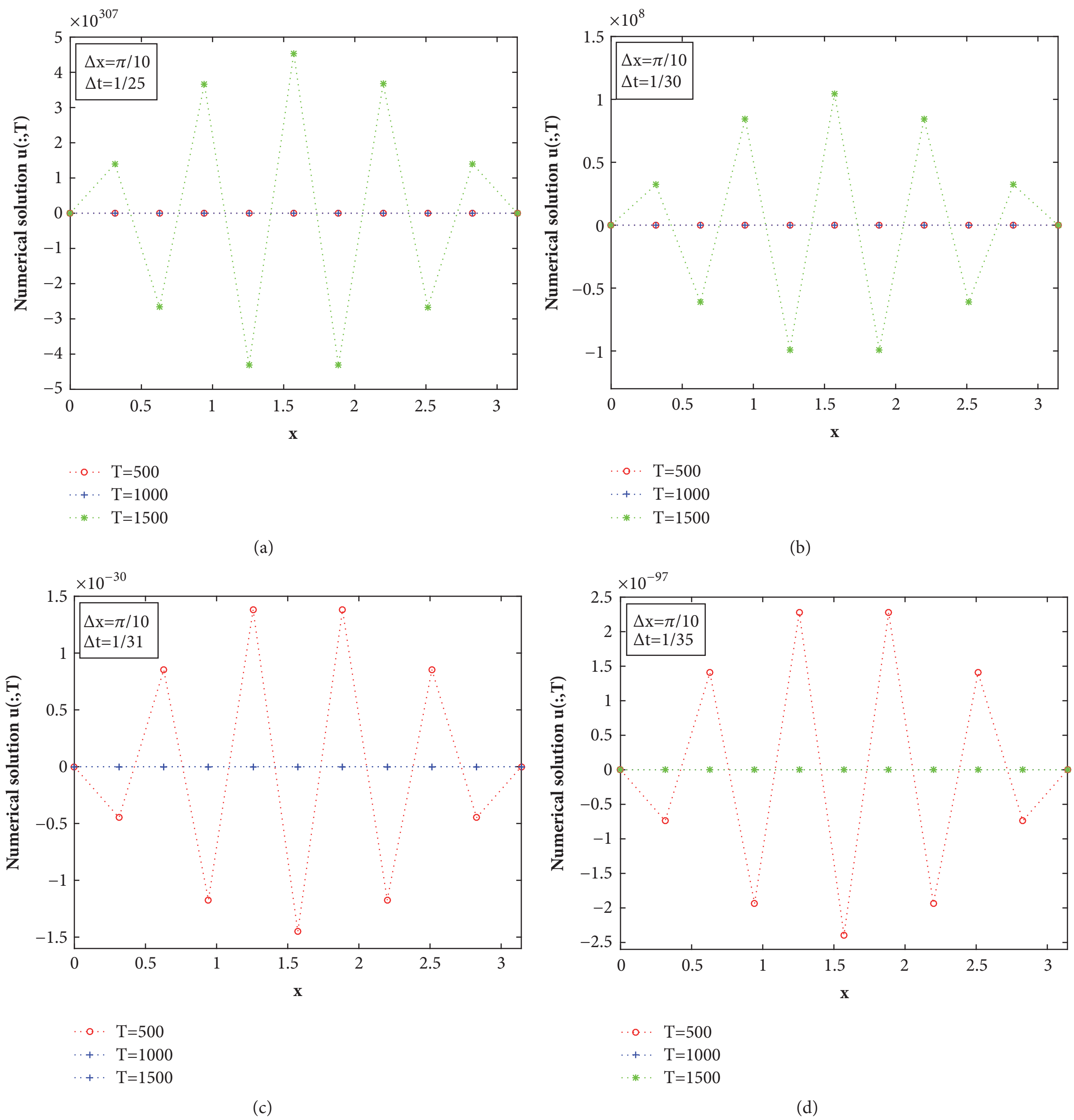

FIGURE 1: **Numerical solution at the different final time T for varying parameter** m ($\theta = 0$ and $\tau = 1$). (a) $m = 25$. (b) $m = 30$. (c) $m = 31$. (d) $m = 35$.

Here, let $\tau = 1$ and $\Delta x = \pi/10$. We set parameters $r_1 = 1.0$, $r_2 = 0.5$, $r_3 = -1.0$, and $r_4 = -0.5$ such that the trivial solution of Problem (1) is asymptotically stable.

5.1.1. Linear θ-Method. First, to verify the effectiveness of the sufficient and necessary condition (16) of the linear θ-method, here we choose the case where $\theta = 0$ to illustrate that. Noting that $\Delta t = \tau/m$, where $m > 0$ is an integer, and substituting parameters $\theta = 0$, $\tau = 1$, $\Delta x = \pi/10$, $r_1 = 1.0$, $r_2 = 0.5$, $r_3 = -1.0$, and $r_4 = -0.5$ into condition (16), we derive that the proposed method is asymptotically stable if $m > 30.4025$. In other words, if $m \geq 31$, then the method is asymptotically stable. Meanwhile, if $m \leq 30$, then the method is not asymptotically stable. In Figure 1, we can get a pictorial understanding of that.

It is easily seen from Figures 1(a) and 1(b) that the numerical solution is unstable as time goes on for $m = 25$ and 30. As shown in Figures 1(c) and 1(d), we know that the numerical solution is asymptotically stable for $m = 31$ and 35. Furthermore, denote the left-hand side of (16) as lhs $= (1-2\theta)[a+b-(c+d)/2(1+\cos(\Delta x))]$, and denote the right-hand side of (16) as rhs $= 1/(1+\cos(\Delta x))$, and Ind $=$ lhs $-$ rhs. From above paragraph, we know that Ind > 0 for $m \leq 30$, and Ind < 0 for $m \geq 31$. The relationship between Ind and m is

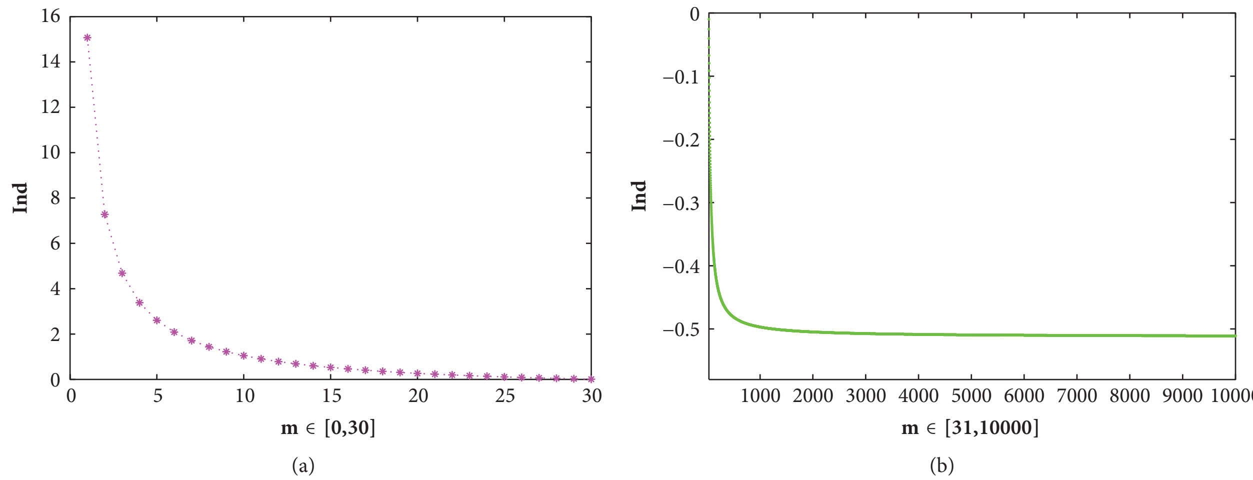

FIGURE 2: **Ind as a function of parameter m** ($\theta = 0$, $\tau = 1$, $\Delta x = \pi/10$, $r_1 = 1.0$, $r_2 = 0.5$, $r_3 = -1.0$, and $r_4 = -0.5$). (a) $m \in [1, 30]$. (b) $m \in [31, 1000]$.

shown in Figures 2(a) and 2(b). All these illustrate the results in Theorem 7.

Next, when $\theta = 1/2$ or 1, we apply the linear θ-method and use different stepsizes to solve problem (42). Theoretically, the numerical solution is asymptotically stable by Theorem 9. Numerically, we know that the numerical solution is asymptotically stable from the plots of Figure 3, which is consistent with the theoretical result.

5.1.2. Compact θ-Method. First, we choose the case $\theta = 0$ to verify the effectiveness of the sufficient and necessary condition (39) of the compact θ-method. Under the case that $\theta = 0$, $\tau = 1$, $\Delta x = \pi/10$, $r_1 = r_3 = 1.0$, and $r_2 = r_4 = 0.5$, and noting that $\Delta t = \tau/m$, by condition (39), the proposed method is asymptotically stable if and only if $m > 45.052$. In other words, if $m \geq 46$, then the proposed method is asymptotically stable. Meanwhile, if $m \leq 45$, then the proposed method is not asymptotically stable. In order to validate it, we give a pictorial understanding of that in Figure 4.

From Figures 4(a) and 4(b), it is easily seen that the numerical solution is unstable as time goes on for $m = 40$ and 45. The numerical solution is asymptotically stable for $m = 46$ and 50 in Figures 4(c) and 4(d), respectively. Furthermore, denote the left-hand side of (39) as $\mathrm{LHS} = 1/6+(1-2\theta)[a+b-(c+d)/2(1+\cos(\Delta x))]$, and denote the right-hand side of (39) as $\mathrm{RHS} = 1/(1+\cos(\Delta x))$, and $\mathrm{Ind} = \mathrm{LHS} - \mathrm{RHS}$. According to the analysis of above paragraph, we know that $\mathrm{Ind} > 0$ for $m \leq 45$, and $\mathrm{Ind} < 0$ for $m \geq 46$. The schematic presentation of the relationship between Ind and positive integer m is given in Figures 5(a) and 5(b). All the numerical results agree well with the findings in Theorem 12.

Then, for $\theta = 1/2$ or 1, we apply the compact θ-method and choose different stepsizes to solve problem (42). The numerical results are shown in Figure 6. Theoretically, according to Theorem 14, the numerical solution is asymptotically stable. Numerically, from these figures, we know that the numerical solution is asymptotically stable, which confirms the theoretical result.

5.2. Convergence Tests of Linear θ-Method and Compact θ-Method. We use the following equation to show our convergence results:

$$\begin{aligned} \frac{\partial}{\partial t} u(x,t) &= \frac{\partial^2}{\partial x^2} u(x,t) + 0.5 \frac{\partial^2}{\partial x^2} u(x,t-\tau) \\ &\quad - u(x,t) - 0.5 u(x,t-\tau) + f(x,t), \\ &\qquad 0 < t \leq T,\ 0 < x < \pi, \\ u(x,t) &= u_0(x,t), \quad -\tau \leq t \leq 0,\ 0 \leq x \leq \pi, \\ u(0,t) &= u(\pi,t) = 0, \quad t \geq -\tau. \end{aligned} \tag{43}$$

Here, the added term $h(x,t)$ and the initial condition $u_0(x,t)$ are specified so that the exact solution is $u(x,t) = e^{-t}\sin(x)$.

We set the parameters $r_1 = 1.0$, $r_2 = 0.5$, $r_3 = -1.0$, $r_4 = -0.5$, $\tau = 0.5$, and $T = 2$ and solve problem (43) with different spatial and temporal stepsizes ($\Delta x = \pi/N$ and $\Delta t = \tau/m$).

For $\theta = 0$, we let $\Delta t = 1e-5$ to guarantee that the linear θ-method (8) is asymptotically stable. When the compact θ-method (36) is applied to solve problem (43), we let $\Delta t \approx \Delta x^4$. The numerical errors and corresponding orders in different sense of norms are displayed in Table 2. Clearly, these results confirm the convergence of the two methods.

For $\theta = 1/2$, when the linear θ-method (8) is applied to solve problem (43), we let $\Delta t \approx \Delta x$, and for the compact θ-method (36), we let $\Delta t \approx \Delta x^2$. For $\theta = 1$, when method (8) is applied to solve problem (43), we let $\Delta t \approx \Delta x^2$, and for method (36), we let $\Delta t \approx \Delta x^4$. The numerical errors and corresponding orders in different sense of norms are listed in Tables 3 and 4, respectively. It is readily found that these results confirm the convergence of the two methods. Obviously, the compact θ-method gives a better convergence result in the space.

Table 2: Errors and convergence orders when $\theta = 0$, $T = 2$, and $\tau = 0.5$.

Linear θ-method					Compact θ-method				
N	L^2-error	Order	L^∞-error	Order	N	L^2-error	Order	L^∞-error	Order
2	$2.40E-02$	–	$1.91E-02$	–	7	$1.15E-03$	–	$8.95E-04$	–
4	$5.62E-03$	2.09	$4.48E-03$	2.09	14	$7.31E-05$	3.98	$5.83E-05$	3.94
8	$1.38E-03$	2.02	$1.10E-03$	2.02	28	$4.58E-06$	4.00	$3.65E-06$	4.00
16	$3.44E-04$	2.01	$2.75E-04$	2.01	56	$2.86E-07$	4.00	$2.28E-07$	4.00
32	$8.57E-05$	2.01	$6.84E-05$	2.01	112	$1.79E-08$	4.00	$1.43E-08$	4.00

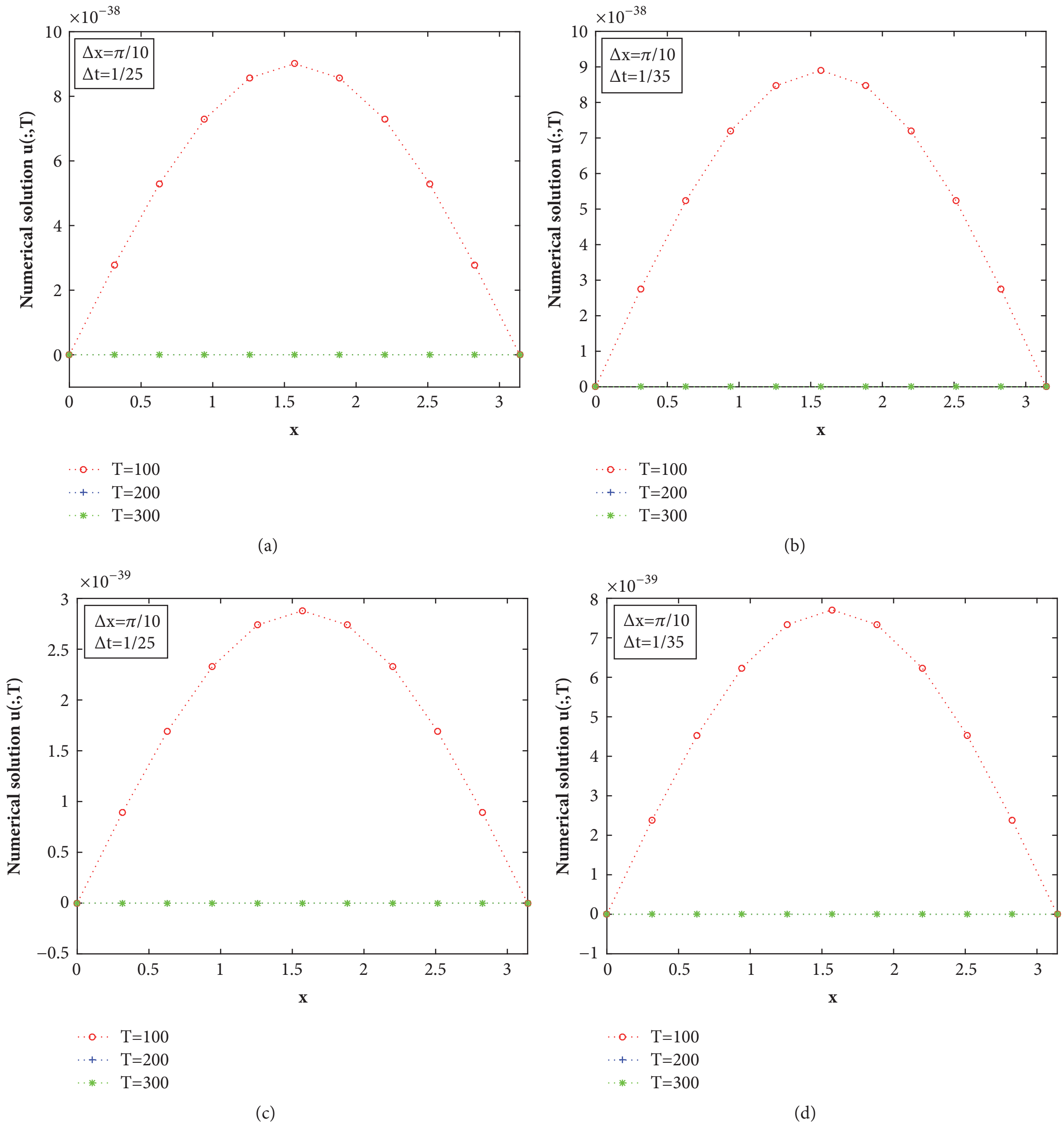

Figure 3: **Numerical solution at the different final time T for varying parameters θ and m** ($\tau = 1$). (a) $\theta = 0.5$ and $m = 25$. (b) $\theta = 0.5$ and $m = 35$. (c) $\theta = 1$ and $m = 25$. (d) $\theta = 1$ and $m = 35$.

Table 3: Errors and convergence orders when $\theta = 0.5$, $T = 2$, and $\tau = 0.5$.

	Linear θ-method				Compact θ-method			
N	L^2-error	Order	L^∞-error	Order	L^2-error	Order	L^∞-error	Order
5	$1.40E-03$	–	$1.06E-03$	–	$7.69E-04$	–	$5.83E-04$	–
10	$3.81E-04$	1.87	$3.04E-04$	1.80	$4.46E-05$	4.11	$3.56E-05$	4.03
20	$9.67E-05$	1.98	$7.72E-05$	1.98	$2.74E-06$	4.03	$2.19E-06$	4.03
40	$2.61E-05$	1.89	$2.09E-05$	1.89	$1.69E-07$	4.01	$1.35E-07$	4.01
80	$6.15E-06$	2.09	$4.91E-06$	2.09	$1.06E-08$	4.00	$8.44E-09$	4.00

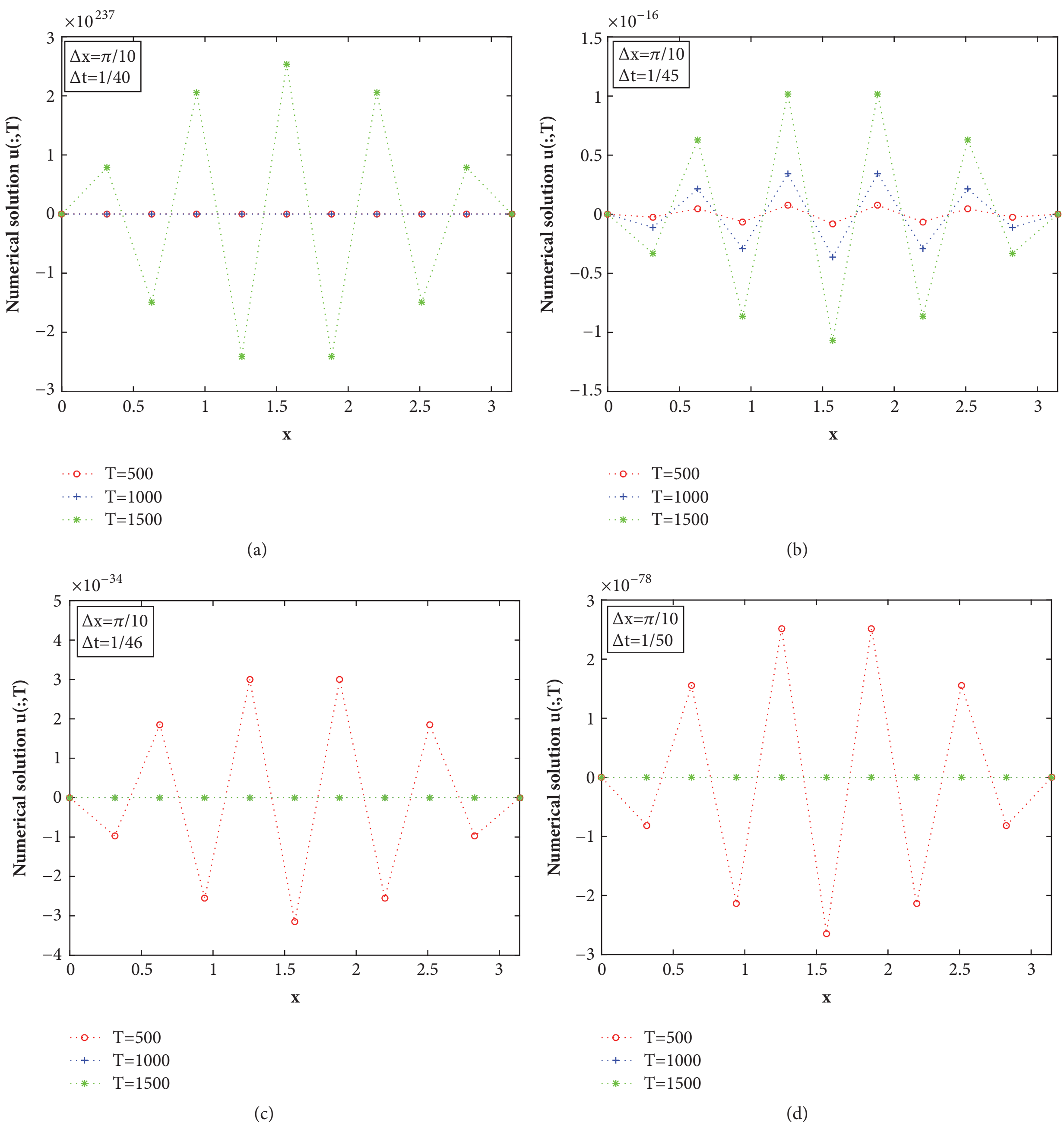

Figure 4: **Numerical solution at the different final time T for varying parameter** m ($\theta = 0$ and $\tau = 1$). (a) $m = 40$. (b) $m = 45$. (c) $m = 46$. (d) $m = 50$.

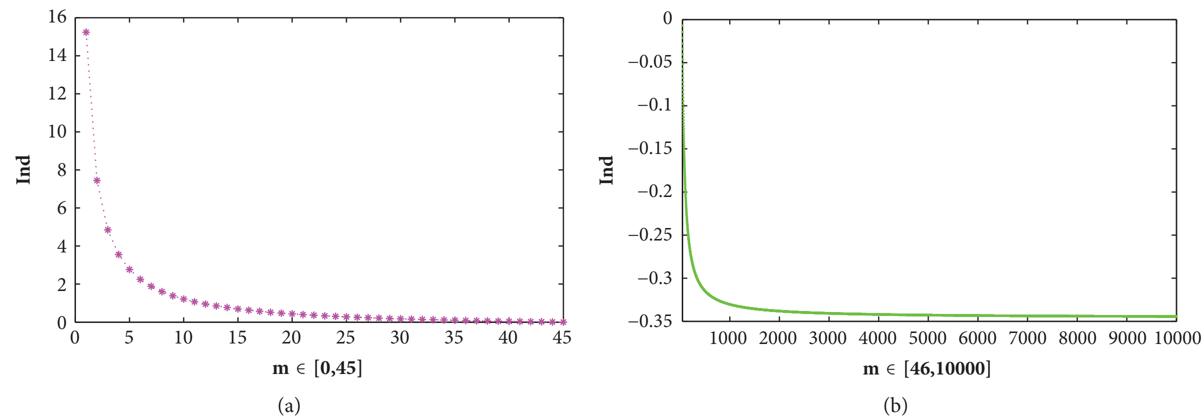

FIGURE 5: **Ind as a function of parameter m** ($\theta = 0$, $\tau = 1$, $\Delta x = \pi/10$, $r_1 = 1.0$, $r_2 = 0.5$, $r_3 = -1.0$, and $r_4 = -0.5$). (a) $m \in [1, 45]$. (b) $m \in [46, 1000]$.

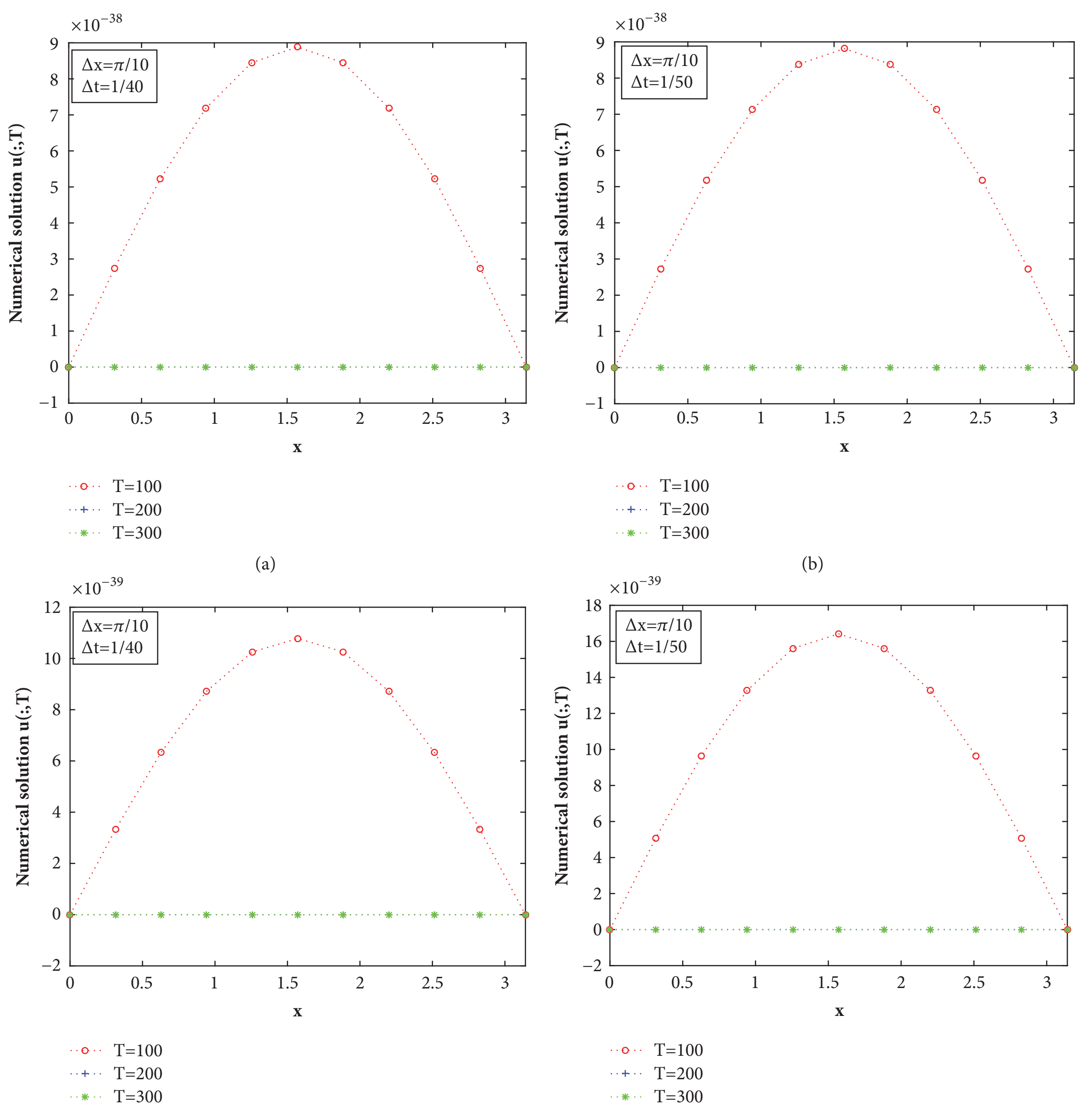

FIGURE 6: **Numerical solution at the different final time T for varying parameters** θ **and** m ($\tau = 1$). (a) $\theta = 0.5$ and $m = 40$. (b) $\theta = 0.5$ and $m = 50$. (c) $\theta = 1$ and $m = 40$. (d) $\theta = 1$ and $m = 50$.

Table 4: Errors and convergence orders when $\theta = 1$, $T = 2$, and $\tau = 0.5$.

	Linear θ-method				Compact θ-method			
N	L^2-error	Order	L^∞-error	Order	L^2-error	Order	L^∞-error	Order
5	$2.16E-02$	–	$1.64E-02$	–	$5.45E-03$	–	$4.14E-03$	–
10	$4.34E-03$	2.31	$3.47E-03$	2.24	$2.96E-04$	4.20	$2.36E-04$	4.13
20	$9.83E-04$	2.14	$7.84E-04$	2.14	$1.82E-05$	4.03	$1.45E-05$	4.03
40	$2.38E-04$	2.04	$1.90E-04$	2.04	$1.13E-06$	4.00	$9.05E-07$	4.00
80	$5.92E-05$	2.01	$4.72E-05$	2.01	$7.09E-08$	4.00	$5.65E-08$	4.00

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work is supported by NSFC (Grants nos. 11571128 and 11771162).

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