Hindawi
Abstract and Applied Analysis
Volume 2017, Article ID 3651271, 12 pages
http://dx.doi.org/10.1155/2017/3651271

Research Article

On the Output Controllability of Positive Discrete Linear Delay Systems

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Received 15 August 2017; Revised 30 September 2017; Accepted 9 October 2017; Published 23 November 2017

Academic Editor: Valery Y. Glizer

Necessary and sufficient conditions for output reachability and null output controllability of positive linear discrete systems with delays in state, input, and output are established. It is also shown that output reachability and null output controllability together imply output controllability.

1. Introduction

The research devoted to controllability was started by Kalman in the 1960s [1] and refers to linear dynamical systems. Controllability is one of the fundamental concepts in the modern mathematical control theory ([2–4],. . .) and continually appears as a necessary condition for the existence of solutions to many control problems, for example, stabilization of unstable system by feedback and optimal control. Basically a system is controllable if it is possible to transfer it around its entire configuration space using only certain admissible controls. There exist many definitions of controllability that depends on the framework or the class of models applied. The following are examples of variations of controllability notions which have been introduced in the control literature: asymptotic controllability [5], relative controllability [6], constrained controllability [7], complete controllability [8], approximate controllability [9], small controllability [10], output controllability [11, 12], and so on.

In most engineering applications, it is needed to direct the output toward some desired value. In fact, having control over the output of the system has a significant importance if not more than the states. For example, the control of a multilink cable-driven manipulator, where the task is typically defined in terms of end effector pose, rather than the joint positions and velocities which can define the system's state [13], also, controlling the output of fixed-speed wind turbines in the electrical network, which can directly affect the behavior of power systems [14]. Output controllability is a property of the impulse response matrix of a linear invariant-time system which reflects the dominant ability of an external input to move the output from any initial condition to any final condition in a finite time [2]. In general, the output controllability means that the system's output can be directed regardless of its state [15]. The necessary and sufficient criterion for output controllability of linear time-invariant systems is addressed in, for example, [12].

Positive systems are a wide class of systems in which state variables and outputs are constrained to be positive, or at least nonnegative for all time whenever the initial conditions and inputs are nonnegative. Since the state variables and outputs of many real-world processes represent quantities that may not have meaning unless they are nonnegative because they measure concentrations, numbers, populations, and so on, positive systems arise frequently in mathematical modeling of engineering problems, management sciences, economics, social sciences, chemistry, biology, ecology, pharmacology, medicine, and so forth.

An excellent survey of positive systems with an emphasis on their applications in the areas of management and social sciences is given by Luenberger in [16]. The more recent monographs by Farina and Rinaldi in [17] and Kaczorek in [18] are devoted entirely to positive linear systems and some of their applications. Since positive systems are confined within a cone located in the positive orthant rather than in the whole space [19, 20], their analysis and synthesis are more complicated and more challenging.

The state controllability of positive linear discrete systems is largely studied by several authors since late 1980s [21–26], the problem of controllability of linear positive discrete systems with delays in state or control was discussed in [27]. The problem of output reachability of positive linear discrete systems is addressed in [28]. The output reachability of positive discrete linear systems with state delay has been studied in [29].

In this paper we examine the issue of output reachability, null output controllability, and output controllability for positive linear systems with multiple delays in state, input, and output. These concepts are equivalent for unconstrained systems. The output reachability of discrete positive linear systems are characterized and proven by a simple algebraic proof. The criteria for the null output controllability will be established. We show that these properties are not equivalent for positive systems. In addition we prove that the positive system is output controllable only if it is output reachable and null output controllable.

The structure of the paper is as follows. In the next section some mathematical preliminaries of positive linear discrete systems with delays are presented. We investigate the output reachability and null output controllability of positive linear discrete systems with delays in state, input, and output, respectively, in Sections 3 and 4. In Section 5, necessary and sufficient conditions for the output controllability of positive delay systems are provided. Numerical examples will be presented in Section 6.

2. Preliminaries

First we introduce some notations. $\mathbb{N}$ is the set of nonnegative integers, $\mathbb{N}_+$ the set of positive integers, $\sigma_s^k = \{s, s+1, \dots, k\}$ the finite subset of $\mathbb{N}$ with $s \le k$, $\mathbb{R}^n$ the set of real vectors with n components, and $\mathbb{R}_+^n$ the set of vectors in $\mathbb{R}^n$ with nonnegative components; that is,

$$\mathbb{R}_+^n = \left\{x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n : x_i \ge 0,\ i \in \sigma_1^n\right\}, \quad (1)$$

where T denotes the transpose, $\mathbb{R}^{n\times m}$ the set of real matrices of order $n \times m$ ($\mathbb{R}^n = \mathbb{R}^{n\times 1}$), I_n the identity matrix in $\mathbb{R}^{n\times n}$, and A^{-1} the inverse of $A \in \mathbb{R}^{n\times n}$.

In this work, we consider the discrete linear delay system

$$\begin{aligned} x_{i+1} &= \sum_{j=0}^{p} A_j x_{i-j} + \sum_{j=0}^{q} B_j u_{i-j}, \quad i \in \mathbb{N}, \\ u_{-j} &\in \mathbb{R}^m \quad \text{for } j \in \sigma_1^q, \\ x_{-j} &\in \mathbb{R}^n \quad \text{for } j \in \sigma_0^p, \end{aligned} \quad (2)$$

with the output equation

$$y_i = \sum_{j=0}^{l} C_j x_{i-j} + \sum_{j=0}^{v} D_j u_{i-j}, \quad i \in \mathbb{N}, \text{ with } l \le p,\ v \le q, \quad (3)$$

where $x_i \in \mathbb{R}^n$ is the system state, $u_i \in \mathbb{R}^m$ is the input (or control), $y_i \in \mathbb{R}^r$, $A_j \in \mathbb{R}^{n\times n}$ ($j \in \sigma_0^p$) are the matrices of the state, $B_j \in \mathbb{R}^{n\times m}$ ($j \in \sigma_0^q$) are the matrices of the input, $C_j \in \mathbb{R}^{r\times n}$ ($j \in \sigma_0^l$) are the matrices of the output and $D_j \in \mathbb{R}^{r\times m}$ ($j \in \sigma_0^v$) are the matrices of the feedthrough (or feedforward), and p, q and v, and l are the nonnegative integer maximal values of delays on state, input, and output, respectively.

Definition 1. The system modeled by (2) and (3) is said to be positive if the state $x_i \in \mathbb{R}_+^n$ and the output $y_i \in \mathbb{R}_+^r$, $i \in \mathbb{N}$, for any initial states $x_{-j} \in \mathbb{R}_+^n$ ($j \in \sigma_0^p$) and for any initial inputs $u_{-j} \in \mathbb{R}_+^m$ ($j \in \sigma_1^q$) and all inputs $u_i \in \mathbb{R}_+^m$, $i \in \mathbb{N}$.

The mathematical theory of positive linear systems is based on the theory of nonnegative matrix developed by Perron and Frobenius (see [16, 30]).

Definition 2. A matrix $A = (a_{ij})$ in $\mathbb{R}^{n\times m}$ is said to be nonnegative and denoted by $A \in \mathbb{R}_+^{n\times m}$, if all of its elements are nonnegative; that is, $a_{ij} \ge 0$ for all $i \in \sigma_1^n$, $j \in \sigma_1^m$.

Remark 3. $A \in \mathbb{R}_+^{n\times m}$ if and only if $Ax \in \mathbb{R}_+^n$ for all $x \in \mathbb{R}_+^m$. Indeed, suppose one of the elements of A, a_{ij}, is negative. Then, for the nonnegative vector $x = (0, \dots, 0, 1, 0, \dots, 0)^T \in \mathbb{R}_+^m$ with the one in the jth component, the ith component of Ax would be a_{ij}, which is negative. It is also easy to verify the converse.

The following proposition provides a necessary and sufficient conditions for positivity of system (2) and (3).

Proposition 4. *System (2) and (3) is positive if and only if*

$$\begin{aligned} A_j &\in \mathbb{R}_+^{n\times n} \quad \left(j \in \sigma_0^p\right), \\ B_j &\in \mathbb{R}_+^{n\times m} \quad \left(j \in \sigma_0^q\right), \end{aligned} \quad (4)$$

$$\begin{aligned} C_j &\in \mathbb{R}_+^{r\times n} \quad \left(j \in \sigma_0^l\right), \\ D_j &\in \mathbb{R}_+^{r\times m} \quad \left(j \in \sigma_0^v\right). \end{aligned} \quad (5)$$

Proof.

Sufficiency. If the condition (4) is satisfied, then

$$x_1 = \sum_{j=0}^{p} A_j x_{-j} + \sum_{j=0}^{q} B_j u_{-j} \in \mathbb{R}_+^n, \quad (6)$$

since $x_{-j} \in \mathbb{R}_+^n$ ($j \in \sigma_0^p$) and $u_{-j} \in \mathbb{R}_+^m$ ($j \in \sigma_0^q$). Assume that $x_k \in \mathbb{R}_+^n$ for $k \in \sigma_1^i$. From (2) we have

$$x_{i+1} = \sum_{j=0}^{p} A_j x_{i-j} + \sum_{j=0}^{q} B_j u_{i-j} \in \mathbb{R}_+^n, \quad (7)$$

since (4) holds and $x_{i-j} \in \mathbb{R}_+^n$ ($j \in \sigma_0^p$), $u_{-j} \in \mathbb{R}_+^m$ ($j \in \sigma_1^q$), and $u_i \in \mathbb{R}_+^m$, $i \in \mathbb{N}$. Hence $x_i \in \mathbb{R}_+^n$ for any $i \in \mathbb{N}$.

Consequently, if condition (5) is satisfied, we get that $y_i \in \mathbb{R}^r_+$ for every $i \in \mathbb{N}$ since $x_{-j} \in \mathbb{R}^n_+$ $(j \in \sigma_0^l)$, $u_{-j} \in \mathbb{R}^m_+$ $(j \in \sigma_1^v)$, and $u_i \in \mathbb{R}^m_+$, $i \in \mathbb{N}$.

Necessity. Assuming that system (2) and (3) is positive, let $u_{-j} = 0$ for $j \in \sigma_0^q$. Then from (2) and (3), for $i = 0$, we have

$$x_1 = \sum_{j=0}^{p} A_j x_{-j} = \overline{A}\overline{x}_0 \in \mathbb{R}^n_+,$$
$$y_0 = \sum_{j=0}^{l} C_j x_{-j} = \overline{C}\overline{x}_1 \in \mathbb{R}^r_+, \quad (8)$$

with

$$\overline{A} = \begin{pmatrix} A_0 & A_1 & \cdots & A_p \end{pmatrix} \in \mathbb{R}^{n\times n(p+1)},$$
$$\overline{C} = \begin{pmatrix} C_0 & C_1 & \cdots & C_l \end{pmatrix} \in \mathbb{R}^{r\times n(l+1)},$$
$$\overline{x}_0 = \left(x_0, x_{-1}, \ldots, x_{-p}\right)^T \in \mathbb{R}^{n(p+1)},$$
$$\overline{x}_1 = \left(x_0, x_{-1}, \ldots, x_{-l}\right)^T \in \mathbb{R}^{n(l+1)}. \quad (9)$$

Hence by Remark 3, we have $\overline{A} \in \mathbb{R}^{n\times n(p+1)}_+$; that is, $A_j \in \mathbb{R}^{n\times n}_+$ $(j \in \sigma_0^p)$ and $\overline{C} \in \mathbb{R}^{r\times n(l+1)}_+$; that is, $C_j \in \mathbb{R}^{r\times n}_+$ $(j \in \sigma_0^l)$ since $\overline{x}_0 \in \mathbb{R}^{n(p+1)}_+$ and $\overline{x}_1 \in \mathbb{R}^{n(l+1)}_+$ are arbitrary. Now, assume that $x_{-j} = 0$ for $j \in \sigma_0^p$, and for $i = 0$, we obtain

$$x_1 = \sum_{j=0}^{q} B_j u_{-j} = \overline{B}\overline{u}_0 \in \mathbb{R}^n_+,$$
$$y_0 = \sum_{j=0}^{v} D_j u_{-j} = \overline{D}\overline{u}_1 \in \mathbb{R}^r_+, \quad (10)$$

with

$$\overline{B} = \begin{pmatrix} B_0 & B_1 & \cdots & B_q \end{pmatrix} \in \mathbb{R}^{n\times m(q+1)},$$
$$\overline{D} = \begin{pmatrix} D_0 & D_1 & \cdots & D_v \end{pmatrix} \in \mathbb{R}^{r\times m(v+1)},$$
$$\overline{u}_0 = \left(u_0, u_{-1}, \ldots, u_{-q}\right)^T \in \mathbb{R}^{m(q+1)},$$
$$\overline{u}_1 = \left(u_0, u_{-1}, \ldots, u_{-v}\right)^T \in \mathbb{R}^{m(v+1)}, \quad (11)$$

which implies that $\overline{B} \in \mathbb{R}^{n\times m(q+1)}_+$; that is, $B_j \in \mathbb{R}^{n\times m}_+$ $(j \in \sigma_0^q)$ and $\overline{D} \in \mathbb{R}^{r\times m(v+1)}_+$, that is, $D_j \in \mathbb{R}^{r\times m}_+$ $(j \in \sigma_0^v)$ since $\overline{u}_0 \in \mathbb{R}^{m(q+1)}_+$ and $\overline{u}_1 \in \mathbb{R}^{m(v+1)}_+$ are arbitrary. This completes the proof. □

In all the sequel, we assume that system (2) and (3) is positive.

In the next proposition, we will present the explicit solution of system (2).

Proposition 5. *The general solution to (2) is given by*

$$x_i = G_i x_0 + \sum_{j=1}^{p}\sum_{k=1}^{p-j+1} G_{i-k}A_{k-1+j}x_{-j} + \sum_{j=1}^{q}\sum_{k=1}^{q-j+1} G_{i-k}B_{k-1+j}u_{-j} + \sum_{j=0}^{i-1}\sum_{k=0}^{q} G_{i-1-j-k}B_k u_j, \quad i \in \mathbb{N}, \quad (12)$$

where the transition matrix $G_i \in \mathbb{R}^{n\times n}$ $(i \in \mathbb{N})$ *is determined by the recurrence relation*

$$G_i = \begin{cases} I_n & \text{for } i = 0, \\ \sum_{k=0}^{p} A_k G_{i-1-k} & \text{for } i \in \mathbb{N}_+, \end{cases} \quad (13)$$

with the assumption

$$G_i = 0 \quad \text{for } i < 0. \quad (14)$$

Proof. The proof is given in [31]. □

We pose $H_i^0 = G_i$, and then

$$H_i^0 = \begin{cases} I_n & \text{for } i = 0, \\ \sum_{k=0}^{p} A_k H_{i-1-k}^0 & \text{for } i \in \mathbb{N}_+, \\ 0 & \text{for } i < 0, \end{cases} \quad (15)$$

and, for all $i \in \mathbb{N}_+$, we pose

$$H_i^j = \sum_{k=1}^{p-j+1} H_{i-k}^0 A_{k-1+j}, \quad j \in \sigma_1^p,$$
$$L_i^j = \sum_{k=1}^{q-j+1} H_{i-k}^0 B_{k-1+j}, \quad j \in \sigma_1^q, \quad (16)$$

with $H_i^j = L_i^j = 0$ for $i \le 0$.

Moreover, for $i \in \mathbb{N}$, we pose

$$K_i = \sum_{k=0}^{q} H_{i-k}^0 B_k, \quad (17)$$

with $K_i = 0$, for $i < 0$.

Clearly by (15), (16), and (17), the solution of (2) is given by the following new formula:

$$x_i = H_i^0 x_0 + \sum_{j=1}^{p} H_i^j x_{-j} + \sum_{j=1}^{q} L_i^j u_{-j} + \sum_{j=0}^{i-1} K_{i-1-j} u_j, \quad i \in \mathbb{N}. \quad (18)$$

In the following and without loss of generality, we assume that $l = v$. Indeed, for example, if $l > v$ we can set $D_j = 0$ for $j \in \sigma_{v+1}^l$.

Now, we introduce the matrices sequence as follows:

$$\begin{aligned}
\mathscr{H}_i^j &= \sum_{k=0}^{l} C_k H_{i-k}^j, \quad j \in \sigma_0^p,\ i \in \mathbb{N}, \\
\mathscr{L}_i^j &= \sum_{k=0}^{l} C_k L_{i-k}^j, \quad j \in \sigma_1^q,\ i \in \mathbb{N}, \\
\mathscr{K}_i &= \sum_{k=0}^{l} C_k K_{i-k}, \quad i \in \mathbb{N}, \\
\overline{\mathscr{K}}_i &= \mathscr{K}_i + D_{i+1}, \quad i \in \sigma_0^{l-1}.
\end{aligned} \tag{19}$$

For $0 \le i < l$, the output equation (3) can be rewritten as

$$\begin{aligned}
y_i &= \sum_{k=0}^{i} C_k x_{i-k} + \sum_{k=i+1}^{l} C_k x_{i-k} + \sum_{k=0}^{i} D_k u_{i-k} + \sum_{k=i+1}^{l} D_k u_{i-k} \\
&= \sum_{k=0}^{i} C_k H_{i-k}^0 x_0 + \sum_{k=0}^{i} C_k \sum_{j=1}^{p} H_{i-k}^j x_{-j} + \sum_{k=0}^{i} C_k \sum_{j=1}^{q} L_{i-k}^j u_{-j} + \sum_{k=i+1}^{l} C_k x_{i-k} + \sum_{k=i+1}^{l} D_k u_{i-k} \\
&\quad + \sum_{k=0}^{i} C_k \sum_{j=0}^{i-k-1} K_{i-k-1-j} u_j + \sum_{k=0}^{i} D_k u_{i-k} \\
&= \sum_{k=0}^{l} C_k H_{i-k}^0 x_0 + \sum_{k=0}^{l} C_k \sum_{j=1}^{p} H_{i-k}^j x_{-j} + \sum_{k=0}^{l} C_k \sum_{j=1}^{q} L_{i-k}^j u_{-j} + \sum_{k=i+1}^{l} C_k x_{i-k} + \sum_{k=i+1}^{l} D_k u_{i-k} \\
&\quad + \sum_{k=0}^{i-1} C_k \sum_{j=0}^{i-k-1} K_{i-k-1-j} u_j + \sum_{k=0}^{i} D_k u_{i-k} \\
&= \left(\sum_{k=0}^{l} C_k H_{i-k}^0\right) x_0 + \sum_{j=1}^{p} \left(\sum_{k=0}^{l} C_k H_{i-k}^j\right) x_{-j} + \sum_{j=1}^{q} \left(\sum_{k=0}^{l} C_k L_{i-k}^j\right) u_{-j} + \sum_{j=i+1}^{l} C_j x_{i-j} \\
&\quad + \sum_{j=i+1}^{l} D_j u_{i-j} + \sum_{j=0}^{i-1} \left(\sum_{k=0}^{i-j-1} C_k K_{i-j-1-k}\right) u_j + \sum_{j=0}^{i-1} D_{i-j} u_j + D_0 u_i \\
&= \mathscr{H}_i^0 x_0 + \sum_{j=1}^{p} \mathscr{H}_i^j x_{-j} + \sum_{j=1}^{q} \mathscr{L}_i^j u_{-j} + \sum_{j=1}^{l-i} C_{i+j} x_{-j} \\
&\quad + \sum_{j=1}^{l-i} D_{i+j} u_{-j} + \sum_{j=0}^{i-1} \left(\mathscr{K}_{i-j-1} + D_{i-j}\right) u_j + D_0 u_i \\
&= \mathscr{H}_i^0 x_0 + \sum_{j=1}^{l-i} \left(\mathscr{H}_i^j + C_{i+j}\right) x_{-j} + \sum_{j=l-i+1}^{p} \mathscr{H}_i^j x_{-j} \\
&\quad + \sum_{j=1}^{l-i} \left(\mathscr{L}_i^j + D_{i+j}\right) u_{-j} + \sum_{j=l-i+1}^{q} \mathscr{L}_i^j u_{-j} \\
&\quad + \sum_{j=0}^{i-1} \overline{\mathscr{K}}_{i-j-1} u_j + D_0 u_i.
\end{aligned} \tag{20}$$

Hence

$$y_i = \mathscr{Q}_{i+1} \tilde{x}_0 + \mathscr{R}_{i+1} u_0^{i+1}, \tag{21}$$

with

$$\mathscr{Q}_{i+1} = \begin{pmatrix} M_{i+1} & O_{i+1} \end{pmatrix} \in \mathbb{R}_+^{r \times (n(p+1)+mq)}, \tag{22}$$

where

$$\begin{aligned}
&M_{i+1} \\
&= \begin{pmatrix} \mathscr{H}_i^0 & \mathscr{H}_i^1 + C_{i+1} & \mathscr{H}_i^2 + C_{i+2} & \cdots & \mathscr{H}_i^{l-i} + C_l & \mathscr{H}_i^{l-i+1} & \cdots & \mathscr{H}_i^p \end{pmatrix} \\
&\in \mathbb{R}_+^{r \times n(p+1)}, \\
&O_{i+1} = \begin{pmatrix} \mathscr{L}_i^1 + D_{i+1} & \cdots & \mathscr{L}_i^{l-i} + D_l & \mathscr{L}_i^{l-i+1} & \cdots & \mathscr{L}_i^q \end{pmatrix} \in \mathbb{R}_+^{r \times mq}, \\
&\tilde{x}_0 = \begin{pmatrix} x_0 \\ x_{-1} \\ \vdots \\ x_{-p} \\ u_{-1} \\ \vdots \\ u_{-q} \end{pmatrix} \in \mathbb{R}_+^{n(p+1)+mq}, \\
&\mathscr{R}_{i+1} = \begin{pmatrix} \overline{\mathscr{K}}_{i-1} & \overline{\mathscr{K}}_{i-2} & \cdots & \overline{\mathscr{K}}_1 & \overline{\mathscr{K}}_0 & D_0 \end{pmatrix} \in \mathbb{R}_+^{r \times (i+1)m}, \\
&u_0^{i+1} = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_i \end{pmatrix} \in \mathbb{R}_+^{(i+1)m}.
\end{aligned} \tag{23}$$

For $i \geq l$, we have

$$\begin{aligned}
y_i &= \sum_{k=0}^{l} C_k H_{i-k}^0 x_0 + \sum_{k=0}^{l} C_k \sum_{j=1}^{p} H_{i-k}^j x_{-j} \\
&\quad + \sum_{k=0}^{l} C_k \sum_{j=1}^{q} L_{i-k}^j u_{-j} + \sum_{k=0}^{l} C_k \sum_{j=0}^{i-k-1} K_{i-k-1-j} u_j \\
&\quad + \sum_{k=0}^{l} D_k u_{i-k} \\
&= \left(\sum_{k=0}^{l} C_k H_{i-k}^0\right) x_0 + \sum_{j=1}^{p} \left(\sum_{k=0}^{l} C_k H_{i-k}^j\right) x_{-j} \\
&\quad + \sum_{j=1}^{q} \left(\sum_{k=0}^{l} C_k L_{i-k}^j\right) u_{-j} \\
&\quad + \sum_{j=0}^{i-l-1} \left(\sum_{k=0}^{l} C_k K_{i-j-1-k}\right) u_j \\
&\quad + \sum_{j=i-l}^{i-1} \left(\sum_{k=0}^{i-j-1} C_k K_{i-j-1-k}\right) u_j + \sum_{j=0}^{l} D_j u_{i-j} \\
&= \left(\sum_{k=0}^{l} C_k H_{i-k}^0\right) x_0 + \sum_{j=1}^{p} \left(\sum_{k=0}^{l} C_k H_{i-k}^j\right) x_{-j} \\
&\quad + \sum_{j=1}^{q} \left(\sum_{k=0}^{l} C_k L_{i-k}^j\right) u_{-j} \\
&\quad + \sum_{j=0}^{i-1} \left(\sum_{k=0}^{l} C_k K_{i-j-1-k}\right) u_j + \sum_{j=0}^{l} D_j u_{i-j} \\
&= \mathscr{H}_i^0 x_0 + \sum_{j=1}^{p} \mathscr{H}_i^j x_{-j} + \sum_{j=1}^{q} \mathscr{L}_i^j u_{-j} + \sum_{j=0}^{i-1} \mathscr{K}_{i-j-1} u_j \\
&\quad + \sum_{j=0}^{l} D_j u_{i-j} \\
&= \mathscr{H}_i^0 x_0 + \sum_{j=1}^{p} \mathscr{H}_i^j x_{-j} + \sum_{j=1}^{q} \mathscr{L}_i^j u_{-j} + \sum_{j=0}^{i-l-1} \mathscr{K}_{i-j-1} u_j \\
&\quad + \sum_{j=i-l}^{i-1} \left(\mathscr{K}_{i-j-1} + D_{i-j}\right) u_j + D_0 u_i \\
&= \mathscr{H}_i^0 x_0 + \sum_{j=1}^{p} \mathscr{H}_i^j x_{-j} + \sum_{j=1}^{p} \mathscr{L}_i^j u_{-j} + \sum_{j=0}^{i-l-1} \mathscr{K}_{i-j-1} u_j \\
&\quad + \sum_{j=i-l}^{i-1} \overline{\mathscr{K}}_{i-j-1} u_j + D_0 u_i.
\end{aligned} \tag{24}$$

Then, we get the linear algebraic equation

$$y_i = \mathscr{Q}_{i+1} \tilde{x}_0 + \mathscr{R}_{i+1} u_0^{i+1}, \tag{25}$$

with

$$\begin{aligned}
\mathscr{Q}_{i+1} &= \begin{pmatrix} \mathscr{H}_i^0 & \mathscr{H}_i^1 & \cdots & \mathscr{H}_i^p & \mathscr{L}_i^1 & \cdots & \mathscr{L}_i^q \end{pmatrix} \\
&\in \mathbb{R}_+^{r\times(n(p+1)+mq)},
\end{aligned} \tag{26}$$

$$\begin{aligned}
\mathscr{R}_{i+1} &= \begin{pmatrix} \mathscr{K}_{i-1} & \mathscr{K}_{i-2} & \cdots & \mathscr{K}_l & \overline{\mathscr{K}}_{l-1} & \cdots & \overline{\mathscr{K}}_0 & D_0 \end{pmatrix} \\
&\in \mathbb{R}_+^{r\times(i+1)m}.
\end{aligned} \tag{27}$$

The following lemmas will be needed in the sequel.

Lemma 6. *For any $i \in \mathbb{N}_+$, we have*

$$H_i^0 = \sum_{k=0}^{p} H_{i-1-k}^0 A_k. \tag{28}$$

Proof. First, for $i = 1$, we have $H_1^0 = A_0 = \sum_{k=0}^{p} H_{-k}^0 A_k$ and (28) holds. Secondly, suppose that (28) holds for $k \in \sigma_1^i$. We prove that it holds for $k = i + 1$.

For $i \in \sigma_1^p$, we have

$$\begin{aligned}
H_{i+1}^0 &= \sum_{k=0}^{i} A_k H_{i-k}^0 = \sum_{k=0}^{i-1} A_k H_{i-k}^0 + A_i \\
&= \sum_{k=0}^{i-1} A_k \left(\sum_{j=0}^{p} H_{i-k-1-j}^0 A_j\right) + A_i \\
&= \sum_{k=0}^{i-1} A_k \left(\sum_{j=0}^{i-k-1} H_{i-k-1-j}^0 A_j\right) + A_i \\
&= \sum_{j=0}^{i-1} \left(\sum_{k=0}^{i-j-1} A_k H_{i-j-1-k}^0\right) A_j + A_i \\
&= \sum_{j=0}^{i-1} H_{i-j}^0 A_j + A_i = \sum_{j=0}^{i} H_{i-j}^0 A_j = \sum_{j=0}^{p} H_{i-j}^0 A_j.
\end{aligned} \tag{29}$$

For $i \geq p + 1$, we have

$$\begin{aligned}
H_{i+1}^0 &= \sum_{k=0}^{p} A_k H_{i-k}^0 = \sum_{k=0}^{p} A_k \left(\sum_{j=0}^{p} H_{i-k-1-j}^0 A_j\right) \\
&= \sum_{j=0}^{p} \left(\sum_{k=0}^{p} A_k H_{i-j-1-k}^0\right) A_j = \sum_{j=0}^{p} H_{i-j}^0 A_j.
\end{aligned} \tag{30}$$

Thus, (28) is satisfied in step $i + 1$. Hence, (28) holds for any $i \in \mathbb{N}_+$. □

Lemma 7. *For all $i \in \mathbb{N}$, we have*

$$\begin{aligned}
H_{i+1}^j &= H_i^{j+1} + H_i^0 A_j, \quad j \in \sigma_0^{p-1}, \\
H_{i+1}^p &= H_i^0 A_p,
\end{aligned} \tag{31}$$

$$L_{i+1}^{j} = L_{i}^{j+1} + H_{i}^{0} B_{j}, \quad j \in \sigma_{1}^{q-1},$$
$$L_{i+1}^{q} = H_{i}^{0} B_{q}. \tag{32}$$

Proof. For $i = 0$, we have

$$H_{0}^{j+1} + H_{0}^{0} A_{j} = A_{j} = H_{1}^{j}, \quad j \in \sigma_{0}^{p-1},$$
$$H_{0}^{0} A_{p} = A_{p} = H_{1}^{p}. \tag{33}$$

Let $i \in \mathbb{N}_{+}$. For $j = 0$, we have

$$H_{i}^{1} + H_{i}^{0} A_{0} = \sum_{k=1}^{p} H_{i-k}^{0} A_{k} + H_{i}^{0} A_{0} = \sum_{k=0}^{p} H_{i-k}^{0} A_{k}; \tag{34}$$

then by Lemma 6, we get

$$H_{i+1}^{0} = H_{i}^{1} + H_{i}^{0} A_{0}. \tag{35}$$

For $j \in \sigma_{1}^{p-1}$, we have

$$\begin{aligned} H_{i+1}^{j} - H_{i}^{j+1} &= \sum_{k=1}^{p-j+1} H_{i+1-k}^{0} A_{k-1+j} - \sum_{k=1}^{p-j} H_{i-k}^{0} A_{k+j} \\ &= \sum_{k=0}^{p-j} H_{i-k}^{0} A_{k+j} - \sum_{k=1}^{p-j} H_{i-k}^{0} A_{k+j} = H_{i}^{0} A_{j}. \end{aligned} \tag{36}$$

And for $j = p$, we have

$$H_{i+1}^{p} = \sum_{k=1}^{1} H_{i+1-k}^{0} A_{k-1+p} = H_{i}^{0} A_{p}. \tag{37}$$

Similarly, we prove that (32) holds. □

Lemma 8. *We have*

$$\mathscr{H}_{i+1}^{0} = \mathscr{H}_{i}^{1} + \mathscr{H}_{i}^{0} A_{0} + C_{i+1}, \quad i < l,$$
$$\mathscr{H}_{i+1}^{0} = \mathscr{H}_{i}^{1} + \mathscr{H}_{i}^{0} A_{0}, \quad i \geq l. \tag{38}$$

And for all $i \in \mathbb{N}$, we have

$$\mathscr{H}_{i+1}^{j} = \mathscr{H}_{i}^{j+1} + \mathscr{H}_{i}^{0} A_{j}, \quad j \in \sigma_{1}^{p-1},$$
$$\mathscr{H}_{i+1}^{p} = \mathscr{H}_{i}^{0} A_{p}, \tag{39}$$

$$\mathscr{L}_{i+1}^{j} = \mathscr{L}_{i}^{j+1} + \mathscr{H}_{i}^{0} B_{j}, \quad j \in \sigma_{1}^{q-1},$$
$$\mathscr{L}_{i+1}^{q} = \mathscr{H}_{i}^{0} B_{q}. \tag{40}$$

Proof. Let $i < l$. For $j = 0$, we have

$$\begin{aligned} \mathscr{H}_{i+1}^{0} &= \sum_{k=0}^{l} C_k H_{i+1-k}^{0} = \sum_{k=0}^{i+1} C_k H_{i+1-k}^{0} \\ &= \sum_{k=0}^{i} C_k H_{i+1-k}^{0} + C_{i+1} \\ &= \sum_{k=0}^{i} C_k \left(H_{i-k}^{1} + H_{i-k}^{0} A_0 \right) + C_{i+1} \\ &= \sum_{k=0}^{i} C_k H_{i-k}^{1} + \left(\sum_{k=0}^{i} C_k H_{i-k}^{0} \right) A_0 + C_{i+1} \\ &= \mathscr{H}_{i}^{1} + \mathscr{H}_{i}^{0} A_0 + C_{i+1}, \end{aligned} \tag{41}$$

for $j \in \sigma_{1}^{p-1}$, we have

$$\begin{aligned} \mathscr{H}_{i+1}^{j} &= \sum_{k=0}^{l} C_k H_{i+1-k}^{j} = \sum_{k=0}^{i} C_k H_{i+1-k}^{j} \\ &= \sum_{k=0}^{i} C_k \left(H_{i-k}^{j+1} + H_{i-k}^{0} A_j \right) = \mathscr{H}_{i}^{j+1} + \mathscr{H}_{i}^{0} A_j, \end{aligned} \tag{42}$$

and, for $j = p$, we have

$$\mathscr{H}_{i+1}^{p} = \sum_{k=0}^{i} C_k H_{i+1-k}^{p} = \left(\sum_{k=0}^{i} C_k H_{i-k}^{0} \right) A_p = \mathscr{H}_{i}^{0} A_p. \tag{43}$$

For $i \geq l$, with $j \in \sigma_{0}^{p-1}$, we have

$$\begin{aligned} \mathscr{H}_{i+1}^{j} &= \sum_{k=0}^{l} C_k H_{i+1-k}^{j} = \sum_{k=0}^{l} C_k \left(H_{i-k}^{j+1} + H_{i-k}^{0} A_j \right) \\ &= \mathscr{H}_{i}^{j+1} + \mathscr{H}_{i}^{0} A_j, \end{aligned} \tag{44}$$

and, for $j = p$, we have

$$\mathscr{H}_{i+1}^{p} = \sum_{k=0}^{l} C_k H_{i+1-k}^{p} = \left(\sum_{k=0}^{l} C_k H_{i-k}^{0} \right) A_p = \mathscr{H}_{i}^{0} A_p. \tag{45}$$

Similarly, we prove that (40) holds. □

3. Output Reachability

In this section we will present necessary and sufficient conditions for output reachability of system (2) and (3). By generalization of definition given in [29] we obtain the following definitions.

Definition 9. The system modeled by (2) and (3) is said to be output reachable in $N \in \mathbb{N}_{+}$ steps if, for any nonnegative final output $y_f \in \mathbb{R}_{+}^{r}$, there exists a nonnegative input sequence $u_i \in \mathbb{R}_{+}^{m}$, $i \in \sigma_{0}^{N-1}$, which steers the output of the system from $x_{-j} = 0$, $j \in \sigma_{0}^{p}$ to y_f, with $u_{-j} = 0$ for $j \in \sigma_{1}^{p}$; that is, $y_f = y_{N-1}$.

Definition 10. The system modeled by (2) and (3) is said to be output reachable if there exists a positive integer $N \in \mathbb{N}_+$ such that the system is output reachable in N steps.

Now, we present a class of nonnegative matrices, called the monomial matrices [18, 30]. The utility of such a matrix will be highlighted in the study of the output reachability of positive linear systems.

A vector $v \in \mathbb{R}_+^n$ with exactly one of its components being nonzero and all the others being zero is called monomial vector or i-monomial if the nonzero component is in the ith position.

Definition 11. A square matrix $A \in \mathbb{R}_+^{n\times n}$ is said to be monomial if it contains n linearly independent monomial columns.

An important property of monomial matrices is given by the following result.

Lemma 12 (see [18]). *Let $A \in \mathbb{R}_+^{n\times n}$. Then A^{-1} exists and is nonnegative if and only if A is a monomial matrix. Furthermore, A^{-1} is also a monomial matrix.*

The characterization of the output reachability is given by the following proposition.

Proposition 13. *The system modeled by (2) and (3) is output reachable if and only if, for some $N \in \mathbb{N}_+$, the output reachability matrix $\mathscr{R}_N$ includes a monomial submatrix of order $r\times r$ $(r \leq Nm)$.*

Proof.

Sufficiency. Let $y_f \in \mathbb{R}_+^r$ be the final output to be reached. From (21) or (25), we have

$$y_{N-1} = \mathscr{Q}_N\widetilde{x}_0 + \mathscr{R}_N u_0^N. \tag{46}$$

With $\widetilde{x}_0 = 0$, this gives

$$y_{N-1} = \mathscr{R}_N u_0^N. \tag{47}$$

The matrix $\mathscr{R}_N$ includes a monomial submatrix of order $r\times r$, and without loss of generality, we can assume that

$$\mathscr{R}_N = \begin{pmatrix} R_1 & R_2 \end{pmatrix} \tag{48}$$

such that $R_1 \in \mathbb{R}_+^{r\times r}$ is a monomial matrix and $R_2 \in \mathbb{R}_+^{r\times(Nm-r)}$. Hence, by Lemma 12, we have $R_1^{-1} \in \mathbb{R}_+^{r\times r}$. Thus, for

$$u_0^N = \begin{pmatrix} R_1^{-1} y_f \\ 0 \end{pmatrix} \in \mathbb{R}_+^{Nm}, \tag{49}$$

we get

$$y_{N-1} = \begin{pmatrix} R_1 & R_2 \end{pmatrix}\begin{pmatrix} R_1^{-1} y_f \\ 0 \end{pmatrix} = y_f; \tag{50}$$

that is, system (2) and (3) is output reachable.

Necessity. Assume that system (2) and (3) is output reachable for some $N \in \mathbb{N}_+$. Thus, for every $z \in \mathbb{R}_+^r$ there exists an input $u^N \in \mathbb{R}_+^{Nm}$ such that

$$z = \mathscr{R}_N u^N, \tag{51}$$

with $\mathscr{R}_N = (r_{ij})_{i\in\sigma_1^r, j\in\sigma_1^{Nm}}$ and $u^N = (u_j)_{j\in\sigma_1^{Nm}}$. In particular, for $z = e_1$, with e_1 being the first column of I_r, we have

$$\sum_{j=1}^{Nm} r_{1j}u_j = 1, \tag{52}$$

and for $i \in \sigma_2^r$, we have

$$\sum_{j=1}^{Nm} r_{ij}u_j = 0. \tag{53}$$

So by (52), there exists $k \in \sigma_1^{Nm}$ such that $u_k \neq 0$, and consequently by equation (53) we have $r_{ik} = 0$ for all $i \in \sigma_2^r$. Hence, if $r_{1k} \neq 0$, then the kth column of $\mathscr{R}_N$ is monomial. If $r_{1k} = 0$, then the kth column of $\mathscr{R}_N$ is null, which implies that

$$\begin{aligned} &\sum_{j=1}^{Nm} r_{1j}u_j = 1, \quad j \neq k, \\ &\sum_{j=1}^{Nm} r_{ij}u_j = 0, \quad j \neq k,\ i \in \sigma_2^r. \end{aligned} \tag{54}$$

The same reasoning gives the existence of a 1-monomial column or another null column of $\mathscr{R}_N$. Since the columns of $\mathscr{R}_N$ are not all null, then $\mathscr{R}_N$ has at least one 1-monomial column.

The same reasoning for $z = e_i$, $i \in \sigma_2^r$, leads to the existence of a i-monomial column. Hence by Definition 11, the matrix $\mathscr{R}_N$ contains a monomial submatrix of order $r\times r$. The proposition is proved. □

Remark 14. If system (2) and (3) is output reachable and

$$\mathscr{R}_N^T\left(\mathscr{R}_N\mathscr{R}_N^T\right)^{-1} \in \mathbb{R}_+^{Nm\times r}, \tag{55}$$

then the nonnegative input $u_0^N \in \mathbb{R}_+^{Nm}$ which steers the output of the system from $x_{-j} = 0$, $j \in \sigma_0^p$, to any desired nonnegative final output $y_f \in \mathbb{R}_+^r$, with $u_{-j} = 0$ for $j \in \sigma_1^q$, can be computed by the formula

$$u_0^N = \mathscr{R}_N^T\left(\mathscr{R}_N\mathscr{R}_N^T\right)^{-1} y_f. \tag{56}$$

4. Null Output Controllability

By generalization of definition given in [11] the precise definitions of the null output controllability of system (2) and (3) are given as follows.

Definition 15. The system modeled by (2) and (3) is said to be null output controllable in $N \in \mathbb{N}_+$ steps if, for any nonnegative initial state sequence $x_{-j} \in \mathbb{R}_+^n$ $(j \in \sigma_0^p)$ and any nonnegative initial input sequence $u_{-j} \in \mathbb{R}_+^m$ $(j \in \sigma_1^q)$, there exists a nonnegative input sequence $u_i \in \mathbb{R}_+^m$, $i \in \sigma_0^{N-1}$, which steers the output of the system from x_{-j} to zero; that is, $y_{N-1} = 0$.

Definition 16. The system modeled by (2) and (3) is said to be null output controllable if there exists a positive integer $N \in \mathbb{N}_+$ such that the system is null output controllable in N steps.

The characterization of the null output controllability is given by the following proposition.

Proposition 17. *The system modeled by (2) and (3) is null output controllable if and only if, for some $N \in \mathbb{N}_+$, the null output controllability matrix $\mathcal{Q}_N$ is null.*

Proof.

Sufficiency. From (21) or (25), at the step $i = N - 1$, we have

$$y_{N-1} = \mathcal{Q}_N \tilde{x}_0 + \mathcal{R}_N u_0^N; \tag{57}$$

since $\mathcal{Q}_N = 0$, then, for $u_0^N = 0$, we have $y_{N-1} = 0$; that is, system (2) and (3) is null output controllable.

Necessity. If system (2) and (3) is null output controllable, then, for some $N \in \mathbb{N}_+$, there exists an input $u_0^N \in \mathbb{R}_+^{Nm}$ such that

$$\mathcal{Q}_N \tilde{x}_0 + \mathcal{R}_N u_0^N = 0. \tag{58}$$

Since $\mathcal{R}_N u_0^N \in \mathbb{R}_+^r$ and $\mathcal{Q}_N \tilde{x}_0 \in \mathbb{R}_+^r$, then $\mathcal{Q}_N \tilde{x}_0 = 0$, which ensures that $\mathcal{Q}_N = 0$ because $\tilde{x}_0 \in \mathbb{R}_+^{n(p+1)+mq}$ by Definition 15, is arbitrary. This finishes the proof. □

System (2) and (3) describes the evolution of the state and output of a system in the nonnegative orthant with delays in the state, input, and output. However, we can rewrite this system in such a way that these delays disappear from the state equation. Let $(x_i)_{i\in\mathbb{N}}$ be the solution of (2) and define a new state variable $\tilde{x}_i \in \mathbb{R}_+^{n(p+1)+mq}$ for $i \in \mathbb{N}$ by

$$\tilde{x}_i = \begin{pmatrix} x_i \\ x_{i-1} \\ \vdots \\ x_{i-p} \\ u_{i-1} \\ \vdots \\ u_{i-q} \end{pmatrix}. \tag{59}$$

It is readily verified that the state $\tilde{x}_i$ satisfies

$$\begin{aligned} \tilde{x}_{i+1} &= A\tilde{x}_i + Bu_i, \quad i \in \mathbb{N}, \\ \tilde{x}_0 &\in \mathbb{R}_+^{n(p+1)+mq}, \end{aligned} \tag{60}$$

and the output y_i satisfies

$$y_i = C\tilde{x}_i + D_0 u_i, \quad i \in \mathbb{N}, \tag{61}$$

where

$$A = \begin{pmatrix} A_0 & A_1 & \cdots & \cdots & A_p & B_1 & \cdots & \cdots & \cdots & B_q \\ I_n & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots & \vdots & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & & & \vdots \\ 0 & \cdots & 0 & I_n & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & \vdots & I_m & \ddots & & & \vdots \\ \vdots & & & & \vdots & 0 & \ddots & \ddots & & \vdots \\ \vdots & & & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & 0 & I_m & 0 \end{pmatrix}, \tag{62}$$

$$B = \begin{pmatrix} B_0 \\ 0 \\ \vdots \\ 0 \\ I_m \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}_+^{(n(p+1)+mq)\times m},$$

$$C = \begin{pmatrix} C_{11} & C_{12} \end{pmatrix},$$

where

$$\begin{aligned} C_{11} &= \begin{pmatrix} C_0 & C_1 & \cdots & C_l & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}_+^{r\times n(p+1)}, \\ C_{12} &= \begin{pmatrix} D_1 & \cdots & D_l & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}_+^{r\times mq}. \end{aligned} \tag{63}$$

Then we have the following result.

Proposition 18. *The system modeled by (2) and (3) is null output controllable if and only if there exists $N \in \mathbb{N}_+$ such that $CA^{N-1} = 0$. In particular, if A is nilpotent, then system (2) and (3) is null output controllable.*

Proof.

Sufficiency. The general solution of (60) is given by

$$\tilde{x}_i = A^i \tilde{x}_0 + \sum_{j=0}^{i-1} A^{i-j-1} Bu_j, \quad i \in \mathbb{N}. \tag{64}$$

For $u_i = 0$, $i \in \sigma_0^{N-1}$, we have $\widetilde{x}_{N-1} = A^{N-1}\widetilde{x}_0$, this implies that

$$y_{N-1} = C\widetilde{x}_{N-1} + D_0 u_{N-1} = CA^{N-1}\widetilde{x}_0 + D_0 u_{N-1} = 0; \quad (65)$$

since $CA^{N-1} = 0$. Hence system (2) and (3) is null output controllable.

Necessity. System (2) and (3) is null output controllable, according to Proposition 17, $\mathcal{Q}_N = 0$ for some $N \in \mathbb{N}_+$. For $u_0^N = 0$, we have

$$y_{N-1} = \mathcal{Q}_N \widetilde{x}_0 + \mathcal{R}_N u_0^N = 0 \quad \forall \widetilde{x}_0 \in \mathbb{R}_+^{n(p+1)+mq}. \quad (66)$$

On the other hand, we have $y_{N-1} = CA^{N-1}\widetilde{x}_0 = 0$; then $CA^{N-1} = 0$ since $\widetilde{x}_0$ is arbitrary. This completes the proof. □

In the remainder of this section and without loss of generality, we assume that $p \geq q$. Indeed, if $p < q$ we can set $A_j = 0$ for $j \in \sigma_{p+1}^q$.

Lemma 19. *For all $i \geq p$, we have*

$$A^i = \begin{pmatrix} H_i^0 & H_i^1 & \cdots & H_i^p & L_i^1 & \cdots & L_i^q \\ H_{i-1}^0 & H_{i-1}^1 & \cdots & H_{i-1}^p & L_{i-1}^1 & \cdots & L_{i-1}^q \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ H_{i-p}^0 & H_{i-p}^1 & \cdots & H_{i-p}^p & L_{i-p}^1 & \cdots & L_{i-p}^q \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}. \quad (67)$$

Proof. Let $u_i = 0$ for $i \in \mathbb{N}$. Then, according to (64), we have

$$\widetilde{x}_i = A^i \widetilde{x}_0, \quad i \in \mathbb{N}. \quad (68)$$

On the other hand, from (18), for all $i \geq p$ we have

$$\widetilde{x}_i = \begin{pmatrix} x_i \\ x_{i-1} \\ \vdots \\ x_{i-p} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} H_i^0 & H_i^1 & \cdots & H_i^p & L_i^1 & \cdots & L_i^q \\ H_{i-1}^0 & H_{i-1}^1 & \cdots & H_{i-1}^p & L_{i-1}^1 & \cdots & L_{i-1}^q \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ H_{i-p}^0 & H_{i-p}^1 & \cdots & H_{i-p}^p & L_{i-p}^1 & \cdots & L_{i-p}^q \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_{-1} \\ \vdots \\ x_{-p} \\ u_{-1} \\ \vdots \\ u_{-q} \end{pmatrix}. \quad (69)$$

Hence by identification between (68) and (69), we get that (67) holds. □

Proposition 20. *If, for some $s \in \sigma_0^l$, C_s is injective, that is, $\operatorname{rank} C_s = n$, then system (2) and (3) is null output controllable implying that A is a nilpotent matrix.*

Proof. System (2) and (3) is null output controllable; then by Proposition 17, for some $N \in \mathbb{N}_+$, we have $\mathcal{Q}_N = 0$. If $N \leq l$, then $\mathscr{H}_{N-1}^j = 0$, $j \in \sigma_0^p$, $C_k = 0$, $k \in \sigma_N^l$ and $\mathscr{L}_{N-1}^k = 0$, $k \in \sigma_1^q$, $D_j = 0$, $j \in \sigma_N^l$. Then $s \in \sigma_0^{N-1}$ and $C_s H_{N-1-s}^j = 0$, $C_s L_{N-1-s}^k = 0$. Since C_s is injective, then $C_s^T C_s$ is invertible, which implies that $H_{N-1-s}^j = 0$ and $L_{N-1-s}^k = 0$. By Lemma 7, for $i \in \sigma_0^p$ we get $H_{N-1-s+i}^j = 0$ and $L_{N-1-s+i}^k = 0$. According to Lemma 19, we have $A^{N-1-s+p} = 0$, that is, A is nilpotent. Similarly, we prove that A is nilpotent if $N \geq 1+l$. This finishes the proof. □

5. Output Controllability

By generalization of definition given in [11] we shall formulate the fundamental definitions for output controllability of system (2) and (3) as follows.

Definition 21. The system modeled by (2) and (3) is said to be output controllable in $N \in \mathbb{N}_+$ steps if for any nonnegative initial state sequence $x_{-j} \in \mathbb{R}_+^n$ ($j \in \sigma_0^p$) and any nonnegative initial input sequence $u_{-j} \in \mathbb{R}_+^m$ ($j \in \sigma_1^q$), there exists a nonnegative input sequence $u_i \in \mathbb{R}_+^m$, $i \in \sigma_0^{N-1}$, which steers the output of the system from x_{-j} to any desired nonnegative final output $y_f \in \mathbb{R}_+^r$, i.e., $y_{N-1} = y_f$.

Definition 22. The system modeled by (2) and (3) is said to be output controllable if there exists a positive integer $N \in \mathbb{N}_+$ such that the system is output controllable in N steps.

The characterization of the output controllability is given by the following proposition.

Proposition 23. *The system modeled by (2) and (3) is output controllable if and only if it is output reachable and null output controllable.*

Proof.

Necessity. It is evident.

Sufficiency. Since system (2) and (3) is output reachable, then, according to Proposition 13, $\mathcal{R}_{N_1}$ for some $N_1 \in \mathbb{N}_+$ includes a monomial submatrix of order $r \times r$. On the other hand, system (2) and (3) is null output controllable; hence, according to Proposition 17, $\mathcal{Q}_{N_2} = 0$ for some $N_2 \in \mathbb{N}_+$. Then, for $N = \max\{N_1, N_2\}$, the matrix

$$\mathcal{R}_N = \begin{pmatrix} \widetilde{\mathcal{R}} & \mathcal{R}_{N_1} \end{pmatrix} \quad (70)$$

contains a monomial submatrix of order $r \times r$, with $\widetilde{\mathcal{R}} \in \mathbb{R}_+^{r\times(N-N_1)m}$. Hence, by proof of Proposition 13, for any $y_f \in$

$\mathbb{R}^r_+$, there exists a nonnegative input $u_0^N \in \mathbb{R}_+^{Nm}$ such that

$$y_f = \mathscr{R}_N u_0^N. \tag{71}$$

And by Lemma 8, we have $\mathscr{Q}_N = 0$. Then for every $\tilde{x}_0 \in \mathbb{R}_+^{n(p+1)+mq}$ we get that

$$y_{N-1} = \mathscr{Q}_N \tilde{x}_0 + \mathscr{R}_N u_0^N = y_f; \tag{72}$$

that is, system (2) and (3) is output controllable. The proposition is proved. □

6. Numerical Examples

Example 1 (output reachability). Suppose that we are given system (2) and (3) with $p = q = l = 2$ and matrices

$$A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix},$$

$$B_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

$$C_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$D_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{73}$$

The conditions of Proposition 13 are satisfied because the output reachability matrix in five steps

$$\mathscr{R}_5 = \begin{pmatrix} \mathscr{K}_3 & \mathscr{K}_2 & \overline{\mathscr{K}}_1 & \overline{\mathscr{K}}_0 & D_0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{74}$$

contains a monomial submatrix of order 2×2.

By simple calculation, we get

$$\mathscr{R}_5^T \left(\mathscr{R}_5 \mathscr{R}_5^T\right)^{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0.4 & 0 \\ 0 & 0 \\ 0.2 & 0 \end{pmatrix} \in \mathbb{R}_+^{5\times 2}. \tag{75}$$

Then the nonnegative input sequence that permitted to transfer the output from the zero initial conditions to the final output $y_f = (1 \;\; 0.5)^T$ according to (56) is

$$u_0^5 = \mathscr{R}_5^T \left(\mathscr{R}_5 \mathscr{R}_5^T\right)^{-1} y_f = (0.5 \;\; 0 \;\; 0.4 \;\; 0 \;\; 0.2)^T. \tag{76}$$

Table 1 gives the values of the output at each step. We see that the final output has been reached within a number of steps of the input data sequence greater than $n + 1 = 4$.

This comes up to be a particularity of discrete delay systems. This is not satisfied in the case of discrete systems without delay where the steps to reach the final output y_f are always less than or equal to $n+1$. This results from the Cayley-Hamilton theorem.

The next two examples study, respectively, the conditions of the null output controllability and output controllability.

Example 2 (null output controllability). Consider the system modeled by (2) and (3) with matrices

$$A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

Table 1: Values of the outputs in the transfer steps.

N	1	2	3	4	5
y_{N-1}	$\begin{pmatrix} 0.5 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0.5 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1.4 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0.5 \end{pmatrix}$

$$B_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, \quad C_0 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{77}$$

System (2) and (3) is null output controllable because the null output controllability matrix in four steps

$$\mathcal{Q}_4 = \begin{pmatrix} \mathcal{H}_3^0 & \mathcal{H}_3^1 & \mathcal{H}_3^2 & \mathcal{L}_3^1 & \mathcal{L}_3^2 \end{pmatrix} \tag{78}$$

is null.

System (2), (3) in this example is null output controllable for any $C_k \in \mathbb{R}_+^{r\times 3}$ $(k \in \sigma_0^2)$ because the matrix A is nilpotent with index $k = 6$; that is, $A^{k-1} \neq 0$ and $A^k = 0$.

Example 3 (output controllability). Consider the system modeled by (2) and (3) with matrices

$$A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0 & 0 \\ 1 & 2 \\ 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 3 \end{pmatrix}, \quad C_0 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$D_0 = \begin{pmatrix} 1 & 4 \\ 0 & 3 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}. \tag{79}$$

System (2) and (3) is output reachable because the output reachability matrix in tree steps

$$\mathcal{R}_3 = \begin{pmatrix} \overline{\mathcal{K}}_1 & \overline{\mathcal{K}}_0 & D_0 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 1 & 1 & 1 & 4 \\ 1 & 1 & 0 & 1 & 0 & 3 \end{pmatrix} \tag{80}$$

contains a monomial submatrix of order 2×2.

The conditions of Proposition 17 are satisfied because the null output controllability matrix in four steps

$$\mathcal{Q}_4 = \begin{pmatrix} \mathcal{H}_3^0 & \mathcal{H}_3^1 & \mathcal{H}_3^2 & \mathcal{L}_3^1 & \mathcal{L}_3^2 \end{pmatrix} \tag{81}$$

is null, so by proof of Proposition 23, the system is output controllable in four steps.

7. Conclusion

The output controllability of positive discrete linear systems with delays in state, control, and output has been considered. Necessary and sufficient conditions for the positivity of discrete systems have been established (Proposition 4). Criteria for output reachability (Proposition 13) and null output controllability (Proposition 17) of the positive discrete systems have been also proved. It has been shown that output reachability and null output controllability together imply output controllability (Proposition 23). Numerical examples were given to illustrate the results.

We think that the techniques used in this paper can be useful to investigate the output reachability, null output controllability, and output controllability problems for different positive dynamical systems such as switched systems, fractional systems with different orders, and fractional switched systems.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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