Hindawi
Journal of Applied Mathematics
Volume 2017, Article ID 3591713, 5 pages
http://dx.doi.org/10.1155/2017/3591713

*Research Article*

# New Integrals Arising in the Samara-Valencia Heat Transfer Model in Grinding

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Received 20 August 2017; Accepted 24 October 2017; Published 14 November 2017

Academic Editor: Ali R. Ashrafi



The Samara-Valencia model for heat transfer in grinding has been recently used for calculating nontabulated integrals. Based on these results, new infinite integrals can be calculated, involving the Macdonald function and the modified Struve function.

## 1. Introduction

Usually, mathematical developments facilitate the computation of mathematical modeling expressions in many different fields. However, mathematical modeling also yields in many cases a good field to develop new mathematical identities and formulas. This is the case of the mathematical modeling of heat transfer in surface grinding. This machining process consists in material removal from a workpiece by an abrasive wheel that rotates at high speed over its surface [1]. Classically, Jaeger's model [2, 3] is used for the calculation of the temperature field in dry grinding. DesRuisseaux's model [4] extends Jaeger's model to include the effect of surface cooling (wet grinding). More recently, the Samara-Valencia model [5] has been proposed. In this model, the two-dimensional convective heat equation is considered. Also, the heat flux profile entering the workpiece and the action of the coolant are considered in the boundary condition. In [5], this boundary-value problem is transformed into an integral equation that is useful for the numerical evaluation of the heat transfer in intermittent wet grinding [6]. However, in the case of dry grinding, this integral equation can be reduced to a two-dimensional integral ($T^{(0)}$ *theorem*) [7].

New mathematical identities have been proved in this framework. For instance, comparing Jaeger's model with Samara-Valencia model, a new Dirac delta representation [8],

$$\lim_{u\to 0}\frac{|u|}{\pi}\frac{K_1\left(\sqrt{x^2+u^2}\right)}{\sqrt{x^2+u^2}}=\delta(x), \tag{1}$$

and two new nontabulated integrals [9] have been obtained. Also, by using $T^{(0)}$ *theorem*, the following integrals have been calculated [10]:

$$\int_0^\infty \cosh(\alpha\xi)\,K_0\left(\beta\sqrt{\xi^2+y^2}\right)d\xi = \frac{\pi\exp\left(-|y|\sqrt{\beta^2-\alpha^2}\right)}{2\sqrt{\beta^2-\alpha^2}},\quad \beta>|\alpha|,\ y\in\mathbb{R}, \tag{2}$$

$$K_0\left(\sqrt{x^2+y^2}\right) = \frac{|y|}{\pi}\int_{-\infty}^{\infty}K_0(|\xi-x|)\frac{K_1\left(\sqrt{\xi^2+y^2}\right)}{\sqrt{\xi^2+y^2}}d\xi,\quad x,y\in\mathbb{R}, \tag{3}$$

$$\int_{-\infty}^{\infty}K_0(|\xi|)\,K_0(|x-\xi|)\,d\xi=\frac{\pi^2}{2}e^{-|x|},\quad x\in\mathbb{R}. \tag{4}$$

It is worth noting that (3) has been calculated also in [9] by using a complex integration contour. Recently, in [11], the following generalization of (4) has been calculated as a finite sum of terms containing beta and hypergeometric functions, by using the convolution theorem of the Fourier transform. This generalization reads as follows:

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left|t'\right|^{\alpha+2n} K_\alpha\left(a\left|t'\right|\right)\left|t-t'\right|^{\beta+2m} K_\beta\left(a\left|t-t'\right|\right) dt' \\
&= \frac{\pi}{a}\frac{(2n)!\,(2m)!}{(2a)^\mu}\Gamma(2\alpha+2n+1)\,\Gamma(2\beta+2m+1) \\
&\times \sum_{k=0}^{n+m}(-4)^k c_k(n,m,\alpha,\beta)\times\Bigg\{\frac{1}{2} \\
&\cdot B\left(k+\frac{1}{2},\mu-k+\frac{1}{2}\right) \\
&\cdot {}_1F_2\left(\begin{matrix} k+\frac{1}{2} \\ \frac{1}{2}, k-\mu+\frac{1}{2}\end{matrix}\,\middle|\,\frac{a^2t^2}{4}\right)+(-1)^{k+1} \\
&\cdot |at|^{2(\mu-k)+1}\sin\pi(\alpha+\beta)\,\Gamma(2(k-\mu)-1) \\
&\times {}_1F_2\left(\begin{matrix} \mu+1 \\ \mu-k+1, \mu-k+\frac{3}{2}\end{matrix}\,\middle|\,\frac{a^2t^2}{4}\right)\Bigg\}, \\
&\qquad \operatorname{Re}\alpha > -\frac{1}{2}-n,\ \operatorname{Re}\beta > -\frac{1}{2}-m,\ a>0,\ t\in\mathbb{R},
\end{aligned} \tag{5}$$

where the following coefficients are defined as

$$c_k(n,m,\alpha,\beta) = \sum_{l=\max(k-m,0)}^{\min(n,k)} b_l(n,\alpha)\, b_{k-l}(m,\beta), \tag{6}$$

with

$$b_l(n,\alpha) = \frac{1}{(n-l)!\,(2l)!\,\Gamma(n+\alpha-l+1)}. \tag{7}$$

The scope of this paper is just to calculate more integrals based on results (2)–(4), which do not seem to be reported in the most common tables of integrals [12–14].

This paper is organized as follows. Section 2 is devoted to the calculation of the new integrals. It is divided into three subsections, each one of them containing one new result. Section 3 collects the conclusions, highlighting the main results obtained in the body of the paper.

## 2. The Integrals

### *2.1. First Result*

**Theorem 1.** *The following integral holds true:*

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{\pm\alpha\xi}\frac{K_1\left(\beta\sqrt{\xi^2+y^2}\right)}{\sqrt{\xi^2+y^2}} d\xi \\
&= 2\int_0^{\infty}\cosh(\alpha\xi)\frac{K_1\left(\beta\sqrt{\xi^2+y^2}\right)}{\sqrt{\xi^2+y^2}} d\xi \\
&= \frac{\pi\exp\left(-|y|\sqrt{\beta^2-\alpha^2}\right)}{\beta|y|}, \quad \beta>|\alpha|,\ y\in\mathbb{R}\setminus\{0\}.
\end{aligned} \tag{8}$$

*Proof.* Expanding the hyperbolic cosine as $\cosh x = (e^x + e^{-x})/2$ [15, Eqn. 8.2] in (2), we have

$$\begin{aligned}
&\frac{\pi\exp\left(-|y|\sqrt{\beta^2-\alpha^2}\right)}{\sqrt{\beta^2-\alpha^2}} \\
&= \int_0^{\infty} e^{\alpha\xi}K_0\left(\beta\sqrt{\xi^2+y^2}\right)d\xi \\
&+ \int_0^{\infty} e^{-\alpha\xi}K_0\left(\beta\sqrt{\xi^2+y^2}\right)d\xi.
\end{aligned} \tag{9}$$

Now, perform the change of variables $\xi \to -\xi$, and rewrite the first or the second integral given in (9) as

$$\int_0^{\infty} e^{\pm\alpha\xi}K_0\left(\beta\sqrt{\xi^2+y^2}\right)d\xi = \int_{-\infty}^{0} e^{\mp\alpha\xi}K_0\left(\beta\sqrt{\xi^2+y^2}\right)d\xi. \tag{10}$$

Therefore, inserting (10) in (9), we can define the following function:

$$\begin{aligned}
F(y) &= \int_{-\infty}^{\infty} e^{\pm\alpha\xi}K_0\left(\beta\sqrt{\xi^2+y^2}\right)d\xi \\
&= \frac{\pi\exp\left(-|y|\sqrt{\beta^2-\alpha^2}\right)}{\sqrt{\beta^2-\alpha^2}}, \quad \beta>|\alpha|,\ y\in\mathbb{R}.
\end{aligned} \tag{11}$$

Let us take temporarily $y > 0$, so that we can drop the absolute value in (11). Then, knowing that $K_0'(x) = -K_1(x)$ [16, Eqn. 51:10:2], we have

$$\begin{aligned}
F'(y) &= \beta y\int_{-\infty}^{\infty} e^{\pm\alpha\xi}\frac{K_1\left(\beta\sqrt{\xi^2+y^2}\right)}{\sqrt{\xi^2+y^2}}d\xi \\
&= \pi\exp\left(-y\sqrt{\beta^2-\alpha^2}\right).
\end{aligned} \tag{12}$$

Recover now the absolute value and take into account again (9)-(10) to obtain (8). □

*Remark 2.* It is worth noting that when $y = 0$, (12) seems to fail. Nonetheless, this is apparent. In order to see it, rewrite the RHS of (12) as follows, knowing that $\beta > 0$ and performing the changes of variables $x = \beta x$ and $u = \beta y$:

$$F'(u) = u\int_{-\infty}^{\infty}\exp\left(\pm\frac{\alpha}{\beta}x\right)\frac{K_1\left(\sqrt{x^2+u^2}\right)}{\sqrt{x^2+u^2}}dx. \tag{13}$$

Therefore, taking the limit $u \to 0$ and applying the Dirac delta representation given in (1), we have

$$\begin{aligned}
&\lim_{u\to 0} F'(u) \\
&= \lim_{u\to 0} u\int_{-\infty}^{\infty}\exp\left(\pm\frac{\alpha}{\beta}x\right)\frac{K_1\left(\sqrt{x^2+u^2}\right)}{\sqrt{x^2+u^2}}dx \\
&= \pi\int_{-\infty}^{\infty}\exp\left(\pm\frac{\alpha}{\beta}x\right)\delta(x)\,dx = \pi,
\end{aligned} \tag{14}$$

which agrees with the LHS of (12), performing the limit $y \to 0$.

We can continue calculating derivatives with respect to $y$ in (12) in order to get new integrals, but the integrands we get are increasingly complex and we omit these results here.

### 2.2. Second Result

**Theorem 3.** *The following integral holds true:*

$$\int_0^\infty K_0(\xi) K_0\left(\sqrt{\xi^2+y^2}\right) d\xi = \frac{\pi^2}{4}\{1 - |y| [K_0(|y|) \mathbf{L}_{-1}(|y|) + K_1(|y|) \mathbf{L}_0(|y|)]\}, \quad y \in \mathbb{R}. \tag{15}$$

*Proof.* Let us define

$$G(y) = \int_0^\infty K_0(\xi) K_0\left(\sqrt{\xi^2+y^2}\right) d\xi \tag{16}$$

$$= \frac{1}{2}\int_{-\infty}^\infty K_0(|\xi|) K_0\left(\sqrt{\xi^2+y^2}\right) d\xi. \tag{17}$$

Notice that

$$G(y) = G(|y|); \tag{18}$$

thereby, hereafter, we will assume that $y > 0$, dropping the absolute value. Performing in (17) the derivative with respect to $y$, we have

$$G'(y) = -\frac{y}{2}\int_{-\infty}^\infty K_0(|\xi|) \frac{K_1\left(\sqrt{\xi^2+y^2}\right)}{\sqrt{\xi^2+y^2}} d\xi. \tag{19}$$

The above integral can be calculated taking $x = 0$ in (3); thus,

$$G'(y) = -\frac{\pi}{2} K_0(y). \tag{20}$$

Now, let us apply the following integral [12, Eqn. 1.12.1(3)]:

$$\int_0^z x^\nu K_\nu(x)\, dx = 2^{\nu-1}\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right) \cdot z [K_\nu(z) \mathbf{L}_{\nu-1}(z) + K_{\nu-1}(z) \mathbf{L}_\nu(z)], \tag{21}$$

where $\mathbf{L}_\nu(z)$ is the modified Struve function, defined as [17, Eqn. 11.2.2]

$$\mathbf{L}_\nu(z) = \sum_{n=0}^\infty \frac{(z/2)^{2n+\nu+1}}{\Gamma(n+3/2)\,\Gamma(n+\nu+3/2)}. \tag{22}$$

Therefore, taking in (21) $\nu = 0$, knowing that $K_{-\nu}(z) = K_\nu(z)$ ([18], Eqn. 5.7.10) and also that [16, Eqn. 43:4:2]

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \tag{23}$$

we calculate $G(y)$ from (20) as

$$G(y) = -\frac{\pi}{4} y [K_0(y) \mathbf{L}_{-1}(y) + K_1(y) \mathbf{L}_\nu(y)] + C, \tag{24}$$

where $C$ is an integration constant. In order to calculate this integration constant, notice, on the one hand, that, from (16), we have

$$G(0) = \int_0^\infty K_0^2(\xi)\, d\xi. \tag{25}$$

The above integral can be calculated taking $x = 0$ in (4):

$$\int_{-\infty}^\infty K_0^2(|\xi|)\, d\xi = 2\int_0^\infty K_0^2(\xi)\, d\xi = \frac{\pi^2}{2}. \tag{26}$$

Thus,

$$G(0) = \frac{\pi^2}{4}. \tag{27}$$

On the other hand, taking limits in (24), we have

$$G(0) = C - \frac{\pi}{4}\lim_{y\to 0} y [K_0(y) \mathbf{L}_{-1}(y) + K_1(y) \mathbf{L}_0(y)]. \tag{28}$$

According to (22), consider the following asymptotic formula:

$$\mathbf{L}_{-1}(y) \approx \frac{1}{\Gamma(3/2)\,\Gamma(1/2)} = \frac{2}{\pi}, \quad y \longrightarrow 0, \tag{29}$$

where we have applied the property of the gamma function $\Gamma(z+1) = z\Gamma(z)$ ([18], Eqn. 1.2.1) and (23). Similarly, we have

$$\mathbf{L}_0(y) \approx \frac{y/2}{\Gamma(3/2)\,\Gamma(1/2)} = \frac{y}{\pi}, \quad y \longrightarrow 0. \tag{30}$$

Also, the asymptotic behavior of the Macdonald function for $y \to 0$ is ([17], Eqn. 10.30.2-3)

$$K_0(y) \approx -\log y, \qquad K_\nu(y) \approx \frac{1}{2}\Gamma(\nu)\left(\frac{y}{2}\right)^{-\nu}, \quad \operatorname{Re}\nu > 0, \tag{31}$$

and thus

$$K_1(y) \approx \frac{1}{y}. \tag{32}$$

Therefore, taking into account (29)–(32), we calculate the limit given in (28) as

$$\lim_{y\to 0} y [K_0(y) \mathbf{L}_{-1}(y) + K_1(y) \mathbf{L}_0(y)] = \frac{1}{\pi}\lim_{y\to 0} y [1 - 2\log y] = 0. \tag{33}$$

Thereby, inserting (33) in (28) and recalling (27), we calculate the integration constant as follows:

$$G(0) = C = \frac{\pi^2}{4}. \tag{34}$$

Finally, substituting (34) into (24) and remembering the definition of $G(y)$, we arrive at (15), where we have considered (18). □

*2.3. Third Result*

**Theorem 4.** *The next integral holds true:*

$$\int_0^\infty \{1 - y[K_0(y)\mathbf{L}_{-1}(y) + K_1(y)\mathbf{L}_0(y)]\}\,dy = \frac{2}{\pi}. \quad (35)$$

*Proof.* For the third result, integrate both sides of (16) as follows:

$$\begin{aligned}\int_0^\infty G(y)\,dy \\ = \int_0^\infty K_0(\xi)\left\{\int_0^\infty K_0\left(\sqrt{\xi^2+y^2}\right)dy\right\}d\xi.\end{aligned} \quad (36)$$

In order to calculate the inner integral given in (36), consider $\alpha = 0$ and $\beta = 1$ in (2); hence,

$$\int_0^\infty K_0\left(\sqrt{\xi^2+y^2}\right)dy = \frac{\pi}{2}e^{-|\xi|}. \quad (37)$$

Inserting (37) into (36), we can drop the absolute value in the exponential, since the variable of integration is positive (i.e., $\xi \in (0,\infty)$); thereby,

$$\int_0^\infty G(y)\,dy = \frac{\pi}{2}\int_0^\infty K_0(\xi)\,e^{-\xi}d\xi. \quad (38)$$

To calculate the above integral, in the literature, we find the following integral, termed *King's integral* [19, Eqn. 11.3.16]:

$$\int_0^x e^{\pm u}K_0(u)\,du = xe^{\pm x}[K_0(x) \pm K_1(x)] \mp 1. \quad (39)$$

Hence, applying to *King's Integral* and the asymptotic formula [18, Eqn. 5.16.5]

$$K_\nu(x) \approx \sqrt{\frac{\pi}{2x}}e^{-x}, \quad x \longrightarrow +\infty, \quad (40)$$

we obtain

$$\begin{aligned}\int_0^\infty K_0(\xi)\,e^{-\xi}d\xi &= \lim_{x\to\infty} xe^{-x}[K_0(x) - K_1(x)] + 1 \\ &= 1.\end{aligned} \quad (41)$$

Recalling now the result given in (15) for $G(y)$ and taking into account (41), (38) finally reads as (35). □

## 3. Conclusions

Based on integrals (2)–(4), calculated in the framework of the Samara-Valencia heat transfer model in surface grinding by using the $T^{(0)}$ *theorem*, new integrals have been derived. From (2), we have derived integrals (11) and (8). Also, applying (3) and (4), we have derived (15). Finally, integrating the result given in (15) and taking into account (2), the integral given in (35) has been obtained. It is worth noting that these results have been confirmed, evaluating numerically the corresponding integrals.

## Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

## Acknowledgments

The author wishes to acknowledge the financial support received from Universidad Católica de Valencia under Grants PRUCV/2015/612 and 2017-160-001.

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