

Research Article

Global Existence of Weak Solutions to a Fractional Model in Magnetoelastic Interactions

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Received 6 July 2016; Accepted 8 September 2016

Academic Editor: Cemil Tunç

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The paper deals with global existence of weak solutions to a one-dimensional mathematical model describing magnetoelastic interactions. The model is described by a fractional Landau-Lifshitz-Gilbert equation for the magnetization field coupled to an evolution equation for the displacement. We prove global existence by using Faedo-Galerkin/penalty method. Some commutator estimates are used to prove the convergence of nonlinear terms.

1. Introduction

The nonlinear parabolic hyperbolic coupled system describing magnetoelastic dynamics in $Q = (0, T) \times \Omega$ ($T > 0$ and Ω is a bounded open set of \mathbb{R}^d , $d \geq 1$) is given by (see [1])

$$\gamma^{-1} \mathbf{m}_t = -\mathbf{m} \times (\mathbf{H}_{\text{eff}} + \mathbf{m}_t). \quad (1)$$

$$\rho \mathbf{u}_{tt} - \text{div} \left(\mathcal{S}(\mathbf{u}) + \frac{1}{2} \mathcal{L}(\mathbf{m}) \right) = 0. \quad (2)$$

Equation (1), well known in the literature, is the Landau-Lifshitz-Gilbert (LLG) equation. The unknown \mathbf{m} , the magnetization vector, is a map from Ω to S^2 (the unit sphere of \mathbb{R}^3) and \mathbf{m}_t is its derivative with respect to time. The symbol \times denotes the vector cross product in \mathbb{R}^3 . Moreover we denote by m_i , $i = 1, 2, 3$, the components of \mathbf{m} . The constant γ represents the damping parameter. \mathbf{H}_{eff} represents the effective field which is given by

$$\mathbf{H}_{\text{eff}} = a \Delta \mathbf{m} + \boldsymbol{\ell}(\mathbf{m}, \mathbf{u}), \quad (3)$$

where a is a positive constant and the components of the vector $\boldsymbol{\ell}(\mathbf{m}, \mathbf{u})$ are given by

$$\ell_i = \lambda_{ijkl} m_j \epsilon_{kl}(\mathbf{u}). \quad (4)$$

Here $\epsilon_{ij}(\mathbf{u}) = (1/2)(\partial_i u_j + \partial_j u_i)$ stand for the components of the linearized strain tensor ϵ , $\lambda_{ijkl} = \lambda_1 \delta_{ijkl} + \lambda_2 \delta_{ij} \delta_{kl} + \lambda_3 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$ with $\delta_{ijkl} = 1$ if $i = j = k = l$ and $\delta_{ijkl} = 0$ otherwise.

Equation (2) describes the evolution of the displacement \mathbf{u} , ρ is a positive constant, and the tensors $\mathcal{S}(\mathbf{u})$, $\mathcal{L}(\mathbf{m})$ are given by

$$\begin{aligned} \mathcal{S}_{kl} &= \sigma_{ijkl} \epsilon_{ij}(\mathbf{u}), \\ \mathcal{L}_{kl} &= \lambda_{ijkl} m_i m_j. \end{aligned} \quad (5)$$

$\sigma = (\sigma_{ijkl})$ is the elasticity tensor satisfying the following symmetry property:

$$\sigma_{ijkl} = \sigma_{klij} = \sigma_{jikl}. \quad (6)$$

Many studies have been done on the fractional Landau-Lifshitz equation; we quote here, for example, [2], where the existence of weak solutions under periodical boundary condition has been proven for equation of a reduced model for thin-film micromagnetics. In [3], the main purpose is to consider the well-posedness of the fractional Landau-Lifshitz equation without Gilbert damping. The global existence of weak solutions is proved by vanishing viscosity method.

Note that the existence and asymptotic behaviors of global weak solutions to the one-dimensional periodical fractional Landau-Lifshitz equation modeling the soft micromagnetic materials are studied in [4]. For the magnetoelasticity coupling, in [1], the authors study the three-dimensional case and establish the existence of weak solutions taking into account three terms of the total free energy. Existence and uniqueness of solutions have been proven in [5] for a simplified model and in [6] a one-dimensional penalty problem is discussed and the gradient flow of the associated type Ginzburg-Landau functional is studied. More precisely the authors prove the existence and uniqueness of a classical solution which tends asymptotically for subsequences to a stationary point of the energy functional. Our aim here is to study the coupled system of magnetoelastic interactions with fractional LLG equation.

The rest of the paper is divided as follows. In the next section we present the model equation we will be interested in. Section 3 recalls some useful lemmas. Finally in Section 4 we prove a global existence result of weak solutions to the considered model.

2. The Model and Main Result

We assume that Ω is a subset of \mathbb{R} and the displacement is only in one direction. Specifically, we consider a simple variable space x and assume that $\Omega = (0, 2\pi)$. We take the following system:

$$\begin{aligned} \gamma^{-1} \mathbf{m}_t &= -\mathbf{m} \times (\mathbf{H}_{\text{eff}} + \mathbf{m}_t) \\ \rho \mathbf{u}_{tt} - \text{div} \left(\mathcal{S}(\mathbf{u}) + \frac{1}{2} \mathcal{L}(\mathbf{m}) \right) &= 0, \end{aligned} \quad (7)$$

with associated initial and boundary conditions

$$\begin{aligned} \mathbf{u}(\cdot, 0) &= \mathbf{u}_0, \\ \mathbf{u}_t(\cdot, 0) &= \mathbf{u}_1, \\ \mathbf{m}(\cdot, 0) &= \mathbf{m}_0, \\ |\mathbf{m}_0| &= 1 \end{aligned} \quad (8)$$

in Ω ,

$$\begin{aligned} \mathbf{u} &= 0, \\ \mathbf{m}(x, t) &= \mathbf{m}(x + 2\pi, t) \end{aligned} \quad (9)$$

on $\Sigma := \partial\Omega \times (0, T)$.

The effective field is given by

$$\mathbf{H}_{\text{eff}} = a\Lambda^{2\alpha} \mathbf{m} + \boldsymbol{\ell}(\mathbf{m}, \mathbf{u}), \quad (10)$$

where $\Lambda = (-\Delta)^{1/2}$ denotes the square root of the Laplacian which can be defined via Fourier transformation [7]. In this paper we are interested in the case $\alpha \in (1/2, 1)$. For the vector \mathbf{u} , we assume that $\mathbf{u} = (0, 0, \omega)$ and we keep the three components of the vector $\mathbf{m} = (m_1, m_2, m_3)$.

It is a common practice (see [5]) to replace the first equation of system (7) by the quasilinear parabolic equation (Ginzburg-Landau type equation):

$$\mathbf{m}_t^\varepsilon + \gamma^{-1} \mathbf{m}_t^\varepsilon \times \mathbf{m}^\varepsilon = -\mathbf{H}_{\text{eff}}^\varepsilon - \frac{|\mathbf{m}^\varepsilon|^2 - 1}{\varepsilon} \mathbf{m}^\varepsilon. \quad (11)$$

Here ε is a positive parameter and $\mathbf{m}^\varepsilon : \Omega \rightarrow \mathbb{R}^3$. ε -penalization in (11) replaces the magnitude constraint $|\mathbf{m}| = 1$.

Throughout, we make use of the following notation. For Ω , an open bounded domain of \mathbb{R}^3 , we denote by $\mathbf{L}^p(\Omega) = (L^p(\Omega))^3$ and $\mathbf{H}^1(\Omega) = (H^1(\Omega))^3$ the classical Hilbert spaces equipped with the usual norm denoted by $\|\cdot\|_{\mathbf{L}^p(\Omega)}$ and $\|\cdot\|_{\mathbf{H}^1(\Omega)}$ (in general, the product functional spaces $(X)^3$ are all simplified to \mathbf{X}). For all $s > 0$, $W^{s,p}$ denotes the usual Sobolev space consisting of all f such that

$$\|f\|_{W^{s,p}} := \left\| \mathcal{F}^{-1} (1 + |\cdot|^2)^{s/2} (\mathcal{F}f)(\cdot) \right\|_{L^p} < \infty, \quad (12)$$

where \mathcal{F} denotes the Fourier transform and \mathcal{F}^{-1} its inverse. Let $\dot{W}^{s,p}$ denote the corresponding homogeneous Sobolev space. When $p = 2$, $W^{s,p}$ corresponds to the usual Sobolev space H^s .

Now we give a definition of the solution in the weak sense of problem (7)-(8)-(9).

Definition 1. Let $\mathbf{m}_0 \in \mathbf{H}^\alpha(\Omega)$, $|\mathbf{m}_0| = 1$ a.e., $\omega_0 \in H_0^1(\Omega)$, and $\omega_1 \in L^2(\Omega)$. One says that the pair (\mathbf{m}, ω) is a weak solution of problem (7)-(8)-(9) if

(i) for all $T > 0$, $\mathbf{m} \in L^\infty(0, T; \mathbf{H}^\alpha(\Omega))$, $\mathbf{m}_t \in L^2(0, T; L^2(\Omega))$, $|\mathbf{m}| = 1$ a.e., $\omega \in L^2(0, T; H_0^1(\Omega))$, and $\omega_t \in L^2(0, T; L^2(\Omega))$;

(ii) for all $\boldsymbol{\varphi} \in L^2(0, T; \mathbf{H}^\alpha(\Omega))$ and $\psi \in H_0^1(Q)$ one has

$$\begin{aligned} \gamma^{-1} \int_Q \mathbf{m}_t \cdot \boldsymbol{\varphi} \, dx \, dt + \int_Q (\mathbf{m} \times \mathbf{m}_t) \cdot \boldsymbol{\varphi} \, dx \, dt \\ = a \int_Q \Lambda^\alpha \mathbf{m} \cdot \Lambda^\alpha (\mathbf{m} \times \boldsymbol{\varphi}) \, dx \, dt \\ + \int_Q (\boldsymbol{\ell}(\mathbf{m}, \omega) \times \mathbf{m}) \cdot \boldsymbol{\varphi} \, dx \, dt \\ - \rho \int_Q \omega_t \psi_t \, dx \, dt + \int_Q \omega_x \psi_x \, dx \, dt \\ + \lambda \int_Q m_1 m_3 \psi_x \, dx \, dt = 0; \end{aligned} \quad (13)$$

(iii) $\mathbf{m}(0, x) = \mathbf{m}_0(x)$ and $\omega(0, x) = \omega_0(x)$ in the trace sense;

(iv) for all $T > 0$ we have

$$\begin{aligned} \frac{a}{2} \int_\Omega |\Lambda^\alpha \mathbf{m}(T)|^2 \, dx + \int_Q |\mathbf{m}_t|^2 \, dx \, dt \\ + \frac{\rho}{2} \int_\Omega |\omega_t(T)|^2 \, dx + \frac{1}{4} \int_\Omega |\omega_x(T)|^2 \, dx \end{aligned}$$

$$\begin{aligned} &\leq \frac{a}{2} \int_{\Omega} |\Lambda^\alpha \mathbf{m}_0|^2 dx + \frac{\rho}{2} \int_{\Omega} |\omega_1|^2 dx \\ &\quad + \frac{1}{4} \int_{\Omega} |\omega_x(0)|^2 dx + C(\Omega, \lambda), \end{aligned} \tag{14}$$

where $C(\Omega, \lambda)$ is a positive constant which depends only on Ω and λ .

The main result of this paper is the following.

Theorem 2. *Let $\alpha \in (1/2, 1)$, $\mathbf{m}_0 \in \mathbf{H}^\alpha(\Omega)$ such that $|\mathbf{m}_0| = 1$ a.e., $\omega_0 \in H_0^1(\Omega)$, and $\omega_1 \in L^2(\Omega)$. Then there exists at least a weak solution for problem (7)-(8)-(9) in the sense of Definition 1.*

The proof of Theorem 2 will be given in Section 4.

3. Some Technical Lemmas

In this section we present some lemmas which will be used in the rest of the paper. We start by recalling the following lemma due to Simon (see [8]).

Lemma 3. *Assume A, B , and C are three Banach spaces and satisfy $A \subset B \subset C$ with compact embedding $A \hookrightarrow B$. Let Θ be bounded in $L^\infty(0, T; A)$ and $\Theta_t := \{f_t; f \in \Theta\}$ be bounded in $L^p(0, T; C)$, $p > 1$. Then Θ is relatively compact in $C([0, T]; B)$.*

There is another lemma whose proof can be found in [[9], page 12].

Lemma 4. *Let Θ be a bounded open set of $\mathbb{R}_x^d \times \mathbb{R}_t$, h_n and h in $L^q(\Theta)$, $1 < q < \infty$ such that $\|h_n\|_{L^q(\Theta)} \leq C$, $h_n \rightarrow h$ a.e. in Θ ; then $h_n \rightharpoonup h$ weakly in $L^q(\Theta)$.*

The following lemma will ensure a compact embedding for the space $W^{s,p}$.

Lemma 5. *Let Θ be a bounded open set of \mathbb{R}^d , which is uniform Lipschitz. Let $s \in [0, 1[$, $p > 1, d \geq 1$. If $sp < d$ then the injection of $W^{s,p}(\Theta)$ in $L^k(\Theta)$ is compact, for any $k < dp/(d - sp)$.*

The proof can be found in [[10], Theorem 4.54., p 216]. We give now a lemma that will play a very important role in the convergence of approximate solutions (see [11–13] for a proof).

Lemma 6 (commutator estimates). *Suppose that $s > 0$ and $p \in (1, +\infty)$. If $f, g \in \mathcal{S}$ (the Schwartz class) then*

$$\begin{aligned} &\| \Lambda^s (fg) - f \Lambda^s g \|_{L^p} \\ &\leq C \left(\| \nabla f \|_{L^{p_1}} \| g \|_{\dot{W}^{s-1, p_2}} + \| f \|_{\dot{W}^{s, p_3}} \| g \|_{L^{p_4}} \right), \end{aligned} \tag{15}$$

$$\| \Lambda^s (fg) \|_{L^p} \leq C \left(\| f \|_{L^{p_1}} \| g \|_{\dot{W}^{s, p_2}} + \| f \|_{\dot{W}^{s, p_3}} \| g \|_{L^{p_4}} \right) \tag{16}$$

with $p_2, p_3 \in (1, +\infty)$ such that $1/p = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4$.

Here is another lemma which can be viewed as a result of the Hardy-Littlewood-Sobolev theorem of fractional integration; see [7] for a detailed proof.

Lemma 7. *Suppose that $p > q > 1$ and $1/p + s = 1/q$. Assume that $f \in L^q$; then $\Lambda^{-s} f \in L^p$ and there is a constant $C > 0$ such that*

$$\| f \|_{\dot{W}^{-s, p}} := \| \Lambda^{-s} f \|_{L^p} \leq C \| f \|_{L^q}. \tag{17}$$

We finish this section with the following result (the proof can be found in [2]).

Lemma 8. *If f and g belong to $H_{per}^{2\alpha}(\Omega) := \{ f \in L^2(\Omega) / \Lambda^{2\alpha} f \in L^2(\Omega) \}$, then*

$$\int_{\Omega} \Lambda^{2\alpha} f \cdot g dx = \int_{\Omega} \Lambda^\alpha f \cdot \Lambda^\alpha g dx. \tag{18}$$

4. Proof of Theorem 2

Our goal is to show global existence of weak solutions for the fractional problem (7)-(8)-(9).

4.1. The Penalty Problem. Let $\varepsilon > 0$ be a fixed parameter. We construct approximated solutions \mathbf{m}^ε converging, as $\varepsilon \rightarrow 0$, to a solution \mathbf{m} of the problem. System (7) is reduced to the following problem:

$$\begin{aligned} &\gamma^{-1} \mathbf{m}_t^\varepsilon \times \mathbf{m}^\varepsilon + \mathbf{m}_t^\varepsilon + a \Lambda^{2\alpha} \mathbf{m}^\varepsilon + \boldsymbol{\ell}(\mathbf{m}^\varepsilon, \omega^\varepsilon) \\ &\quad + \frac{|\mathbf{m}^\varepsilon|^2 - 1}{\varepsilon} \mathbf{m}^\varepsilon = 0 \\ &\rho \omega_{tt}^\varepsilon - \omega_{xx}^\varepsilon - \lambda (m_1^\varepsilon m_3^\varepsilon)_x = 0 \end{aligned} \tag{19}$$

in $Q = \Omega \times (0, T)$, where the vector $\boldsymbol{\ell}(\mathbf{m}, \omega)$ is given by $\boldsymbol{\ell}(\mathbf{m}, \omega) = (\lambda m_3 \omega_x, 0, \lambda m_1 \omega_x)$, $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = \lambda$, and $\sigma_{1313} = 1$.

System (19) is supplemented with initial and boundary conditions

$$\begin{aligned} &\omega^\varepsilon(\cdot, 0) = \omega_0, \\ &\omega_t^\varepsilon(\cdot, 0) = \omega_1, \\ &\mathbf{m}^\varepsilon(\cdot, 0) = \mathbf{m}_0, \\ &|\mathbf{m}_0| = 1 \\ &\omega^\varepsilon = 0, \\ &\mathbf{m}^\varepsilon(x, t) = \mathbf{m}^\varepsilon(x + 2\pi, t) \end{aligned} \tag{20}$$

a.e. in Ω ,
on Σ .

We apply Faedo-Galerkin method: let $\{f_i\}_{i \in \mathbb{N}}$ be an orthonormal basis of $L^2(\Omega)$ consisting of all the eigenfunctions for the operator $\Lambda^{2\alpha}$ (the existence of such a basis can be proved as in [14], Ch. II),

$$\begin{aligned} \Lambda^{2\alpha} f_i &= \alpha_i f_i, \quad i = 1, 2, \dots, \\ f_i(0) &= f_i(2\pi), \end{aligned} \tag{21}$$

and let $\{g_i\}_{i \in \mathbb{N}}$ be an orthonormal basis of $L^2(\Omega)$ consisting of all the eigenfunctions for the operator $-\Delta$:

$$\begin{aligned} -\Delta g_i &= \beta_i g_i, \quad i = 1, 2, \dots, \\ g_i &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{22}$$

We then consider the following problem in $Q = \Omega \times (0, T)$:

$$\begin{aligned} \gamma^{-1} \mathbf{m}_t^{\varepsilon, N} \times \mathbf{m}^{\varepsilon, N} + \mathbf{m}_t^{\varepsilon, N} + a\Lambda^{2\alpha} \mathbf{m}^{\varepsilon, N} + \boldsymbol{\ell}(\mathbf{m}^{\varepsilon, N}, \omega^{\varepsilon, N}) \\ + \frac{|\mathbf{m}^{\varepsilon, N}|^2 - 1}{\varepsilon} \mathbf{m}^{\varepsilon, N} = 0 \\ \rho \omega_{tt}^{\varepsilon, N} - \omega_{xx}^{\varepsilon, N} - \lambda(m_1^{\varepsilon, N} m_3^{\varepsilon, N})_x = 0 \end{aligned} \tag{23}$$

with initial and boundary conditions,

$$\begin{aligned} \omega^{\varepsilon, N}(\cdot, 0) &= \omega^N(\cdot, 0), \\ \omega_t^{\varepsilon, N}(\cdot, 0) &= \omega_t^N(\cdot, 0), \\ \mathbf{m}^{\varepsilon, N}(\cdot, 0) &= \mathbf{m}^N(\cdot, 0), \\ &\text{in } \Omega, \\ \omega^{\varepsilon, N} &= 0, \\ \mathbf{m}^{\varepsilon, N}(x, t) &= \mathbf{m}^{\varepsilon, N}(x + 2\pi, t) \\ &\text{on } \Sigma = \partial\Omega \times (0, T), \end{aligned} \tag{24}$$

$$\int_{\Omega} \omega^N(x, 0) g_i(x) dx = \int_{\Omega} \omega_0(x) g_i(x) dx,$$

$$\int_{\Omega} \omega_t^N(x, 0) g_i(x) dx = \int_{\Omega} \omega_1(x) g_i(x) dx,$$

$$\int_{\Omega} \mathbf{m}^N(x, 0) f_i(x) dx = \int_{\Omega} \mathbf{m}_0(x) f_i(x) dx.$$

We are looking for approximate solutions $(\mathbf{m}^{\varepsilon, N}, \omega^{\varepsilon, N})$ to (23) under the form

$$\begin{aligned} \mathbf{m}^{\varepsilon, N} &= \sum_{i=1}^N \mathbf{a}_i(t) f_i(x), \\ \omega^{\varepsilon, N} &= \sum_{i=1}^N b_i(t) g_i(x). \end{aligned} \tag{25}$$

If we multiply each scalar equation of the first equation of (23) by f_i and the second by g_i and integrate in Ω we get

to a system of ordinary differential equations in the unknown $(\mathbf{a}_i(t), b_i(t)), i = 1, 2, \dots, N$. We observe that we can write the first equation in the form

$$\begin{aligned} -a\Lambda^{2\alpha} \mathbf{m}^{\varepsilon, N} - \boldsymbol{\ell}(\mathbf{m}^{\varepsilon, N}, \omega^{\varepsilon, N}) - \frac{|\mathbf{m}^{\varepsilon, N}|^2 - 1}{\varepsilon} \mathbf{m}^{\varepsilon, N} \\ = \mathbb{A}(\mathbf{m}^{\varepsilon, N}) \mathbf{m}_t^{\varepsilon, N} \end{aligned} \tag{26}$$

with

$$\mathbb{A} = \begin{pmatrix} 1 & \gamma^{-1} m_3^{\varepsilon, N} & -\gamma^{-1} m_2^{\varepsilon, N} \\ -\gamma^{-1} m_3^{\varepsilon, N} & 1 & \gamma^{-1} m_1^{\varepsilon, N} \\ \gamma^{-1} m_2^{\varepsilon, N} & -\gamma^{-1} m_1^{\varepsilon, N} & 1 \end{pmatrix}. \tag{27}$$

It is clear that the matrix \mathbb{A} is invertible which implies the system of first-order ordinary differential equations is Lipschitz locally; then there exists a local solution to the problem that we can extend on $[0, T]$ using a priori estimates. For this, we multiply the first equation of (23) by $\mathbf{m}_t^{\varepsilon, N}$ and the second by $\omega_t^{\varepsilon, N}$; integrating in Ω , we obtain (by using Lemma 8)

$$\begin{aligned} \int_{\Omega} |\mathbf{m}_t^{\varepsilon, N}|^2 dx + \frac{a}{2} \frac{d}{dt} \int_{\Omega} |\Lambda^{\alpha} \mathbf{m}^{\varepsilon, N}|^2 dx \\ + \lambda \int_{\Omega} (m_1^{\varepsilon, N} m_3^{\varepsilon, N})_t \omega_x^{\varepsilon, N} dx \\ + \frac{1}{4\varepsilon} \frac{d}{dt} \int_{\Omega} (|\mathbf{m}^{\varepsilon, N}|^2 - 1)^2 dx = 0 \\ \frac{\rho}{2} \frac{d}{dt} \int_{\Omega} |\omega_t^{\varepsilon, N}|^2 dx + \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\omega_x^{\varepsilon, N}|^2 dx \\ - \lambda \int_{\Omega} (m_1^{\varepsilon, N} m_3^{\varepsilon, N})_x \omega_t^{\varepsilon, N} dx = 0. \end{aligned} \tag{28}$$

We find after summing

$$\begin{aligned} \int_{\Omega} |\mathbf{m}_t^{\varepsilon, N}|^2 dx + \frac{a}{2} \frac{d}{dt} \int_{\Omega} |\Lambda^{\alpha} \mathbf{m}^{\varepsilon, N}|^2 dx \\ + \frac{1}{4\varepsilon} \frac{d}{dt} \int_{\Omega} (|\mathbf{m}^{\varepsilon, N}|^2 - 1)^2 dx + \frac{\rho}{2} \frac{d}{dt} \int_{\Omega} |\omega_t^{\varepsilon, N}|^2 dx \\ + \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\omega_x^{\varepsilon, N}|^2 dx \\ + \lambda \int_{\Omega} ((m_1^{\varepsilon, N} m_3^{\varepsilon, N})_t \omega_x^{\varepsilon, N} - (m_1^{\varepsilon, N} m_3^{\varepsilon, N})_x \omega_t^{\varepsilon, N}) dx \\ = 0. \end{aligned} \tag{29}$$

On the other hand

$$\begin{aligned}
 & \frac{d}{dt} \int_{\Omega} m_1^{\varepsilon,N} m_3^{\varepsilon,N} \omega_x^{\varepsilon,N} dx \\
 &= \int_{\Omega} (m_1^{\varepsilon,N} m_3^{\varepsilon,N})_t \omega_x^{\varepsilon,N} dx + \int_{\Omega} m_1^{\varepsilon,N} m_3^{\varepsilon,N} \omega_{x,t}^{\varepsilon,N} dx \\
 &= \int_{\Omega} (m_1^{\varepsilon,N} m_3^{\varepsilon,N})_t \omega_x^{\varepsilon,N} dx \\
 &\quad - \int_{\Omega} (m_1^{\varepsilon,N} m_3^{\varepsilon,N})_x \omega_t^{\varepsilon,N} dx \\
 &= \int_{\Omega} ((m_1^{\varepsilon,N} m_3^{\varepsilon,N})_t \omega_x^{\varepsilon,N} - (m_1^{\varepsilon,N} m_3^{\varepsilon,N})_x \omega_t^{\varepsilon,N}) dx.
 \end{aligned} \tag{30}$$

Now integrating in time,

$$\begin{aligned}
 & \int_Q |\mathbf{m}_t^{\varepsilon,N}|^2 dx + \frac{a}{2} \int_{\Omega} |\Lambda^\alpha \mathbf{m}^{\varepsilon,N}(T)|^2 dx \\
 &\quad + \frac{1}{4\varepsilon} \int_{\Omega} (|\mathbf{m}^{\varepsilon,N}(T)|^2 - 1)^2 dx \\
 &\quad + \frac{\rho}{2} \int_{\Omega} |\omega_t^{\varepsilon,N}(T)|^2 dx + \frac{1}{2} \int_{\Omega} |\omega_x^{\varepsilon,N}(T)|^2 dx \\
 &\quad + \lambda \int_{\Omega} (m_1^{\varepsilon,N} m_3^{\varepsilon,N} \omega_x^{\varepsilon,N})(T) dx \\
 &= \frac{a}{2} \int_{\Omega} |\Lambda^\alpha \mathbf{m}^N(0)|^2 dx \\
 &\quad + \frac{1}{4\varepsilon} \int_{\Omega} (|\mathbf{m}^N(0)|^2 - 1)^2 dx \\
 &\quad + \frac{\rho}{2} \int_{\Omega} |\omega_t^N(0)|^2 dx + \frac{1}{2} \int_{\Omega} |\omega_x^N(0)|^2 dx \\
 &\quad + \lambda \int_{\Omega} (m_1^N m_3^N \omega_x^N)(0) dx.
 \end{aligned} \tag{31}$$

Omitting superscripts, we obtain for all $b > 0$,

$$\begin{aligned}
 \left| \int_{\Omega} m_1 m_3 \omega_x dx \right| &\leq \int_{\Omega} \left(\frac{1}{2b} (m_1 m_3)^2 + \frac{b}{2} \omega_x^2 \right) dx \\
 &\leq \frac{1}{2b} \int_{\Omega} |\mathbf{m}|^4 dx + \frac{b}{2} \int_{\Omega} \omega_x^2 dx \\
 &= \frac{1}{2b} \int_{\Omega} (|\mathbf{m}|^2 - 1 + 1)^2 dx \\
 &\quad + \frac{b}{2} \int_{\Omega} \omega_x^2 dx \\
 &\leq \frac{1}{b} \int_{\Omega} (|\mathbf{m}|^2 - 1)^2 dx + \frac{2\pi}{b} \\
 &\quad + \frac{b}{2} \int_{\Omega} \omega_x^2 dx \\
 &= \frac{1}{b} \int_{\Omega} (|\mathbf{m}|^2 - 1)^2 dx + \frac{2\pi}{b} \\
 &\quad + \frac{b}{2} \|\omega_x\|_{L^2(\Omega)}^2.
 \end{aligned} \tag{32}$$

Hence, taking into account (31),

$$\begin{aligned}
 & \int_Q |\mathbf{m}_t^{\varepsilon,N}|^2 dx + \frac{a}{2} \int_{\Omega} |\Lambda^\alpha \mathbf{m}^{\varepsilon,N}(T)|^2 dx \\
 &\quad + \left(\frac{1}{4\varepsilon} - \frac{\lambda}{b} \right) \int_{\Omega} (|\mathbf{m}^{\varepsilon,N}(T)|^2 - 1)^2 dx \\
 &\quad + \frac{\rho}{2} \int_{\Omega} |\omega_t^{\varepsilon,N}(T)|^2 dx - \frac{2\lambda\pi}{b} \\
 &\quad + \frac{1 - \lambda b}{2} \int_{\Omega} |\omega_x^{\varepsilon,N}(T)|^2 dx \\
 &\leq \frac{a}{2} \int_{\Omega} |\Lambda^\alpha \mathbf{m}^N(0)|^2 dx \\
 &\quad + \left(\frac{1}{4\varepsilon} + \frac{\lambda}{b} \right) \int_{\Omega} (|\mathbf{m}^N(0)|^2 - 1)^2 dx \\
 &\quad + \frac{\rho}{2} \int_{\Omega} |\omega_t^N(0)|^2 dx + \frac{2\lambda\pi}{b} \\
 &\quad + \frac{1 + \lambda b}{2} \int_{\Omega} |\omega_x^N(0)|^2 dx
 \end{aligned} \tag{33}$$

and hence for ε small enough ($\varepsilon < 1/16\lambda^2$) and $b = 1/2\lambda$ one has

$$\begin{aligned}
 & \frac{a}{2} \int_{\Omega} |\Lambda^\alpha \mathbf{m}^{\varepsilon,N}(T)|^2 dx + \int_Q |\mathbf{m}_t^{\varepsilon,N}|^2 dx dt \\
 &\quad + \frac{1}{8\varepsilon} \int_{\Omega} (|\mathbf{m}^{\varepsilon,N}(T)|^2 - 1)^2 dx \\
 &\quad + \frac{\rho}{2} \int_{\Omega} |\omega_t^{\varepsilon,N}(T)|^2 dx + \frac{1}{4} \int_{\Omega} |\omega_x^{\varepsilon,N}(T)|^2 dx \\
 &\leq \frac{a}{2} \int_{\Omega} |\Lambda^\alpha \mathbf{m}^N(0)|^2 dx \\
 &\quad + \frac{3}{8\varepsilon} \int_{\Omega} (|\mathbf{m}^N(0)|^2 - 1)^2 dx + 8\pi\lambda^2 \\
 &\quad + \frac{\rho}{2} \int_{\Omega} |\omega_t^N(0)|^2 dx + \frac{3}{4} \int_{\Omega} |\omega_x^N(0)|^2 dx.
 \end{aligned} \tag{34}$$

The right-hand side is uniformly bounded. Indeed, for $\alpha \in (1/2, 1)$, $\mathbf{H}^\alpha(\Omega) \hookrightarrow \mathbf{L}^4(\Omega)$; furthermore (for constants C_1, C_2 , and C independent of N)

$$\begin{aligned}
 & \int_{\Omega} (|\mathbf{m}^N(0)|^2 - 1)^2 dx \\
 &= \int_{\Omega} |\mathbf{m}^N(0)|^4 dx - 2 \int_{\Omega} |\mathbf{m}^N(0)|^2 dx + 2\pi \\
 &\leq \|\mathbf{m}^N(0)\|_{\mathbf{L}^4(\Omega)}^4 + 2\pi \leq C_1 \|\mathbf{m}_0^N\|_{\mathbf{H}^\alpha(\Omega)}^4 + C_2, \\
 & \int_{\Omega} |\omega_x^N(0)|^2 dx = \int_{\Omega} |\omega_x^N(0) - \omega_x^0 + \omega_x^0|^2 dx
 \end{aligned}$$

$$\begin{aligned}
&\leq 2 \int_{\Omega} \left| \omega_x^N(0) - \omega_x^0 \right|^2 dx + 2 \int_{\Omega} \left| \omega_x^0 \right|^2 dx \\
&\leq 2 \left\| \omega^N(0) - \omega_0 \right\|_{H_0^1(\Omega)}^2 + 2 \left\| \omega_0 \right\|_{H_0^1(\Omega)}^2 \leq C,
\end{aligned} \tag{35}$$

thanks to the strong convergence $\omega^N(\cdot, 0) \rightarrow \omega_0$ in $H_0^1(\Omega)$. For the other term $(\omega_t^N(0))$, the estimate can be carried out in an analogous way using the strong convergence $\omega_t^N(\cdot, 0) \rightarrow \omega_1$ in $L^2(\Omega)$. Moreover, noting that (for a constant C independent of ε and N)

$$\begin{aligned}
\int_{\Omega} \left| \mathbf{m}^{\varepsilon, N} \right|^2 dx &= \int_{\Omega} \left(\left| \mathbf{m}^{\varepsilon, N} \right|^2 - 1 + 1 \right) dx \\
&\leq \frac{1}{2} \int_{\Omega} \left(\left| \mathbf{m}^{\varepsilon, N} \right|^2 - 1 \right)^2 dx + C,
\end{aligned} \tag{36}$$

therefore, for fixed $\varepsilon > 0$, we have

$$\begin{aligned}
&\left(\mathbf{m}^{\varepsilon, N} \right)_N \text{ is bounded in } L^\infty(0, T; \mathbf{H}^\alpha(\Omega)), \\
&\left(\mathbf{m}_t^{\varepsilon, N} \right)_N \text{ is bounded in } L^2(0, T; \mathbf{L}^2(\Omega)), \\
&\left(\left| \mathbf{m}^{\varepsilon, N} \right|^2 - 1 \right)_N \text{ is bounded in } L^\infty(0, T; L^2(\Omega)), \\
&\left(\omega^{\varepsilon, N} \right)_N \text{ is bounded in } L^2(0, T; H_0^1(\Omega)), \\
&\left(\omega_t^{\varepsilon, N} \right)_N \text{ is bounded in } L^2(0, T; L^2(\Omega)).
\end{aligned} \tag{37}$$

Note that (37) is due to the Poincaré lemma. Now, from classical compactness results there exist two subsequences which we still denote by $(\mathbf{m}^{\varepsilon, N})$ and $(\omega^{\varepsilon, N})$ such that for fixed $\varepsilon > 0$ and for any $1 < p < \infty$

$$\begin{aligned}
&\mathbf{m}^{\varepsilon, N} \rightharpoonup \mathbf{m}^\varepsilon \\
&\text{weakly in } L^p(0, T; \mathbf{H}^\alpha(\Omega)), \\
&\mathbf{m}^{\varepsilon, N} \longrightarrow \mathbf{m}^\varepsilon \\
&\text{strongly in } C([0, T], \mathbf{H}^\beta(\Omega)), \text{ a.e. for } 0 \leq \beta < \alpha \\
&\mathbf{m}_t^{\varepsilon, N} \rightharpoonup \mathbf{m}_t^\varepsilon \\
&\text{weakly in } L^2(0, T; \mathbf{L}^2(\Omega)), \\
&\left| \mathbf{m}^{\varepsilon, N} \right|^2 - 1 \rightharpoonup \zeta \\
&\text{weakly in } L^p(0, T; L^2(\Omega)), \\
&\omega^{\varepsilon, N} \rightharpoonup \omega^\varepsilon \\
&\text{weakly in } L^2(0, T; H_0^1(\Omega)), \\
&\omega_t^{\varepsilon, N} \rightharpoonup \omega_t^\varepsilon \\
&\text{weakly in } L^2(0, T; L^2(\Omega)),
\end{aligned}$$

$$\begin{aligned}
&\omega^{\varepsilon, N} \longrightarrow \omega^\varepsilon \\
&\text{strongly in } L^2(Q).
\end{aligned} \tag{38}$$

Convergence (38) is due to Lemma 3 and thanks to Lemma 4 it can be shown that $\zeta = |\mathbf{m}^\varepsilon|^2 - 1$. Moreover from the Sobolev embedding (Lemma 5) $H^\alpha(Q) \hookrightarrow L^4(Q)$, the further compactness result follows:

$$m_i^{\varepsilon, N} m_j^{\varepsilon, N} \longrightarrow m_i^\varepsilon m_j^\varepsilon \text{ strongly in } L^2(Q). \tag{39}$$

The above estimates allow us to pass to the limit as N goes to infinity and to get the desired result. Indeed consider the variational formulation of (23):

$$\begin{aligned}
&\gamma^{-1} \int_Q \mathbf{m}_t^{\varepsilon, N} \times \mathbf{m}^{\varepsilon, N} \cdot \boldsymbol{\phi} dx dt + \int_Q \mathbf{m}_t^{\varepsilon, N} \cdot \boldsymbol{\phi} dx dt \\
&+ a \int_Q \Lambda^\alpha \mathbf{m}^{\varepsilon, N} \cdot \Lambda^\alpha \boldsymbol{\phi} dx dt \\
&+ \int_Q \boldsymbol{\ell}(\mathbf{m}^{\varepsilon, N}, \omega^{\varepsilon, N}) \cdot \boldsymbol{\phi} dx dt \\
&+ \int_Q \frac{\left| \mathbf{m}^{\varepsilon, N} \right|^2 - 1}{\varepsilon} \mathbf{m}^{\varepsilon, N} \cdot \boldsymbol{\phi} dx dt = 0 \\
&- \rho \int_Q \omega_t^{\varepsilon, N} \psi_t dx dt + \int_Q \omega_x^{\varepsilon, N} \psi_x dx dt \\
&+ \lambda \int_Q m_1^{\varepsilon, N} m_3^{\varepsilon, N} \psi_x dx dt = 0,
\end{aligned} \tag{40}$$

for any $\boldsymbol{\phi} \in L^2(0, T; \mathbf{H}^\alpha(\Omega))$ and $\psi \in H_0^1(Q)$. Taking $N \rightarrow \infty$ in (40), we find

$$\begin{aligned}
&\gamma^{-1} \int_Q \mathbf{m}_t^\varepsilon \times \mathbf{m}^\varepsilon \cdot \boldsymbol{\phi} dx dt + \int_Q \mathbf{m}_t^\varepsilon \cdot \boldsymbol{\phi} dx dt \\
&+ a \int_Q \Lambda^\alpha \mathbf{m}^\varepsilon \cdot \Lambda^\alpha \boldsymbol{\phi} dx dt + \int_Q \boldsymbol{\ell}(\mathbf{m}^\varepsilon, \omega^\varepsilon) \cdot \boldsymbol{\phi} dx dt \\
&+ \int_Q \frac{\left| \mathbf{m}^\varepsilon \right|^2 - 1}{\varepsilon} \mathbf{m}^\varepsilon \cdot \boldsymbol{\phi} dx dt = 0, \\
&- \rho \int_Q \omega_t^\varepsilon \psi_t dx dt + \int_Q \omega_x^\varepsilon \psi_x dx dt \\
&+ \lambda \int_Q m_1^\varepsilon m_3^\varepsilon \psi_x dx dt = 0,
\end{aligned} \tag{41}$$

for any $\boldsymbol{\phi} \in L^2(0, T; \mathbf{H}^\alpha(\Omega))$ and $\psi \in H_0^1(Q)$. We proved the following result.

Proposition 9. *Given $\mathbf{m}_0 \in \mathbf{H}^\alpha(\Omega)$ such that $|\mathbf{m}_0| = 1$ a.e., $\omega_0 \in H_0^1(\Omega)$, and $\omega_1 \in L^2(\Omega)$, then there exists a solution \mathbf{m}^ε*

to problem (19) in the sense of distributions. Moreover we have the following energy estimate:

$$\begin{aligned} & \frac{a}{2} \int_{\Omega} |\Lambda^\alpha \mathbf{m}^\varepsilon(T)|^2 dx + \int_Q |\mathbf{m}_t^\varepsilon(T)|^2 dx dt \\ & + \frac{1}{8\varepsilon} \int_{\Omega} (|\mathbf{m}^\varepsilon(T)|^2 - 1)^2 dx \\ & + \frac{\rho}{2} \int_{\Omega} |\omega_t^\varepsilon(T)|^2 dx + \frac{1}{4} \int_{\Omega} |\omega_x^\varepsilon(T)|^2 dx \quad (42) \\ & \leq \frac{a}{2} \int_{\Omega} |\Lambda^\alpha \mathbf{m}_0|^2 dx + 8\pi\lambda^2 + \frac{\rho}{2} \int_{\Omega} |\omega_1|^2 dx \\ & + \frac{1}{4} \int_{\Omega} |\omega_x(0)|^2 dx. \end{aligned}$$

4.2. *Convergence of Approximate Solutions.* Our aim here is to pass to the limit as $\varepsilon \rightarrow 0$. For this, we will use estimate (42), from which we have the following:

$$\begin{aligned} & (\mathbf{m}^\varepsilon)_\varepsilon \text{ is bounded in } L^\infty(0, T; \mathbf{H}^\alpha(\Omega)), \\ & (\mathbf{m}_t^\varepsilon)_\varepsilon \text{ is bounded in } L^2(0, T; L^2(\Omega)), \\ & (|\mathbf{m}^\varepsilon|^2 - 1)_\varepsilon \text{ is bounded in } L^\infty(0, T; L^2(\Omega)), \quad (43) \\ & (\omega^\varepsilon)_\varepsilon \text{ is bounded in } L^2(0, T; H_0^1(\Omega)), \\ & (\omega_t^\varepsilon)_\varepsilon \text{ is bounded in } L^2(0, T; L^2(\Omega)). \end{aligned}$$

Then there exist two subsequences which we still denote by (\mathbf{m}^ε) and (ω^ε) such that for any $1 < p < \infty$

$$\begin{aligned} & \mathbf{m}^\varepsilon \rightharpoonup \mathbf{m} \\ & \text{weakly in } L^p(0, T; \mathbf{H}^\alpha(\Omega)), \\ & \mathbf{m}^\varepsilon \rightarrow \mathbf{m} \\ & \text{strongly in } C([0, T], \mathbf{H}^\beta(\Omega)), \text{ a.e. for } 0 \leq \beta < \alpha \\ & \mathbf{m}_t^\varepsilon \rightharpoonup \mathbf{m}_t \\ & \text{weakly in } L^2(0, T; L^2(\Omega)), \\ & |\mathbf{m}^\varepsilon|^2 - 1 \rightarrow 0 \\ & \text{strongly in } L^p(0, T; L^2(\Omega)), \text{ a.e.} \quad (44) \\ & \omega^\varepsilon \rightharpoonup \omega \\ & \text{weakly in } L^2(0, T; H_0^1(\Omega)), \\ & \omega_t^\varepsilon \rightharpoonup \omega_t \\ & \text{weakly in } L^2(0, T; L^2(\Omega)), \\ & \omega^\varepsilon \rightarrow \omega \\ & \text{strongly in } L^2(Q). \end{aligned}$$

It can be shown from convergence (44) that $|\mathbf{m}| = 1$ a.e. Now in order to pass to the limit $\varepsilon \rightarrow 0$ in (41), let $\boldsymbol{\varphi} \in C^\infty([0, T] \times \overline{\Omega})$, and let $\boldsymbol{\phi} = \mathbf{m}^\varepsilon \times \boldsymbol{\varphi}$. We first recall the identity $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$ for all $\mathbf{a}, \mathbf{b}, \mathbf{c}$, and \mathbf{d} in \mathbb{R}^3 .

As $\boldsymbol{\phi}$ is in $L^2(0, T; \mathbf{H}^\alpha(\Omega))$, the following holds:

$$\begin{aligned} & \gamma^{-1} \int_Q (\mathbf{m}_t^\varepsilon \cdot \mathbf{m}^\varepsilon) (\mathbf{m}^\varepsilon \cdot \boldsymbol{\varphi}) dx dt \\ & - \gamma^{-1} \int_Q (\mathbf{m}_t^\varepsilon \cdot \boldsymbol{\varphi}) |\mathbf{m}^\varepsilon|^2 dx dt \\ & + \int_Q (\mathbf{m}_t^\varepsilon \times \mathbf{m}^\varepsilon) \cdot \boldsymbol{\varphi} dx dt \\ & + a \int_Q \Lambda^\alpha \mathbf{m}^\varepsilon \cdot \Lambda^\alpha (\mathbf{m}^\varepsilon \times \boldsymbol{\varphi}) dx dt \quad (45) \\ & + \int_Q (\boldsymbol{\ell}(\mathbf{m}^\varepsilon, \omega^\varepsilon) \times \mathbf{m}^\varepsilon) \cdot \boldsymbol{\varphi} dx dt = 0 \\ & - \rho \int_Q \omega_t^\varepsilon \psi_t dx dt + \int_Q \omega_x^\varepsilon \psi_x dx dt \\ & + \lambda \int_Q m_1^\varepsilon m_3^\varepsilon \psi_x dx dt = 0. \end{aligned}$$

Note that for this choice we have $\Lambda^\alpha(\mathbf{m}^\varepsilon \times \boldsymbol{\varphi}) \in L^2(Q)$, indeed applying the multiplicative estimates (16) in Lemma 6 to \mathbf{m}^ε and $\boldsymbol{\varphi}$ (for $s = \alpha, p = 2, p_1 = \infty, p_2 = 2, p_3 = 2$, and $p_4 = \infty$); we find for different constants C independent of ε :

$$\begin{aligned} & \|\Lambda^\alpha(\mathbf{m}^\varepsilon \times \boldsymbol{\varphi})\|_{L^2(Q)} \\ & \leq C (\|\mathbf{m}^\varepsilon\|_{L^\infty(Q)} \|\boldsymbol{\varphi}\|_{\mathbf{H}^\alpha(Q)} + \|\mathbf{m}^\varepsilon\|_{\mathbf{H}^\alpha(Q)} \|\boldsymbol{\varphi}\|_{L^\infty(Q)}) \quad (46) \\ & \leq C; \end{aligned}$$

since $2\alpha > 1$ (1 here is the dimension) then $\mathbf{H}^\alpha(\Omega) \hookrightarrow L^\infty(\Omega)$ and consequently $(\mathbf{m}^\varepsilon)_\varepsilon$ is bounded in $L^\infty(Q)$.

Taking $\varepsilon \rightarrow 0$ and following the idea introduced in [4] we have

$$\begin{aligned} & \int_Q (\mathbf{m}_t^\varepsilon \cdot \mathbf{m}^\varepsilon) (\mathbf{m}^\varepsilon \cdot \boldsymbol{\varphi}) dx dt \\ & = \int_Q (\mathbf{m}_t^\varepsilon \cdot \mathbf{m}^\varepsilon) (\mathbf{m}^\varepsilon \cdot \boldsymbol{\varphi}) dx dt \\ & - \int_Q (\mathbf{m}_t \cdot \mathbf{m}) (\mathbf{m} \cdot \boldsymbol{\varphi}) dx dt \\ & = \int_Q (\mathbf{m}_t^\varepsilon \cdot \mathbf{m}^\varepsilon) (\mathbf{m}^\varepsilon - \mathbf{m}) \cdot \boldsymbol{\varphi} dx dt \\ & + \int_Q (\mathbf{m}^\varepsilon \cdot (\mathbf{m}_t^\varepsilon - \mathbf{m}_t)) (\mathbf{m} \cdot \boldsymbol{\varphi}) dx dt \\ & + \int_Q (\mathbf{m}_t \cdot (\mathbf{m}^\varepsilon - \mathbf{m})) (\mathbf{m} \cdot \boldsymbol{\varphi}) dx dt \rightarrow 0 \\ & \int_Q |\mathbf{m}^\varepsilon|^2 \mathbf{m}_t^\varepsilon \cdot \boldsymbol{\varphi} dx dt \end{aligned}$$

$$\begin{aligned}
&= \int_Q (|\mathbf{m}^\varepsilon|^2 - 1) \mathbf{m}_t^\varepsilon \cdot \boldsymbol{\varphi} \, dx \, dt + \int_Q \mathbf{m}_t^\varepsilon \cdot \boldsymbol{\varphi} \, dx \, dt \\
&\longrightarrow \int_Q \mathbf{m}_t \cdot \boldsymbol{\varphi} \, dx \, dt.
\end{aligned} \tag{47}$$

Now for the fourth term of the first equation, we introduce the commutator (see [4]):

$$\Gamma_\boldsymbol{\varphi}(\mathbf{m}) := \Lambda^\alpha(\mathbf{m} \times \boldsymbol{\varphi}) - \boldsymbol{\varphi} \times \Lambda^\alpha \mathbf{m} \tag{48}$$

Let $\mathfrak{F}_\varepsilon = \int_Q \Lambda^\alpha \mathbf{m}^\varepsilon \cdot \Lambda^\alpha(\mathbf{m}^\varepsilon \times \boldsymbol{\varphi}) \, dx \, dt$ and $\mathfrak{F} = \int_Q \Lambda^\alpha \mathbf{m} \cdot \Lambda^\alpha(\mathbf{m} \times \boldsymbol{\varphi}) \, dx \, dt$. We will show that $\mathfrak{F}_\varepsilon \rightarrow \mathfrak{F}$.

First, note that $\Gamma_\boldsymbol{\varphi}(\mathbf{m}) \in \mathbf{L}^2(Q)$. Indeed, applying (15) for $p_1 = 1/(1-\alpha)$, $p_2 = 2/(2\alpha-1)$, p_3 and p_4 in $(2, +\infty)$, we find (for different constants C)

$$\begin{aligned}
\|\Gamma_\boldsymbol{\varphi}(\mathbf{m})\|_{\mathbf{L}^2(\Omega)} &\leq C \left(\|\nabla \boldsymbol{\varphi}\|_{\mathbf{L}^{p_1}(\Omega)} \|\mathbf{m}\|_{\dot{\mathbf{W}}^{\alpha-1, p_2}(\Omega)} \right. \\
&\quad \left. + \|\boldsymbol{\varphi}\|_{\dot{\mathbf{W}}^{\alpha, p_3}(\Omega)} \|\mathbf{m}\|_{\mathbf{L}^{p_4}(\Omega)} \right) \\
&\leq C \left(\|\nabla \boldsymbol{\varphi}\|_{\mathbf{L}^{p_1}(\Omega)} \|\mathbf{m}\|_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\varphi}\|_{\dot{\mathbf{W}}^{\alpha, p_3}(\Omega)} \|\mathbf{m}\|_{\mathbf{L}^2(\Omega)} \right) \\
&\leq C \|\mathbf{m}\|_{\mathbf{L}^2(\Omega)},
\end{aligned} \tag{49}$$

thanks to Lemma 7. Once again

$$\begin{aligned}
&\|\Gamma_\boldsymbol{\varphi}(\mathbf{m}^\varepsilon - \mathbf{m})\|_{\mathbf{L}^2(\Omega)} \\
&\leq C \left(\|\nabla \boldsymbol{\varphi}\|_{\mathbf{L}^{p_1}(\Omega)} \|\mathbf{m}^\varepsilon - \mathbf{m}\|_{\dot{\mathbf{W}}^{\alpha-1, p_2}(\Omega)} \right. \\
&\quad \left. + \|\boldsymbol{\varphi}\|_{\dot{\mathbf{W}}^{\alpha, p_3}(\Omega)} \|\mathbf{m}^\varepsilon - \mathbf{m}\|_{\mathbf{L}^{p_4}(\Omega)} \right) \\
&\|\Gamma_\boldsymbol{\varphi}(\mathbf{m}^\varepsilon - \mathbf{m})\|_{\mathbf{L}^2(\Omega)} \\
&\leq C \left(\|\nabla \boldsymbol{\varphi}\|_{\mathbf{L}^\infty(0, T; \mathbf{L}^{p_1}(\Omega))} \|\mathbf{m}^\varepsilon - \mathbf{m}\|_{\mathbf{L}^2(Q)} \right. \\
&\quad \left. + \|\boldsymbol{\varphi}\|_{\mathbf{L}^\infty(0, T; \dot{\mathbf{W}}^{\alpha, p_3}(\Omega))} \|\mathbf{m}^\varepsilon - \mathbf{m}\|_{\mathbf{L}^2(0, T; \mathbf{H}^\beta(\Omega))} \right),
\end{aligned} \tag{50}$$

where the right-hand side of the last inequality tends to 0.

Finally, we have

$$\begin{aligned}
|\mathfrak{F}_\varepsilon - \mathfrak{F}| &= \left| \int_Q \Lambda^\alpha \mathbf{m}^\varepsilon \cdot \Gamma_\boldsymbol{\varphi}(\mathbf{m}^\varepsilon) \, dx \, dt \right. \\
&\quad \left. - \int_Q \Lambda^\alpha \mathbf{m} \cdot \Gamma_\boldsymbol{\varphi}(\mathbf{m}) \, dx \, dt \right| \\
&= \left| \int_Q \Lambda^\alpha \mathbf{m}^\varepsilon \cdot \Gamma_\boldsymbol{\varphi}(\mathbf{m}^\varepsilon - \mathbf{m}) \, dx \, dt \right. \\
&\quad \left. + \int_Q \Lambda^\alpha(\mathbf{m}^\varepsilon - \mathbf{m}) \cdot \Gamma_\boldsymbol{\varphi}(\mathbf{m}) \, dx \, dt \right| \\
&\leq \int_Q |\Lambda^\alpha \mathbf{m}^\varepsilon \cdot \Gamma_\boldsymbol{\varphi}(\mathbf{m}^\varepsilon - \mathbf{m})| \, dx \, dt \\
&\quad + \left| \int_Q \Lambda^\alpha(\mathbf{m}^\varepsilon - \mathbf{m}) \cdot \Gamma_\boldsymbol{\varphi}(\mathbf{m}) \, dx \, dt \right|
\end{aligned}$$

$$\begin{aligned}
&\leq C \|\Gamma_\boldsymbol{\varphi}(\mathbf{m}^\varepsilon - \mathbf{m})\|_{\mathbf{L}^2(Q)} \\
&\quad + \left| \int_Q \Lambda^\alpha(\mathbf{m}^\varepsilon - \mathbf{m}) \cdot \Gamma_\boldsymbol{\varphi}(\mathbf{m}) \, dx \, dt \right| \longrightarrow 0.
\end{aligned} \tag{51}$$

Therefore

$$\begin{aligned}
&\gamma^{-1} \int_Q \mathbf{m}_t \cdot \boldsymbol{\varphi} \, dx \, dt + \int_Q (\mathbf{m} \times \mathbf{m}_t) \cdot \boldsymbol{\varphi} \, dx \, dt \\
&= a \int_Q \Lambda^\alpha \mathbf{m} \cdot \Lambda^\alpha(\mathbf{m} \times \boldsymbol{\varphi}) \, dx \, dt \\
&\quad + \int_Q (\boldsymbol{\ell}(\mathbf{m}, \omega) \times \mathbf{m}) \cdot \boldsymbol{\varphi} \, dx \, dt, \\
&\quad - \rho \int_Q \omega_t \psi_t \, dx \, dt + \int_Q \omega_x \psi_x \, dx \, dt \\
&\quad + \lambda \int_Q m_1 m_3 \psi_x \, dx \, dt = 0.
\end{aligned} \tag{52}$$

This being true for every $\boldsymbol{\varphi} \in \mathbf{C}^\infty([0, T] \times \bar{\Omega})$ and by a standard density argument, it is true for any $\boldsymbol{\varphi} \in \mathbf{L}^2(0, T; \mathbf{H}^\alpha(\Omega))$. Note that, from (42), one can easily get (14). Hence (\mathbf{m}, ω) is a solution of problem (7)-(8)-(9) in the sense of Definition 1. The proof of Theorem 2 is complete.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The research is supported by the PHC Volubilis Program MA/14/301 ‘‘Elaboration et analyse de mod\`eles asymptotiques en micro-magn\`etisme, magn\`eto-\`elasticit\`e et \`electro-\`elasticit\`e’’ with joint financial support from the French Ministry of Foreign Affairs and the Moroccan Ministry of Higher Education and Scientific Research.

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