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Research Article

Unbounded Solutions for Functional Problems on the Half-Line

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This paper presents an existence and localization result of unbounded solutions for a second-order differential equation on the half-line with functional boundary conditions. By applying unbounded upper and lower solutions, Green's functions, and Schauder fixed point theorem, the existence of at least one solution is shown for the above problem. One example and one application to an Emden-Fowler equation are shown to illustrate our results.

1. Introduction

The authors consider the following boundary value problem composed by the differential equation:

$$u''(t) = f\left(t, u(t), u'(t)\right), \quad t \geq 0, \tag{1}$$

where $f : [0, +\infty[\times \mathbb{R}^2 \to \mathbb{R}$ is continuous and bounded by some L^1 function, and the functional boundary conditions on the half-line are as follows:

$$\begin{gathered} L\left(u, u(0), u'(0)\right) = 0, \\ u'(+\infty) = B, \end{gathered} \tag{2}$$

with $B \in \mathbb{R}$ and $L : C[0, +\infty[\times \mathbb{R}^2 \to \mathbb{R}$ a continuous function verifying some monotone assumption:

$$u'(+\infty) := \lim_{t \to +\infty} u'(t). \tag{3}$$

Boundary value problems on the half-line arise naturally in the study of radially symmetric solutions of nonlinear elliptic equations, and many works have been done in this area; see [1]. The functional dependence on the boundary conditions allows that problem (1), (2) covers a huge variety of boundary value problems such as separated, multipoint, nonlocal, integrodifferential, with maximum or minimum arguments, as it can be seen, for instance, in [2–11] and the references therein. However, to the best of our knowledge, it is the first time where this type of functional boundary conditions are applied to the half-line.

Lower and upper solutions method is a very adequate technique to deal with boundary value problems as it provides not only the existence of bounded or unbounded solutions but also their localization and, from that, some qualitative data about solutions, their variation and behavior (see [12–14]). Some results are concerned with the existence of bounded or positive solutions, as in [15, 16], and the references therein. For problem (1), (2) we prove the existence of two types of solution, depending on B: if $B \neq 0$ the solution is unbounded and if $B = 0$ the solution is bounded. In this way, we gather different strands of boundary value problems and types of solutions in a single method.

The paper is organized as follows. In Section 2 some auxiliary results are defined such as the adequate space functions, some weighted norms, a criterion to overcome the lack of compactness, and the definition of lower and upper solutions. Section 3 contains the main result: an existence and localization theorem, which proof combines lower and upper solution technique with the fixed-point theory. Finally, last two sections contain, to illustrate our results, an example and an application to some problem composed by a discontinuous Emden-Fowler-type equation with a infinite multipoint conditions, which are not covered by the existent literature.

2. Definitions and Auxiliary Results

Consider the space

$$X = \left\{ x \in C^1 [0, +\infty[: \lim_{t \to +\infty} \frac{x(t)}{1+t} \in \mathbb{R}, \lim_{t \to +\infty} x'(t) \in \mathbb{R} \right\} \tag{4}$$

with the norm $\|x\|_X = \max\{\|x\|_0, \|x'\|_1\}$, where

$$\|\omega\|_0 := \sup_{0 \le t < +\infty} \frac{|\omega(t)|}{1+t}, \qquad \|\omega'\|_1 := \sup_{0 \le t < +\infty} |\omega'(t)|. \tag{5}$$

In this way $(X, \|\cdot\|_X)$ is a Banach space.

Solutions of the linear problem associated to (1) and usual boundary conditions are defined with Green's function, which can be obtained by standard calculus.

Lemma 1. *Let th, $h \in L^1[0, +\infty[$. Then the linear boundary value problem composed by*

$$\begin{aligned} u''(t) &= h(t), \quad t \ge 0, \\ u(0) &= A, \\ u'(+\infty) &= B, \end{aligned} \tag{6}$$

for $A, B \in \mathbb{R}$, has a unique solution in X, given by

$$u(t) = A + Bt + \int_0^{+\infty} G(t,s)\, h(s)\, ds, \tag{7}$$

where

$$G(t,s) = \begin{cases} -s, & 0 \le s \le t \\ -t, & t \le s < +\infty. \end{cases} \tag{8}$$

Proof. If u is a solution of problem (6), then the general solution for the differential equation is

$$u(t) = c_1 + c_2 t + \int_0^t (t-s)\, h(s)\, ds, \tag{9}$$

where c_1, c_2 are constants. Since $u(t)$ should satisfy the boundary conditions, we get

$$c_1 = A, \qquad c_2 = B - \int_0^{+\infty} h(s)\, ds. \tag{10}$$

The solution becomes

$$u(t) = A + Bt - t \int_0^{+\infty} h(s)\, ds + \int_0^t (t-s)\, h(s)\, ds. \tag{11}$$

And by computation

$$u(t) = A + Bt + \int_0^{+\infty} G(t,s)\, h(s)\, ds, \tag{12}$$

with G given by (8).

Conversely, if u is a solution of (7), it is easy to show that it satisfies the differential equation in (6). Also $u(0) = A$ and $u'(+\infty) = B$. □

The lack of compactness of X is overcome by the following lemma which gives a general criterion for relative compactness, referred to in [1].

Lemma 2. *A set $M \subset X$ is relatively compact if the following conditions hold:*

(1) *all functions from M are uniformly bounded;*

(2) *all functions from M are equicontinuous on any compact interval of $[0, +\infty[$;*

(3) *all functions from M are equiconvergent at infinity; that is, for any given $\epsilon > 0$, there exists a $t_\epsilon > 0$ such that*

$$\left| \frac{x(t)}{1+t} - \lim_{t \to +\infty} \frac{x(t)}{1+t} \right| < \epsilon, \qquad \left| x'(t) - \lim_{t \to +\infty} x'(t) \right| < \epsilon \qquad \forall t > t_\epsilon, \ x \in M. \tag{13}$$

The existence tool will be Schauder's fixed point theorem.

Theorem 3 (see [17]). *Let Y be a nonempty, closed, bounded, and convex subset of a Banach space X, and suppose that $P : Y \to Y$ is a compact operator. Then P is at least one fixed point in Y.*

The functions considered as lower and upper solutions for the initial problem are defined as follows.

Definition 4. Given $B \in \mathbb{R}$, a function $\alpha \in X$ is a lower solution of problem (1), (2) if

$$\begin{aligned} \alpha''(t) &\ge f\left(t, \alpha(t), \alpha'(t)\right), \quad t \ge 0, \\ L\left(\alpha, \alpha(0), \alpha'(0)\right) &\ge 0, \\ \alpha'(+\infty) &< B. \end{aligned} \tag{14}$$

A function β is an upper solution if it satisfies the reverse inequalities.

3. Existence and Localization Results

In this section we prove the existence of at least one solution for the problem (1), (2), and, moreover, some localization data.

Theorem 5. *Let* $f : [0,+\infty[\times \mathbb{R}^2 \to \mathbb{R}$ *be a continuous function, verifying that, for each* $\rho > 0$*, there exists a positive function* φ_ρ *with* $\varphi_\rho, t\varphi_\rho \in L^1[0,+\infty[$ *such that for* $(x(t), y(t)) \in \mathbb{R}^2$ *with* $\sup_{0\le t<+\infty}\{|x(t)|/(1+t), |y(t)|\} < \rho$*,*

$$|f(t,x,y)| \le \varphi_\rho(t), \quad t \ge 0. \tag{15}$$

Moreover, if $L(x_1,x_2,x_3)$ *is nondecreasing on* x_1 *and* x_3 *and there are* α, β*, lower and upper solutions of (1), (2), respectively, such that*

$$\alpha(t) \le \beta(t), \quad \forall t \ge 0, \tag{16}$$

then problem (1), (2) has at least one solution $u \in X$*, with* $\alpha(t) \le u(t) \le \beta(t)$*, for* $t \ge 0$*.*

Proof. Let α, β be, respectively, lower and upper solutions of (1), (2) verifying (16). Consider the modified problem

$$\begin{aligned} u''(t) &= f\left(t, \delta(t,u(t)), u'(t)\right) \\ &\quad + \frac{1}{1+t^3}\frac{u(t)-\delta(t,u(t))}{1+|u(t)-\delta(t,u(t))|}, \quad t \ge 0, \\ u(0) &= \delta\left(0, u(0) + L\left(u, u(0), u'(0)\right)\right), \\ u'(+\infty) &= B, \end{aligned} \tag{17}$$

where $\delta : [0,+\infty[\times \mathbb{R} \to \mathbb{R}$ is given by

$$\delta(t,x) = \begin{cases} \beta(t), & x > \beta(t) \\ x, & \alpha(t) \le x \le \beta(t) \\ \alpha(t), & x < \alpha(t). \end{cases} \tag{18}$$

For clearness, the proof will follow several steps.

Step 1 (if u is a solution of (17), then $\alpha(t) \le u(t) \le \beta(t)$, $\forall t \ge 0$). Let u be a solution of the modified problem (17) and suppose, by contradiction, that there exists $t \ge 0$ such that $\alpha(t) > u(t)$. Therefore

$$\inf_{0\le t<+\infty}(u(t)-\alpha(t)) < 0. \tag{19}$$

If there is $t_* \in]0,+\infty[$ such that

$$\min_{0\le t<+\infty}(u(t)-\alpha(t)) := u(t_*)-\alpha(t_*) < 0, \tag{20}$$

we have $u'(t_*) = \alpha'(t_*)$ and $u''(t_*) - \alpha''(t_*) \ge 0$. By Definition 4 we get the contradiction

$$\begin{aligned} 0 &\le u''(t_*) - \alpha''(t_*) \\ &= f\left(t_*, \delta(t_*, u(t_*)), u'(t_*)\right) \\ &\quad + \frac{1}{1+t_*^3}\frac{u(t_*)-\delta(t_*,u(t_*))}{1+|u(t_*)-\delta(t_*,u(t_*))|} - \alpha''(t_*) \\ &= f\left(t_*, \alpha(t_*), \alpha'(t_*)\right) \\ &\quad + \frac{1}{1+t_*^3}\frac{u(t_*)-\alpha(t_*)}{1+|u(t_*)-\alpha(t_*)|} - \alpha''(t_*) \\ &\le \frac{u(t_*)-\alpha(t_*)}{1+|u(t_*)-\alpha(t_*)|} < 0. \end{aligned} \tag{21}$$

So $u(t) \ge \alpha(t)$, $\forall t > 0$.

If the infimum is attained at $t = 0$ then

$$\min_{0\le t<+\infty}(u(t)-\alpha(t)) := u(0)-\alpha(0) < 0. \tag{22}$$

As u is solution of (17), by the definition of δ, the following contradiction is achieved

$$\begin{aligned} 0 &> u(0)-\alpha(0) \\ &= \delta\left(0, u(0)+L\left(u,u(0),u'(0)\right)\right) - \alpha(0) \\ &\ge \alpha(0)-\alpha(0) = 0. \end{aligned} \tag{23}$$

If

$$\inf_{0\le t<+\infty}(u(t)-\alpha(t)) := u(+\infty)-\alpha(+\infty) < 0, \tag{24}$$

then $u'(+\infty) - \alpha'(+\infty) \le 0$. As u is solution of (17), by Definition 4, this contradiction holds

$$0 \ge u'(+\infty) - \alpha'(+\infty) = B - \alpha'(+\infty) > 0. \tag{25}$$

Therefore $u(t) \le \alpha(t)$, $\forall t \ge 0$.

In a similar way we can prove that $u(t) \ge \beta(t)$, $\forall t \ge 0$.

Step 2 (problem (17) has at least one solution). Let $u \in X$ and define the operator $T : X \to X$

$$Tu(t) = \Delta + Bt + \int_0^{+\infty} G(t,s) F_u(s)\, ds, \tag{26}$$

with

$$\begin{aligned} F_u(s) &:= f\left(s, \delta(s,u(s)), u'(s)\right) \\ &\quad + \frac{1}{1+s^3}\frac{u(s)-\delta(s,u(s))}{1+|u(s)-\delta(s,u(s))|}, \end{aligned} \tag{27}$$

$\Delta := \delta(0, u(0)+L(u,u(0),u'(0)))$, and G is the Green function given by (8).

Therefore, problem (17) becomes

$$\begin{aligned} u''(t) &= F_u(t), \quad t \geq 0, \\ u(0) &= \Delta, \\ u'(+\infty) &= B, \end{aligned} \tag{28}$$

and if $tF_u(t), F_u(t) \in L^1[0,+\infty[$, by Lemma 1 it is enough to prove that T has a fixed point.

Step 2.1 (T is well defined). As f is a continuous function, $Tu \in C^1[0,+\infty[$ and, by (15), for any $u \in X$ with $\rho > \max\{\|u\|_X, \|\alpha\|_X, \|\beta\|_X\}$

$$\int_0^{+\infty} |F_u(s)|\, ds \leq \int_0^{+\infty} \phi_\rho(s) + \frac{1}{1+s^3} ds < +\infty. \tag{29}$$

That is $F_u(t)$ and $tF_u(t) \in L^1[0,+\infty[$. By Lebesgue Dominated Convergence Theorem,

$$\begin{aligned} \lim_{t\to+\infty} \frac{(Tu)(t)}{1+t} &= \lim_{t\to+\infty} \frac{\Delta + Bt}{1+t} \\ &\quad + \int_0^{+\infty} \lim_{t\to+\infty} \frac{G(t,s)}{1+t} F_u(s)\, ds \\ &\leq B + \int_0^{+\infty} \phi_\rho(s) + \frac{1}{1+s^3} ds < +\infty, \end{aligned} \tag{30}$$

and analogously for

$$\begin{aligned} \lim_{t\to+\infty} (Tu)'(t) &= B - \lim_{t\to+\infty} \int_t^{+\infty} F_u(s)\, ds = B \\ &< +\infty. \end{aligned} \tag{31}$$

Therefore $Tu \in X$.

Step 2.2 (T is continuous). Consider a convergent sequence $u_n \to u$ in X; there exists $\rho_1 > 0$ such that $\max\{\sup_n \|u_n\|_X, \|\alpha\|_X, \|\beta\|_X\} < \rho_1$.

With $M := \sup_{0\leq t<+\infty} |G(t,s)|/(1+t)$, we have

$$\begin{aligned} &\|Tu_n - Tu\|_X \\ &\quad = \max\left\{\|Tu_n - Tu\|_0, \left\|(Tu_n)' - (Tu)'\right\|_1\right\} \\ &\quad \leq \int_0^{+\infty} M \left|F_{u_n}(s) - F_u(s)\right| ds \\ &\qquad + \int_t^{+\infty} \left|F_{u_n}(s) - F_u(s)\right| ds \longrightarrow 0, \end{aligned} \tag{32}$$

as $n \to +\infty$.

Step 2.3 (T is compact). Let $B \subset X$ be any bounded subset. Therefore there is $r > 0$ such that $\|u\|_X < r$, $\forall u \in B$.

For each $u \in B$, and for $\max\{r, \|\alpha\|_X, \|\beta\|_X\} < r_1$,

$$\begin{aligned} \|Tu\|_0 &= \sup_{0\leq t<+\infty} \frac{|Tu(t)|}{1+t} \\ &\leq \sup_{0\leq t<+\infty} \frac{|\Delta + Bt|}{1+t} \\ &\quad + \int_0^{+\infty} \sup_{0\leq t<+\infty} \frac{|G(t,s)|}{1+t} \left|F_u(s)\right| ds \\ &\leq \sup_{0\leq t<+\infty} \frac{|\Delta + Bt|}{1+t} \\ &\quad + \int_0^{+\infty} M\left(\phi_{r_1}(s) + \frac{1}{1+s^3}\right) ds < +\infty, \\ \left\|(Tu)'\right\|_1 &= \sup_{0\leq t<+\infty} \left|(Tu)'(t)\right| \leq |B| + \int_t^{+\infty} \left|F_u(s)\right| ds \\ &\leq |B| + \int_t^{+\infty} \phi_{r_1}(s) + \frac{1}{1+s^3} ds < +\infty. \end{aligned} \tag{33}$$

So $\|Tu\|_X = \max\{\|Tu\|_0, \|(Tu)'\|_1\} < +\infty$; that is, TB is uniformly bounded in X.

TB is equicontinuous, because, for $L > 0$ and $t_1, t_2 \in [0, L]$, we have, as $t_1 \to t_2$,

$$\begin{aligned} &\left|\frac{Tu(t_1)}{1+t_1} - \frac{Tu(t_2)}{1+t_2}\right| \leq \left|\frac{\Delta + Bt_1}{1+t_1} - \frac{\Delta + Bt_2}{1+t_2}\right| \\ &\quad + \int_0^{+\infty} \left|\frac{G(t_1,s)}{1+t_1} - \frac{G(t_2,s)}{1+t_2}\right| |F(u(s))|\, ds \\ &\leq \left|\frac{\Delta + Bt_1}{1+t_1} - \frac{\Delta + Bt_2}{1+t_2}\right| \\ &\quad + \int_0^{+\infty} \left|\frac{G(t_1,s)}{1+t_1} - \frac{G(t_2,s)}{1+t_2}\right| \left(\phi_{r_1}(s) + \frac{1}{1+s^3}\right) ds \\ &\longrightarrow 0, \\ &\left|(Tu)'(t_1) - (Tu)'(t_2)\right| \\ &= \left|\int_{t_1}^{+\infty} F_u(s)\, ds - \int_{t_2}^{+\infty} F_u(s)\, ds\right| \leq \int_{t_1}^{t_2} \left|F_u(s)\right| ds \\ &\leq \int_{t_1}^{t_2} \phi_{r_1}(s) + \frac{1}{1+s^3} ds \longrightarrow 0. \end{aligned} \tag{34}$$

So TB is equicontinuous.

Moreover TB is equiconvergent at infinity, because, as $t \to +\infty$,

$$\begin{aligned} &\left|\frac{Tu(t)}{1+t} - \lim_{t\to+\infty} \frac{Tu(t)}{1+t}\right| \\ &\quad \leq \left|\frac{\Delta + Bt}{1+t} - B\right| + \int_0^{+\infty} \left|\frac{G(t,s)}{1+t} + 1\right| \left|F_u(s)\right| ds \end{aligned}$$

$$\leq \left|\frac{\Delta + Bt}{1+t} - B\right| + \int_0^{+\infty} \left|\frac{G(t,s)}{1+t} + 1\right| \left(\phi_{\rho_1} + \frac{1}{1+s^3}\right) ds \longrightarrow 0,$$

$$\left|(Tu)'(t) - \lim_{t\to+\infty} (Tu)'(t)\right| = \int_t^{+\infty} \left|F_u(s)\right| ds \leq \int_t^{+\infty} \left(\phi_{\rho_1} + \frac{1}{1+s^3}\right) ds \longrightarrow 0, \quad \text{as } t \longrightarrow +\infty. \tag{35}$$

So, by Lemma 2, TB is relatively compact.

Step 2.4. Let $D \subset X$ be a nonempty, closed, bounded, and convex subset. Then $TD \subset D$.

Let $D \subset X$ defined by

$$D := \{u \in X : \|u\|_X \leq \rho_2\} \tag{36}$$

with

$$\rho_2 := \max \left\{ \begin{array}{c} \rho_1, |\beta(0)| + |B| + \int_0^{+\infty} M\left(\phi_{\rho_1}(s) + \frac{1}{1+s^3}\right) ds, \\ |B| + \int_t^{+\infty} \left(\phi_{\rho_1}(s) + \frac{1}{1+s^3}\right) ds \end{array} \right\}, \tag{37}$$

with ρ_1 given by Step 2.1.

For $u \in D$ and $t \in [0, +\infty[$, we get

$$\|Tu\|_0 \leq \sup_{0\leq t<+\infty} \frac{|\beta(0)| + |Bt|}{1+t} + \int_0^{+\infty} M\left(\phi_{\rho_1}(s) + \frac{1}{1+s^3}\right) ds \leq |\beta(0)| + |B| + M\int_0^{+\infty} \left(\phi_{\rho_1}(s) + \frac{1}{1+s^3}\right) ds \leq \rho_2, \tag{38}$$

$$\left\|(Tu)'\right\|_1 \leq |B| + \int_t^{+\infty} \left(\phi_{\rho_1}(s) + \frac{1}{1+s^3}\right) ds \leq \rho_2. \tag{39}$$

Then $\|Tu\|_X \leq \rho_2$; that is, $TD \subset D$.

Then, by Schauder's Fixed Point Theorem, T has at least one fixed point $u_1 \in X$.

Step 3 (u_1 is a solution of (1), (2)). By Step 1, as u_1 is a solution of (17) then $\alpha(t) \leq u_1(t) \leq \beta(t)$, $\forall t \in [0, +\infty[$. So, the differential equation (1) is obtained. It remains to prove that $\alpha(0) \leq u_1(0) + L(u_1, u_1(0), u_1'(0)) \leq \beta(0)$.

Suppose, by contradiction, that $\alpha(0) > u_1(0) + L(u_1, u_1(0), u_1'(0))$. Then

$$u_1(0) = \delta\left(0, u_1(0) + L\left(u_1, u_1(0), u_1'(0)\right)\right) = \alpha(0) \tag{40}$$

and by the monotony of L and Definition 4, the following contradiction holds

$$0 > u_1(0) + L\left(u_1, u_1(0), u_1'(0)\right) - \alpha(0) = L\left(u_1, \alpha(0), u_1'(0)\right) \geq L\left(\alpha, \alpha(0), \alpha'(0)\right) \geq 0. \tag{41}$$

So $\alpha(0) \leq u_1(0) + L(u_1, u_1(0), u_1'(0))$ and in a similar way we can prove that $u_1(0) + L(u_1, u_1(0), u_1'(0)) \leq \beta(0)$.

Therefore, u_1 is a solution of (1), (2). □

A similar result can be obtained if f is a L^1-Carathéodory function and

$$u''(t) = f\left(t, u(t), u'(t)\right), \quad \text{a.e. } t \geq 0. \tag{42}$$

Definition 6. A function $f : [0, +\infty[\times \mathbb{R}^2 \to \mathbb{R}$ is said to be L^1-Carathéodory if it verifies the following:

(1) for each $(x, y) \in \mathbb{R}^2$, $t \mapsto f(t, x, y)$ is measurable on $[0, +\infty[$;

(2) for almost every $t \in [0, +\infty[$, $(x, y) \mapsto f(t, x, y)$ is continuous in $\mathbb{R}^2$;

(3) for each $\rho > 0$, there exists a positive function φ_ρ with $\varphi_\rho, t\varphi_\rho \in L^1[0, +\infty[$ such that, for $(x(t), y(t)) \in \mathbb{R}^2$ with $\sup_{0\leq t<+\infty}\{|x(t)|/(1+t), |y(t)|\} < \rho$,

$$\left|f(t, x, y)\right| \leq \varphi_\rho(t), \quad \text{a.e. } t \in [0, +\infty[. \tag{43}$$

However in this case an extra assumption on f must be assumed.

Theorem 7. *Let $f : [0, +\infty[\times \mathbb{R}^2 \to \mathbb{R}$ be a L^1-Carathéodory function such that $f(t, x, y)$ is monotone on y.*

If there are α, β, lower and upper solutions of (42), (2), respectively, such that

$$\alpha(t) \leq \beta(t), \quad \forall t \geq 0, \tag{44}$$

and $L(x_1, x_2, x_3)$ is nondecreasing on x_1 and x_3, then problem (42), (2) has at least one solution $u \in X$ with $\alpha(t) \leq u(t) \leq \beta(t)$, $\forall t \geq 0$.

Proof. The proof is similar to Theorem 5 except the first step.

Let $u(t)$ be a solution of the modified problem composed by

$$u''(t) = f\left(t, \delta(t, u(t)), u'(t)\right) + \frac{1}{1+t^3}\frac{u(t) - \delta(t, u(t))}{1 + |u(t) - \delta(t, u(t))|}, \quad \text{a.e. } t > 0, \tag{45}$$

and the boundary conditions

$$u(0) = \delta\left(0, u(0) + L\left(u, u(0), u'(0)\right)\right), \quad u'(+\infty) = B. \tag{46}$$

If, by contradiction, there is $t_* \in]0, +\infty[$ such that

$$\min_{0\leq t<+\infty} (u(t) - \alpha(t)) := u(t_*) - \alpha(t_*) < 0, \tag{47}$$

then $u'(t_*) = \alpha'(t_*)$, $u''(t_*) - \alpha''(t_*) \geq 0$, and there exists an interval $I_- :=]t_-, t_*[$ where $u(t) < \alpha(t)$, $u'(t) \leq \alpha'(t)$, $\forall t \in I_-$.

By Definition 4 and if $f(t,x,y)$ is nondecreasing on y, this contradiction holds for $t \in I_-$:

$$\begin{aligned} 0 &\leq u''(t) - \alpha''(t) \\ &= f\left(t, \delta(t, u(t)), u'(t)\right) \\ &\quad + \frac{1}{1+t^3} \frac{u(t) - \delta(t,u(t))}{1 + |u(t) - \delta(t,u(t))|} - \alpha''(t) \\ &\leq f\left(t, \alpha(t), \alpha'(t)\right) + \frac{1}{1+t^3}\frac{u(t)-\alpha(t)}{1+|u(t)-\alpha(t)|} \\ &\quad - \alpha''(t) \leq \frac{u(t)-\alpha(t)}{1+|u(t)-\alpha(t)|} < 0. \end{aligned} \tag{48}$$

The same remains valid if f is nonincreasing, considering an interval $I_+ :=]t_*, t_+[$ where $u(t) < \alpha(t)$, $u'(t) \geq \alpha'(t)$, $\forall t \in I_+$.

So in both cases $u(t) \geq \alpha(t)$, $\forall t \in [0, +\infty[$.

The remaining steps are identical to the proof of Theorem 5, and we omit them. □

4. Example

Consider the second-order problem in the half-line with one functional boundary condition:

$$\begin{aligned} &u''(t) = \frac{\sin(u(t)+1) + \left(u'(t)\right)^3 + u(t)e^{-t}}{1+t^3}, \quad t > 0, \\ &4u^2(0) + \min_{0 \leq t < +\infty} u(t) + u'(0) - 2 = 0, \\ &u'(+\infty) = 0,5. \end{aligned} \tag{49}$$

Remark that the above problem is a particular case of (1), (2) with

$$\begin{aligned} f(t,x,y) &= \frac{\sin(x+1) + y^3 + xe^{-t}}{1+t^3}, \\ B &= 0,5, \\ L(a,b,c) &= 4b^2 + \min_{0 \leq t < +\infty} a(t) + c - 2. \end{aligned} \tag{50}$$

f is continuous in $[0, +\infty[$, and, for $u \in X$, assumption (15) holds with $\varphi_\rho = k/(1+t^3)$, for some $k > 0$ and $\rho > 1$.

As $L(a,b,c)$ is not decreasing in a and c, and the functions $\alpha(t) \equiv -1$ and $\beta(t) = t$ are lower and upper solutions for (49), respectively, then, by Theorem 5, there is at least an unbounded solution u of (49) such that

$$-1 \leq u(t) \leq t, \quad \forall t \in [0, +\infty[. \tag{51}$$

5. Application

Emden-Fowler-types equations (see [18]) can model, for example, the heat diffusion perpendicular to parallel planes by

$$\begin{aligned} &\frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\alpha}{x}\frac{\partial u(x,t)}{\partial x} + af(x,t)g(u) + h(x,t) \\ &\quad = \frac{\partial u(x,t)}{\partial t}, \quad 0 < x < t, \end{aligned} \tag{52}$$

where $f(x,t)g(u) + h(x,t)$ means the nonlinear heat source and $u(x,t)$ is the temperature.

In the steady-state case, and with $h(x,t) \equiv 0$, last equation becomes

$$u''(x) + \frac{\alpha}{x}u'(x) + af(x)g(u) = 0, \quad x \geq 0. \tag{53}$$

If $f(x) \equiv 1$ and $g(u) = u^n$, (53) is called the Lane-Emden equation of the first kind, whereas in the second kind one has $g(u) = e^u$. Both cases are used in the study of thermal explosions. For more details see [19].

In the literature, Emden-Fowler-types equations are associated to Dirichlet or Neumann boundary conditions (see [20, 21]).

To the best of our knowledge, it is the first time where some Emden-Fowler is considered together with functional boundary conditions on the half-line.

Consider that we are looking for nonnegative solutions for the problem composed by the discontinuous differential equation

$$u''(x) = \frac{u'(x)}{1+x^3} + \frac{u^4(x)}{e^x}, \quad \text{a.e. } x > 0, \tag{54}$$

coupled with the infinite multipoint conditions

$$\begin{aligned} &\sum_{n=1}^{+\infty} a_n u(\eta_n) - u(0) + u'(0) = 0, \\ &u'(+\infty) = \delta, \quad (0 < \delta < 1), \end{aligned} \tag{55}$$

where a_n and η_n are nonnegative sequences such that $a_1\eta_1 \geq a_2\eta_2 \geq \cdots \geq a_n\eta_n \geq \cdots$, $\sum_{n=1}^{+\infty} a_n u(\eta_n)$, and $\sum_{n=1}^{+\infty} a_n\eta_n$ are convergent with $\sum_{n=1}^{+\infty} a_n(\eta_n + k) \leq 1 - k$, $(0 < k < 1)$.

This is a particular case of (42), (2), where

$$\begin{aligned} f(x,y,z) &= \frac{z}{1+x^3} + \frac{y^4}{e^x}, \\ B &= \delta, \\ L(v,y,z) &= \sum_{n=1}^{+\infty} a_n v(\eta_n) - y + z. \\ \left|f(x,y,z)\right| &\leq \frac{k_1}{1+x^3} + \frac{k_2}{e^x} := \varphi_r(x), \\ &\quad k_1, k_2 > 0, \; r > 1. \end{aligned} \tag{56}$$

As $\varphi_r(x), x\varphi_r(x) \in L^1[0,+\infty[$ thus f is L^1-Carathéodory. Also f is monotone on z; more precisely f is nondecreasing on z. As $L(v, y, z)$ is not decreasing in v and z, and functions $\alpha(x) \equiv 0$ and $\beta(x) = x + k$ are lower and upper solutions for problem (54), (55), respectively, then, by Theorem 7, there is at least an unbounded and nonnegative solution u of (54), (55) such that

$$0 \leq u(x) \leq x + k, \quad \forall x \in [0, +\infty[. \tag{57}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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