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Research Article

A Computational Study of the Boundary Value Methods and the Block Unification Methods for $y'' = f(x, y, y')$

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We derive a new class of linear multistep methods (LMMs) via the interpolation and collocation technique. We discuss the use of these methods as boundary value methods and block unification methods for the numerical approximation of the general second-order initial and boundary value problems. The convergence of these families of methods is also established. Several test problems are given to show a computational comparison of these methods in terms of accuracy and the computational efficiency.

1. Introduction

Linear multistep methods (LMMs) are widely used for the numerical integration of ordinary differential equations. They are a class of k-step difference equations of the form

$$\sum_{r=0}^{k}\alpha_r y_{n+r} = h^{\mu}\sum_{r=0}^{k}\beta_r f_{n+r}, \tag{1}$$

where α_r, β_r are coefficients to be uniquely determined, μ is the order of the differential equation whose solution is being sought, h is the constant stepsize, $y_{n+i} \equiv y(x_{n+i})$, and $f_{n+i} \equiv f(x_{n+i}, y_{n+i}, y'_{n+i}, \ldots, y^{(\mu-1)}_{n+i})$.

To be able to use (1), we need to impose k additional conditions. Initial value methods (IVMs) are methods whose additional conditions are specified as initial conditions so that they form discrete initial value problems. The IVMs are used for the numerical integration of initial value problems [1–4]. However, if these additional conditions are specified as initial and final conditions (or methods) so that they form a discrete analog of the continuous boundary value problems, we have the boundary value methods (BVMs). They are used for the approximation of both initial and boundary value problems [5–11]. The BVMs are a larger class of methods that contains the IVMs since the IVMs are BVMs with zero final conditions. Sometimes the additional conditions are given as a set of LMMs which together with the main method (1) forms the block methods. If the union of the methods in the block is obtained for $n = 0(k)(N-k)$, where N is the number of grid points, so that we have N difference equations in N unknowns (grid values) which can be easily solved, the resulting approach is termed the block unification methods (BUMs) [12]. The union of the methods in the block is taken to have a consistent equation.

In what follows, we will consider the general second-order system of the form

$$y'' = f\left(x, y, y'\right), \quad x \in [a, b], \tag{2}$$

coupled with any of the initial or boundary conditions

$$y(a) = y_0$$
$$y'(a) = y'_0,$$
$$y(a) = y_0$$
$$y(b) = y_N,$$
$$y'(a) = y'_0$$
$$y(b) = y_N,$$
$$y(a) = y_0$$
$$y'(b) = y'_N,$$

$$y'(a) = y'_0$$
$$y'(b) = y'_N, \tag{3}$$

where $f : \mathbb{R}^m \to \mathbb{R}^m$ are continuous functions. The existence and uniqueness of the solutions of (2) subject to any of (3) have been given in Wend [13] or Ascher et al. [14].

The BVMs have been used for the numerical integration of first-order initial and boundary value problems and their convergence and stability analysis have been fully discussed [5–10]. Aceto et al. [15] constructed P-stable LMMs which were used as BVMs for the special second-order problem $y'' = f(x, y)$. Recently, Biala and Jator [11] developed BVMs for the direct solution of the general second-order initial and boundary value problems arising from the semidiscretization of three-dimensional partial differential equations.

The BUMs have also been successfully applied to solve initial and boundary value problems [12]. In this paper, we construct a new class of LMMs which we implement as boundary value methods and block unification methods. We compare the results of these two classes of methods in terms of accuracy and CPU time.

The outline of the paper is as follows: In Section 2, we derive a (2ν)-step continuous LMM (CLMM) via the interpolation and collocation technique [2–4, 11, 12]. In Section 3, we construct the BVMs by using the CLMM to derive a (2ν)-step discrete LMM which is to be used with some initial and final methods (also obtained from the CLMM). We also discuss the convergence and the use of the BVMs in this section. Section 4 details the BUMs. Their convergence analysis is carried out and an algorithm for their implementation is also discussed. In Section 5, we give several numerical test problems which were solved using both the BVMs and the BUMs. Their comparison in terms of accuracy and computational efficiency (CPU Time) was also shown. Finally, we give some concluding remarks on the methods in Section 6.

2. Derivation of the CLMM

In this section, we will construct a 2ν-step CLMM using the interpolation and collocation technique. The CLMM will be used to generate the BVMs and the BUMs.

We begin by constructing the CLMM of the form

$$U(x) = \alpha_\nu(x)\, y_{n+\nu} + \alpha_{\nu-1}(x)\, y_{n+\nu-1} + \alpha_0(x)\, y_n + h^2 \sum_{r=0}^{2\nu} \beta_r(x)\, f_{n+r}, \tag{4}$$

where $\alpha_0(x)$, $\alpha_{\nu-1}(x)$, $\alpha_\nu(x)$, and $\beta_r(x)$ are continuous coefficients. The next theorem discusses the construction of the CLMM.

Theorem 1. *Let (4) satisfy the following equations:*

$$U(x_{n+j}) = y_{n+j} \quad j = 0, \nu-1, \nu,$$
$$U''(x_{n+j}) = f_{n+j} \quad j = 0(1)(2\nu); \tag{5}$$

then the continuous representation (4) is equivalent to

$$U(x) = \sum_{j=0}^{2\nu+3} \frac{\det(W_j)}{\det(W)} P_j(x), \tag{6}$$

where $P_j(x) = x^j$; $j = 0(1)(2\nu+3)$ are basis functions and the matrix W is defined as follows:

$$W = \begin{pmatrix} P_0(x_n) & P_1(x_n) & \cdots & P_{2\nu+3}(x_n) \\ P_0(x_{n+\nu-1}) & P_1(x_{n+\nu-1}) & \cdots & P_{2\nu+3}(x_{n+\nu-1}) \\ P_0(x_{n+\nu}) & P_1(x_{n+\nu}) & \cdots & P_{2\nu+3}(x_{n+\nu}) \\ P''_0(x_n) & P''_1(x_n) & \cdots & P''_{2\nu+3}(x_n) \\ P''_0(x_{n+1}) & P''_1(x_{n+1}) & \cdots & P''_{2\nu+3}(x_{n+1}) \\ \vdots & \vdots & \vdots & \vdots \\ P''_0(x_{n+2\nu}) & P''_1(x_{n+2\nu}) & \cdots & P''_{2\nu+3}(x_{n+2\nu}) \end{pmatrix}; \tag{7}$$

W_j is obtained by replacing the jth column of W by V, where

$$V = (y_n, y_{n+\nu-1}, y_{n+\nu}, f_n, f_{n+1}, \ldots, f_{n+2\nu})^T, \tag{8}$$

where T denotes the transpose.

Proof. We begin the proof by assuming polynomial basis functions of the form

$$\alpha_j(x) = \sum_{i=0}^{2\nu+3} \alpha_{i+1,j} P_i(x), \quad j = 0, \nu-1, \nu,$$
$$h^2 \beta_j(x) = \sum_{i=0}^{2\nu+3} h^2 \beta_{i+1,j} P_i(x), \quad j = 0(1)(2\nu), \tag{9}$$

where $\alpha_{i+1,j}$, $h^2\beta_{i+1,j}$ are coefficients to be determined.

By substituting (9) into (4), we have

$$U(x) = \sum_{i=0}^{2\nu+3} \alpha_{i+1,0} P_i(x)\, y_n + \sum_{i=0}^{2\nu+3} \alpha_{i+1,\nu-1} P_i(x)\, y_{n+\nu-1} + \sum_{i=0}^{2\nu+3} \alpha_{i+1,\nu} P_i(x)\, y_{n+\nu} + \sum_{j=0}^{2\nu} \sum_{i=0}^{2\nu+3} h^2 \beta_{i+1,j} P_i(x)\, f_{n+j} \tag{10}$$

which is simplified to

$$U(x) = \sum_{i=0}^{2\nu+3} \left\{ \alpha_{i+1,0}\, y_n + \alpha_{i+1,\nu-1}\, y_{n+\nu-1} + \alpha_{i+1,\nu}\, y_{n+\nu} + \sum_{j=0}^{2\nu} h^2 \beta_{i+1,j} f_{n+j} \right\} P_i(x) \tag{11}$$

and expressed in the form

$$U(x) = \sum_{i=0}^{2v+3} \ell_i P_i(x), \tag{12}$$

where

$$\ell_i = \alpha_{i+1,0} y_n + \alpha_{i+1,v-1} y_{n+v-1} + \alpha_{i+1,v} y_{n+v} + \sum_{j=0}^{2v} h^2 \beta_{i+1,j} f_{n+j}. \tag{13}$$

Imposing conditions (5) on (12), we obtain a system of $(2v+4)$ equations which can be expressed as $WL = V$, where $L = (\ell_0, \ell_1, \ldots, \ell_{2v+3})^T$ is a vector of $(2v+4)$ undetermined coefficients.

Using Cramer's rule, the elements of L are determined and given as

$$\ell_i = \frac{\det(W_j)}{\det(W)}, \quad j = 0(1)(2v+3), \tag{14}$$

where W_j is obtained by replacing the jth column of W by V. We rewrite (12) as (6) using the newly found elements of L. □

Remark 2. It has been shown in [11] that symmetric schemes are the best candidates to be used as final methods. Thus, CLMM (4) is chosen to ensure that we have discrete symmteric schemes by evaluation at some points x_{n+j}.

3. The Boundary Value Methods

CLMM (4) is evaluated at x_{n+j}, $j = 1(1)(2v)$, $j \neq v-1, v$, to obtain the BVMs. The main method of the BVM, that is, $U(x_{n+2v})$, is of the form

$$y_{n+2v} + \alpha_v y_{n+v} + \alpha_{v-1} y_{n+v-1} + \alpha_0 y_n = h^2 \sum_{r=0}^{2v} \beta_r f_{n+r} \tag{15}$$

whose derivative formula, obtained by evaluating $U'(x)$ at x_{n+2v}, is

$$h y'_{n+2v} + \alpha'_v y_{n+v} + \alpha'_{v-1} y_{n+v-1} + \alpha'_0 y_n = h^2 \sum_{r=0}^{2v} \beta'_r f_{n+r}. \tag{16}$$

3.1. Convergence of the BVMs. In this section, we will discuss the convergence of the BVMs. We emphasize that (4) is evaluated at x_{n+j}, $j = 1(1)(2v)$, $j \neq v-1, v$, to obtain

$$\begin{aligned}
&y_{n+1} + \alpha_{v-1}^{(1)} y_{n+v-1} + \alpha_v^{(1)} y_{n+v} + \alpha_0^{(1)} y_n = h^2 \sum_{i=0}^{2v} \beta_i^{(1)} f_{n+i} \\
&y_{n+2} + \alpha_{v-1}^{(2)} y_{n+v-1} + \alpha_v^{(2)} y_{n+v} + \alpha_0^{(2)} y_n = h^2 \sum_{i=0}^{2v} \beta_i^{(2)} f_{n+i} \\
&\quad\vdots \\
&y_{n+v-2} + \alpha_{v-1}^{(v-2)} y_{n+v-1} + \alpha_v^{(v-2)} y_{n+v} + \alpha_0^{(v-2)} y_n = h^2 \sum_{i=0}^{2v} \beta_i^{(v-1)} f_{n+i} \\
&y_{n+v+1} + \alpha_{v-1}^{(v+1)} y_{n+v-1} + \alpha_v^{(v+1)} y_{n+v} + \alpha_0^{(v+1)} y_n = h^2 \sum_{i=0}^{2v} \beta_i^{(v+1)} f_{n+i} \\
&\quad\vdots \\
&y_{n+2v} + \alpha_{v-1}^{(2v)} y_{n+v-1} + \alpha_v^{(2v)} y_{n+v} + \alpha_0^{(2v)} y_n = h^2 \sum_{i=0}^{2v} \beta_i^{(2v)} f_{n+i}
\end{aligned} \tag{17}$$

and also, by evaluating $U'(x)$ at x_{n+i}, $i = 0(1)(2v)$, we obtain the derivative formulas

$$\begin{aligned}
&h y'_n + \alpha_{v-1}'^{(0)} y_{n+v-1} + \alpha_v'^{(0)} y_{n+v} + \alpha_0'^{(0)} y_n = h^2 \sum_{i=0}^{2v} \beta_i'^{(0)} f_{n+i} \\
&h y'_{n+1} + \alpha_{v-1}'^{(1)} y_{n+v-1} + \alpha_v'^{(1)} y_{n+v} + \alpha_0'^{(1)} y_n = h^2 \sum_{i=0}^{2v} \beta_i'^{(1)} f_{n+i} \\
&\quad\vdots \\
&h y'_{n+2v} + \alpha_{v-1}'^{(2v)} y_{n+v-1} + \alpha_v'^{(2v)} y_{n+v} + \alpha_0'^{(2v)} y_n = h^2 \sum_{i=0}^{2v} \beta_i'^{(2v)} f_{n+i}.
\end{aligned} \tag{18}$$

We note that the formulas in (17) and (18) are of $O(h^{2v+4})$. We establish the convergence of the BVMs in the following theorem.

Theorem 3. *Let* $\mathbf{Y}$ *be an approximation of the solution vector* $\overline{\mathbf{Y}}$ *for the system obtained on a partition* $\pi_N := \{a = x_0 < x_1 < \cdots < x_N = b,\ x_n = x_{n-1} + h\}$ *from methods (17) and (18). If* $e_n = |y(x_n) - y_n|$, $he'_n = |hy'(x_n) - hy'_n|$, *where the exact solution* $y(x)$ *is several times differentiable on* $[a, b]$, *and if* $\|\mathbf{E}\| = \|\mathbf{Y} - \overline{\mathbf{Y}}\|$, *then the BVM is convergent and of order* $2v+2$, *which implies that* $\|\mathbf{E}\| = O(h^{2v+2})$.

Proof. We compactly write (17) and (18) in matrix form by introducing the following matrix notations. Let A be a $2N \times 2N$ matrix defined by

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{19}$$

where A_{ij} are $N \times N$ matrices given as

$$A_{11} = \begin{bmatrix} & \alpha'^{(0)}_{\nu-1} & \alpha'^{(0)}_{\nu} & \alpha'^{(0)}_{0} & & & & \\ 1 & \alpha^{(1)}_{\nu-1} & \alpha^{(1)}_{\nu} & \alpha^{(1)}_{0} & & & & \\ \vdots & \vdots & \vdots & \vdots & & & & \\ 1 & \alpha^{(\nu-2)}_{\nu-1} & \alpha^{(\nu-2)}_{\nu} & \alpha^{(\nu-2)}_{0} & & & & \\ 1 & \alpha^{(\nu+1)}_{\nu-1} & \alpha^{(\nu+1)}_{\nu} & \alpha^{(\nu+1)}_{0} & & & & \\ \vdots & \vdots & \vdots & \vdots & & & & \\ 1 & \alpha^{(2\nu)}_{\nu-1} & \alpha^{(2\nu)}_{\nu} & \alpha^{(2\nu)}_{0} & & & & \\ & \ddots & \ddots & \ddots & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & 1 & \alpha^{(2\nu)}_{\nu-1} & \alpha^{(2\nu)}_{\nu} & \cdots & \alpha^{(2\nu)}_{0} \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} \alpha'^{(1)}_{\nu-1} & \alpha'^{(1)}_{\nu} & \alpha'^{(1)}_{0} & & & \\ \alpha'^{(2)}_{\nu-1} & \alpha'^{(2)}_{\nu} & \alpha'^{(2)}_{0} & & & \\ \vdots & \vdots & \vdots & & & \\ \alpha'^{(2\nu)}_{\nu-1} & \alpha'^{(2\nu)}_{\nu} & \alpha'^{(2\nu)}_{0} & & & \\ & & \alpha'^{(2\nu)}_{\nu-1} & \alpha'^{(2\nu)}_{\nu} & \cdots & \alpha'^{(2\nu)}_{0} \\ & & \ddots & \ddots & \ddots & \\ & & \alpha'^{(2\nu)}_{\nu-1} & \alpha'^{(2\nu)}_{\nu} & & \alpha'^{(2\nu)}_{0} \end{bmatrix}, \tag{20}$$

A_{12} is an $N \times N$ null matrix, and A_{22} is an $N \times N$ identity matrix.

Similarly, let B be a $2N \times 2N$ matrix defined by

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \tag{21}$$

where B_{ij} are $N \times N$ matrices given as

$$B_{11} = h^2 \begin{bmatrix} \beta'^{(0)}_{1} & \beta'^{(0)}_{2} & \cdots & \beta'^{(0)}_{2\nu} & & & & & \\ \beta^{(1)}_{1} & \beta^{(1)}_{2} & \cdots & \beta^{(1)}_{2\nu} & & & & & \\ \vdots & \vdots & \vdots & \vdots & & & & & \\ \beta^{(\nu-1)}_{1} & \beta^{(\nu-1)}_{2} & \cdots & \beta^{(\nu-1)}_{2\nu} & & & & & \\ \beta^{(\nu+1)}_{1} & \beta^{(\nu+1)}_{2} & \cdots & \beta^{(\nu+1)}_{2\nu} & & & & & \\ \vdots & \vdots & \vdots & \vdots & & & & & \\ \beta^{(2\nu)}_{1} & \beta^{(2\nu)}_{2} & \cdots & \beta^{(2\nu)}_{2\nu} & & & & & \\ & & & & \beta'^{(0)}_{0} & \beta'^{(0)}_{1} & \cdots & \beta'^{(0)}_{2\nu} & \\ & & & & \beta^{(1)}_{0} & \beta'^{(1)}_{1} & \cdots & \beta'^{(1)}_{2\nu} & \\ & & & & \vdots & \vdots & \vdots & \vdots & \\ & & & & \beta^{(\nu-1)}_{0} & \beta^{(\nu-1)}_{1} & \cdots & \beta^{(\nu-1)}_{2\nu} & \\ & & & & \beta^{(\nu+1)}_{0} & \beta^{(\nu+1)}_{1} & \cdots & \beta^{(\nu+1)}_{2\nu} & \\ & & & & \vdots & \vdots & \vdots & \vdots & \\ & & & & \beta^{(2\nu)}_{0} & \beta^{(2\nu)}_{1} & \cdots & \beta^{(2\nu)}_{2\nu} & \\ & & & & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & & & & \beta^{(2\nu)}_{0} & \beta^{(2\nu)}_{1} & \cdots \; \beta^{(2\nu)}_{2\nu} \end{bmatrix}, \tag{22}$$

$$B_{21} = h^2 \begin{bmatrix} \beta'^{(1)}_{1} & \beta'^{(1)}_{2} & \cdots & \beta'^{(1)}_{2\nu} & & & & \\ \vdots & \vdots & \vdots & \vdots & & & & \\ \beta'^{(2\nu)}_{1} & \beta'^{(2\nu)}_{2} & \cdots & \beta'^{(2\nu)}_{2\nu} & & & & \\ & & & & \beta'^{(1)}_{0} & \beta'^{(1)}_{1} & \cdots & \beta'^{(1)}_{2\nu} \\ & & & & \vdots & \vdots & \vdots & \vdots \\ & & & & \beta'^{(1)}_{0} & \beta'^{(1)}_{1} & \cdots & \beta'^{(1)}_{2\nu} \\ & & & & & \ddots & \ddots & \ddots \\ & & & & & \beta'^{(1)}_{0} & \beta'^{(1)}_{1} & \cdots \; \beta'^{(1)}_{2\nu} \end{bmatrix}$$

and B_{12}, B_{22} are $N \times N$ null matrices.

We also define the following vectors:

$$\overline{\mathbf{Y}} = \left(y_1, \ldots, y_N, hy'_1, \ldots, hy'_N\right)^T,$$
$$\mathbf{Y} = \left(y(x_1), \ldots, y(x_N), hy'(x_1), \ldots, hy'(x_N)\right)^T,$$
$$\mathbf{F} = \left(f_1, \ldots, f_N, hf'_1, \ldots, hf'_N\right)^T,$$
$$\mathbf{L}(h) = \left(l_1, \ldots, l_n, l'_1, \ldots, l'_N\right)^T, \tag{23}$$
$$\mathbf{C} = \left(\beta_0'^{(0)} h^2 f_0 - hy'_0, \beta_0^{(0)} h^2 f_0 - y_0, \beta_0^{(1)} h^2 f_0, \ldots, \beta^{(\nu-1)} h^2 f_0, \beta_0^{(\nu+1)} h^2 f_0, \ldots, \beta_0^{(2\nu)} h^2 f_0, 0, \ldots, 0, \beta_0'^{(0)} h^2 f_0, \ldots, \beta_0'^{(2\nu)} h^2 f_0, 0, \ldots, 0\right)^T.$$

The exact form of the system formed by (17) and (18) is given by

$$A\mathbf{Y} - B\mathbf{F}(\mathbf{Y}) + \mathbf{C} + \mathbf{L}(h) = 0, \tag{24}$$

where $\mathbf{L}(h)$ is the truncation error vector of the formulas in (17) and (18). The approximate form of the system is given by

$$A\overline{\mathbf{Y}} - B\mathbf{F}\left(\overline{\mathbf{Y}}\right) + \mathbf{C} = 0, \tag{25}$$

where $\overline{\mathbf{Y}}$ is the approximate solution of vector $\mathbf{Y}$.

Subtracting (24) from (25) and letting $\mathbf{E} = \overline{\mathbf{Y}} - \mathbf{Y} = (e_1, \ldots, e_N, e'_1, \ldots, e'_N)^T$ and using the mean value theorem, we have the error system

$$(A - BJ)\mathbf{E} = \mathbf{L}(h), \tag{26}$$

where J is the Jacobian matrix and its entries J_{11}, J_{12}, J_{21}, and J_{22} are defined as

$$J_{11} = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_N} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_N}{\partial y_1} & \cdots & \frac{\partial f_N}{\partial y_N} \end{bmatrix},$$

$$J_{12} = \begin{bmatrix} \frac{\partial f_1}{\partial y'_1} & \cdots & \frac{\partial f_1}{\partial y'_N} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_N}{\partial y'_1} & \cdots & \frac{\partial f_N}{\partial y'_N} \end{bmatrix},$$

$$J_{21} = h\begin{bmatrix} \frac{\partial f'_1}{\partial y_1} & \cdots & \frac{\partial f'_1}{\partial y_N} \\ \vdots & \vdots & \vdots \\ \frac{\partial f'_N}{\partial y_1} & \cdots & \frac{\partial f'_N}{\partial y_N} \end{bmatrix},$$

$$J_{22} = h\begin{bmatrix} \frac{\partial f'_1}{\partial y'_1} & \cdots & \frac{\partial f'_1}{\partial y'_N} \\ \vdots & \vdots & \vdots \\ \frac{\partial f'_N}{\partial y'_1} & \cdots & \frac{\partial f'_N}{\partial y'_N} \end{bmatrix}. \tag{27}$$

Let $N = -BJ$ be a matrix of dimension $2N$ so that (26) becomes

$$(A + N)\mathbf{E} = \mathbf{L}(h), \tag{28}$$

and, for sufficiently small h, $A + N$ is a monotone matrix and thus nonsingular (see [16]). Therefore

$$\begin{aligned} (A+N)^{-1} &= D = \left(d_{ij}\right) \geq 0, \\ \sum_{j=1}^{2N} d_{ij} &= O\left(h^{-2}\right), \\ \mathbf{E} &= D\mathbf{L}(h), \\ \|\mathbf{E}\| &= \|D\mathbf{L}(h)\| = O\left(h^{-2}\right) O\left(h^{2\nu+4}\right) \\ &= O\left(h^{2\nu+2}\right), \end{aligned} \tag{29}$$

which shows that the methods are convergent and the global error is of order $O(h^{2\nu+2})$. □

3.2. Use of Methods. The BVMs can only be successfully implemented if used together with appropriate additional methods [5]. In this regard, we have proposed a main method and additional methods which are obtained from the same continuous scheme (the CLMM).

To use LMM (15) as BVMs, we rewrite main method (15) as

$$\begin{aligned} &y_{n+\nu-1} + \alpha_\nu y_{n-1} + \alpha_{\nu-1} y_{n-2} + \alpha_0 y_{n-\nu-1} \\ &\quad = h^2 \sum_{i=-\nu-1}^{\nu-1} \beta_{i+\nu+1} f_{n+i}, \quad n = \nu+1, \ldots, N-\nu+1, \end{aligned} \tag{30}$$

with the derivative formula

$$\begin{aligned} &hy'_{n+\nu-1} + \alpha'_\nu y_{n-1} + \alpha'_{\nu-1} y_{n-2} + \alpha'_0 y_{n-\nu-1} \\ &\quad = h^2 \sum_{i=-\nu-1}^{\nu-1} \beta'_{i+\nu+1} f_{n+i}, \quad n = \nu+1, \ldots, N-\nu+1, \end{aligned} \tag{31}$$

which are to be used with some boundary conditions and $U'(x_r)$, $r = 0(1)(2\nu-1)$.

The discrete solutions

$$\begin{aligned} &y_0, \ldots, y_\nu, y_{N-\nu+2}, \ldots, y_N \\ &y'_0, \ldots, y'_\nu, y'_{N-\nu+2}, \ldots, y'_N \end{aligned} \tag{32}$$

are to be obtained for methods (30) and (31) to be useful. However, (3) provides two solution values so that we have to impose $4v-2$ additional conditions of which $U'(x_r)$, $r=0(1)(2v-1)$, gives $2v$ methods and the remaining $2v-2$ methods are given as a set of $v-1$ initial and $v-1$ final methods which is readily obtained from CLMM (4). Approximations (32) need be at least of order $O(h^{p+1})$ accuracy if BVM (30) is of order p in order to have a solution accuracy of $O(h^{p+1})$. Equations (30) and (31) with the $4v-2$ additional methods give a set of $2N$ equations in $2N$ unknowns which can be easily solved. We give below the BVMs of orders 6 and 8.

BVM of Order 6 ($v=2$)

$$y_{n+1}-2y_{n-1}+y_{n-3}=\frac{h^2}{15}\left(f_{n-3}+16f_{n-2}+26f_{n-1}+16f_n+f_{n+1}\right),\quad n=3,\ldots,N-1, \tag{33}$$

with the derivative formulas

$$\begin{aligned}
hy'_{n+1}&=\frac{-107}{42}y_{n-1}+\frac{128}{21}y_{n-2}-\frac{149}{42}y_{n-3}+\frac{h^2}{1260}\left(325f_{n-3}+4048f_{n-2}+1106f_{n-1}+1744f_n+397f_{n+1}\right),\\
hy'_0&=\frac{-107}{42}y_2+\frac{128}{21}y_1-\frac{149}{42}y_0+\frac{h^2}{1260}\left(-67f_0+2256f_1+434f_2-48f_3+5f_4\right),\\
hy'_1&=\frac{41}{21}y_2-\frac{61}{21}y_1+\frac{20}{21}y_0+\frac{h^2}{10080}\left(-613f_0+11464f_1-2870f_2+344f_3-37f_4\right),\\
hy'_2&=\frac{5}{42}y_2+\frac{16}{21}y_1-\frac{37}{42}y_0+\frac{h^2}{1260}\left(73f_0+1136f_1+574f_2-48f_3+5f_4\right),\\
hy'_3&=\frac{41}{21}y_2-\frac{61}{21}y_1+\frac{20}{21}y_0+\frac{h^2}{10080}\left(-725f_0-7656f_1+9898f_2+410f_3-149f_4\right)
\end{aligned}\tag{34}$$

which are to be used with the initial method

$$y_3-2y_2+y_1=\frac{h^2}{240}\left(-f_0+24f_1+194f_2+24f_3-f_4\right) \tag{35}$$

and the final method

$$y_{N-3}-2y_{N-2}+y_{N-1}=\frac{h^2}{240}\left(-f_N+24f_{N-1}+194f_{N-2}+24f_{N-3}-f_{N-4}\right). \tag{36}$$

BVM of Order 8 ($v=3$)

$$y_{n+2}-2y_{n-1}+y_{n-4}=\frac{h^2}{2240}\left(141f_{n-4}+2430f_{n-3}+4131f_{n-2}+6756f_{n-1}+4131f_n+2430f_{n+1}+141f_{n+2}\right),\quad n=4,\ldots,N-2, \tag{37}$$

with the derivative formulas

$$\begin{aligned}
hy'_{n+2}&=\frac{-233}{30}y_{n-1}+\frac{243}{20}y_{n-2}-\frac{263}{60}y_{n-3}+\frac{h^2}{44800}\left(128089f_{n-4}+216774f_{n-3}+305847f_{n-2}+122452f_{n-1}+6327f_n+69318f_{n+1}+13113f_{n+2}\right),\\
hy'_0&=\frac{-233}{30}y_3+\frac{243}{20}y_2-\frac{263}{60}y_0+\frac{h^2}{44800}\left(-1031f_0+147654f_1+297207f_2+35412f_3-2313f_4+198f_5-7f_6\right),\\
hy'_1&=\frac{5713}{1920}y_3-\frac{5073}{1280}y_2+\frac{3793}{3840}y_0+\frac{h^2}{77414400}\left(-3977083f_0-113321f_1-208639509f_2-20080444f_3+389931f_4+130254f_5-23611f_6\right),\\
hy'_2&=\frac{7}{30}y_3+\frac{3}{20}y_2-\frac{23}{60}y_0+\frac{h^2}{1209600}\left(26803f_0+534018f_1+323229f_2-123236f_3+31389f_4+6654f_5+691f_6\right),\\
hy'_3&=\frac{3313}{1920}y_3-\frac{2673}{1280}y_2+\frac{1292}{3840}y_0+\frac{h^2}{2867200}\left(-60609f_0-1189494f_1-1182447f_2+820748f_3-90927f_4+17802f_5-1793f_6\right),\\
hy'_4&=\frac{7}{30}y_3+\frac{3}{20}y_2-\frac{23}{60}y_0+\frac{h^2}{1209600}\left(28403f_0+510978f_1+804189f_2+1376924f_3+512349f_4-29694f_5+2291f_6\right),\\
hy'_5&=\frac{5713}{19200}y_3-\frac{5073}{1280}y_2+\frac{3793}{3840}y_0+\frac{h^2}{77414400}\left(-4632443f_0-85305138f_1-108369429f_2+34314436f_3-100660011f_4+28146894f_5-678971f_6\right)
\end{aligned}\tag{38}$$

which are to be used with the initial methods

$$y_1 + \frac{95}{64}y_3 - \frac{349}{128}y_2 + \frac{31}{128}y_0 = \frac{h^2}{77414400}\big(91775f_0 + 2787594f_1 + 9305553f_2 + 127768f_3 + 109167f_4 + 13578f_5 - 961f_6\big),$$
$$y_4 - 2y_3 + y_2 = \frac{h^2}{60480}\big(31f_0 - 438f_1 + 6513f_2 + 48268f_3 + 6513f_4 - 438f_5 + 31f_6\big) \quad (39)$$

and the final methods

$$y_{N-4} - 2y_{N-3} + y_{N-2} = \frac{h^2}{60480}\big(31f_N - 438f_{N-1} + 6513f_{N-2} + 48268f_{N-3} + 6513f_{N-4} - 438f_{N-5} + 31f_{N-6}\big),$$
$$y_{N-5} - \frac{223}{64}y_{N-3} + \frac{349}{128}y_{N-2} - \frac{31}{128}y_{N-2} = \frac{h^2}{25804800}\big(-f_N - 724398f_{N-1} - 431259f_{N-2} - 4156156f_{N-3} + 2706981f_{N-4} + 200274f_{N-5} - 5141f_{N-6}\big). \quad (40)$$

4. The Block Unification Methods

The BUMs are also a class of methods for the numerical integration of both initial and boundary value problems. CLMM (4) is a $2v$-step continuous scheme and as such the BUM requires a set of $4v$ methods so that, on the partition $\pi_N, h > o$, $x_n = x_0 + nh$, $n = 0(1)N$, the solution of the $2v$ step $[x_n, y_n, y_n'] \mapsto [x_{n+2v}, y_{n+2v}, y_{n+2v}']$ is obtained. CLMM (4) is used to generate $(4v - 1)$ methods by evaluating (4) at $x = \{x_{n+1}, \ldots, x_{n+v-2}, x_{n+v+1}, \ldots, x_{n+2v}\}$ and also evaluating $U'(x)$ at $x = \{x_n, x_{n+1}, \ldots, x_{n+2v}\}$. CLMM (4) can only be used to construct $4v - 1$ methods and as such we give the last method as (since it is also of $O(h^{2v+4})$)

$$y_{n+2v} + \alpha_v y_{n+v} + \alpha_{v-1} y_{n+v-1} = h^2 \sum_{i=1}^{2v} \beta_i f_{n+i}. \quad (41)$$

The BUM ($4v$ methods) is of the form

$$y_{n+1} + \alpha_v y_{n+v} + \alpha_{v-1} y_{n+v-1} + \alpha_0 y_n = h^2 \sum_{i=0}^{2v} \beta_i f_{n+i}$$
$$y_{n+2} + \alpha_v y_{n+v} + \alpha_{v-1} y_{n+v-1} + \alpha_0 y_n = h^2 \sum_{i=0}^{2v} \beta_i f_{n+i}$$
$$\vdots$$
$$y_{n+v-2} + \alpha_v y_{n+v} + \alpha_{v-1} y_{n+v-1} + \alpha_0 y_n = h^2 \sum_{i=0}^{2v} \beta_i f_{n+i}$$
$$y_{n+v+1} + \alpha_v y_{n+v} + \alpha_{v-1} y_{n+v-1} + \alpha_0 y_n = h^2 \sum_{i=0}^{2v} \beta_i f_{n+i}$$
$$\vdots$$
$$y_{n+2v} + \alpha_v y_{n+v} + \alpha_{v-1} y_{n+v-1} + \alpha_0 y_n = h^2 \sum_{i=0}^{2v} \beta_i f_{n+i}$$
$$y_{n+2v} + \alpha_v y_{n+v} + \alpha_{v-1} y_{n+v-1} = h^2 \sum_{i=0}^{2v} \beta_i f_{n+i}$$
$$hy_n' + \alpha_v' y_{n+v} + \alpha_{v-1}' y_{n+v-1} + \alpha_0' y_n = h^2 \sum_{i=0}^{2v} \beta_i' f_{n+i}$$
$$hy_{n+1}' + \alpha_v' y_{n+v} + \alpha_{v-1}' y_{n+v-1} + \alpha_0' y_n = h^2 \sum_{i=0}^{2v} \beta_i' f_{n+i}$$
$$\vdots$$
$$hy_{n+2v}' + \alpha_v' y_{n+v} + \alpha_{v-1}' y_{n+v-1} + \alpha_0' y_n = h^2 \sum_{i=0}^{2v} \beta_i' f_{n+i},$$
$$n = 0\,(2v)\,(N - 2v). \quad (42)$$

Formulas (42) that form the BUM are all weighted the same unlike the BVMs that have main methods (15) and (16). We give below the BUMs of orders 6 and 8.

BUM of Order 6

$$y_{n+3} - 2y_{n+2} + y_{n+1} = \frac{h^2}{240}\big(-f_n + 24f_{n+1} + 194f_{n+2} + 24f_{n+3} - f_{n+4}\big),$$
$$y_{n+4} - 2y_{n+2} + y_n = \frac{h^2}{15}\big(f_n + 16f_{n+1} + 26f_{n+2} + 16f_{n+3} + f_{n+4}\big),$$
$$y_{n+4} - 3y_{n+2} + 2y_{n+1} = \frac{h^2}{240}\big(-3f_n + 52f_{n+1} + 402f_{n+2} + 252f_{n+3} + 17f_{n+4}\big),$$
$$hy_n' = \frac{-107}{42}y_{n+2} + \frac{128}{21}y_{n+1} - \frac{149}{42}y_n + \frac{h^2}{1260}\big(-67f_n + 2256f_{n+1} + 434f_{n+2} - 48f_{n+3} + 5f_{n+4}\big),$$
$$hy_{n+1}' = \frac{41}{21}y_{n+2} - \frac{61}{21}y_{n+1} + \frac{20}{21}y_n + \frac{h^2}{10080}\big(-613f_n + 11464f_{n+1} - 2870f_{n+2} + 344f_{n+3} - 37f_{n+4}\big),$$
$$hy_{n+2}' = \frac{5}{42}y_{n+2} + \frac{16}{21}y_{n+1} - \frac{37}{42}y_n + \frac{h^2}{1260}\big(73f_n + 1136f_{n+1} + 574f_{n+2} - 48f_{n+3} + 5f_{n+4}\big),$$

$$hy'_{n+3} = \frac{41}{21}y_{n+2} - \frac{61}{21}y_{n+1} + \frac{20}{21}y_n + \frac{h^2}{10080}\left(-725f_n - 7656f_{n+1} + 9898f_{n+2} + 410f_{n+3} - 149f_{n+4}\right),$$

$$hy'_{n+4} = \frac{-107}{42}y_{n+2} + \frac{128}{21}y_{n+1} - \frac{149}{42}y_n + \frac{h^2}{1260}\left(325f_n + 4048f_{n+1} + 1106f_{n+2} + 1744f_{n+3} + 397f_{n+4}\right),$$

$$n = 0\,(4)\,(N-4). \tag{43}$$

BUM of Order 8

$$y_{n+1} + \frac{95}{64}y_{n+3} - \frac{349}{128}y_{n+2} + \frac{31}{128}y_n = \frac{h^2}{77414400}\left(91775f_n + 2787594f_{n+1} + 9305553f_{n+2} + 127768f_{n+3} + 109167f_{n+4} + 13578f_{n+5} - 961f_{n+6}\right),$$

$$y_{n+4} - 2y_{n+3} + y_{n+2} = \frac{h^2}{60480}\left(31f_n - 438f_{n+1} + 6513f_{n+2} + 48268f_{n+3} + 6513f_{n+4} - 438f_{n+5} + 31f_{n+6}\right),$$

$$y_{n+5} - \frac{223}{64}y_{n+3} + \frac{349}{128}y_{n+2} - \frac{31}{128}y_n = \frac{h^2}{25804800}\left(-f_n - 724398f_{n+1} - 431259f_{n+2} - 4156156f_{n+3} + 2706981f_{n+4} + 200274f_{n+5} - 5141f_{n+6}\right),$$

$$y_{n+6} - 2y_{n+3} + y_n = \frac{h^2}{2240}\left(141f_n + 2430f_{n+1} + 4131f_{n+2} + 6756f_{n+3} + 4131f_{n+4} + 2430f_{n+5} + 141f_{n+6}\right),$$

$$y_{n+6} - 4y_{n+3} + 3y_{n+2} = \frac{h^2}{10080}\left(-11f_n - 18f_{n+1} + 2523f_{n+2} + 27268f_{n+3} + 19323f_{n+4} + 10734f_{n+5} + 661f_{n+6}\right),$$

$$hy'_n = \frac{-233}{30}y_{n+3} + \frac{243}{20}y_{n+2} - \frac{263}{60}y_n + \frac{h^2}{44800}\left(-1031f_n + 147654f_{n+1} + 297207f_{n+2} + 35412f_{n+3} - 2313f_{n+4} + 198f_{n+5} - 7f_{n+6}\right),$$

$$hy'_{n+1} = \frac{5713}{1920}y_{n+3} - \frac{5073}{1280}y_{n+2} + \frac{3793}{3840}y_n + \frac{h^2}{77414400}\left(-3977083f_n - 113321f_{n+1} - 208639509f_{n+2} - 20080444f_{n+3} + 389931f_{n+4} + 130254f_{n+5} - 23611f_{n+6}\right),$$

$$hy'_{n+2} = \frac{7}{30}y_{n+3} + \frac{3}{20}y_{n+2} - \frac{23}{60}y_n + \frac{h^2}{1209600}\left(26803f_n + 534018f_{n+1} + 323229f_{n+2} - 123236f_{n+3} + 31389f_{n+4} + 6654f_{n+5} + 691f_{n+6}\right),$$

$$hy'_{n+3} = \frac{3313}{1920}y_{n+3} - \frac{2673}{1280}y_{n+2} + \frac{1292}{3840}y_n + \frac{h^2}{2867200}\left(-60609f_n - 1189494f_{n+1} - 1182447f_{n+2} + 820748f_{n+3} - 90927f_{n+4} + 17802f_{n+5} - 1793f_{n+6}\right),$$

$$hy'_{n+4} = \frac{7}{30}y_{n+3} + \frac{3}{20}y_{n+2} - \frac{23}{60}y_n + \frac{h^2}{1209600}\left(28403f_n + 510978f_{n+1} + 804189f_{n+2} + 1376924f_{n+3} + 512349f_{n+4} - 29694f_{n+5} + 2291f_{n+6}\right),$$

$$hy'_{n+5} = \frac{5713}{19200}y_{n+3} - \frac{5073}{1280}y_{n+2} + \frac{3793}{3840}y_n + \frac{h^2}{77414400}\left(-4632443f_n - 85305138f_{n+1} - 108369429f_{n+2} + 34314436f_{n+3} - 100660011f_{n+4} + 28146894f_{n+5} - 678971f_{n+6}\right),$$

$$hy'_{n+6} = \frac{-233}{30}y_{n+3} + \frac{243}{20}y_{n+2} - \frac{263}{60}y_n + \frac{h^2}{44800}\left(128089f_n + 216774f_{n+1} + 305847f_{n+2} + 122452f_{n+3} + 6327f_{n+4} + 69318f_{n+5} + 13113f_{n+6}\right),$$

$$n = 0\,(6)\,(N-6). \tag{44}$$

4.1. Convergence and Use of the BUMs. BUMs (42) are weighted the same as each formula in (42) is used the same number of times as others. Their convergence was established in a similar way to Theorem 3 with some changes in the coefficients of the matrices and the global error is also of $O(h^{2v+2})$.

The BUM is implemented efficiently by using the following algorithm.

Step 1. Use the block unification of (42) for $n = 0$ to obtain $\mathbf{Y}_1$ in the interval $[y_n, y_{n+2v}]$; for $n = 1$, $\mathbf{Y}_2$ is obtained in the interval $[y_{n+2v}, y_{n+4v}]$; and in the intervals $[y_{n+4v}, y_{n+6v}]$, $[y_{n+6v}, y_{n+8v}], \ldots, [y_{N-2v}, y_N]$ for $n = 2, 3, \ldots, (\Gamma - 1)$, we obtain $\mathbf{Y}_3, \ldots, \mathbf{Y}_\Gamma$ where $N = 2v \times \Gamma$.

Step 2. The unified block given by the system $\mathbf{Y}_1 \cup \mathbf{Y}_2 \cup \cdots \cup \mathbf{Y}_{\Gamma-1} \cup \mathbf{Y}_\Gamma$ obtained in Step 1 results in a system of $2N$ equations in $2N$ unknowns which can be easily solved.

Step 3. The values of the solution and the first derivatives of (2) are generated by the sequence of $\{y_n\}$, $\{y'_n\}$, $n = 0, \ldots, N$, obtained as the solution in Step 2.

5. Test Problems

We consider five numerical examples. The examples were solved using the BVMs and the BUMs of different order derived in this paper. Comparisons are made between the BVMs and BUMs by obtaining the maximum errors in the interval of integration. We note that the number of function evaluations (NFEs) involved in implementing the two methods is $N \times 2v$ in the entire range of integration. In order to show the competitiveness of the derived methods with some existing methods in the literature, we compared our methods with the Extended Trapezoidal Rules (ETRs), Extended Trapezoidal Rules of the second kind (ETR_2s), and the Top Order Methods (TOMs) of orders 6, 8, and 10, respectively, given in [6]. For linear problems, we solve the resulting system of equations using Gaussian elimination with partial pivoting and, for nonlinear problems, we use a modified Newton-Raphson method.

Example 1. We consider the boundary value problem given in [6]:

$$\begin{aligned} &\left(x^3 u''\right)'' = 1, \quad 1 < x < 2, \\ &u(1) = u''(1) = u(2) = u''(2) = 0, \\ &\text{Exact: } u(x) = \frac{1}{4}\left(10\log(2) - 3\right)(1 - x) \\ &\qquad + \frac{1}{2}\left(x^{-1} + (3 + x)\log(x) - x\right). \end{aligned} \tag{45}$$

Example 2. We consider the nonlinear Fehlberg problem given in [12]:

$$\begin{aligned} &y_1'' = -4x^2 y_1 - \frac{2}{\sqrt{y_1^2 + y_2^2}} y_2, \\ &\qquad \sqrt{\frac{\pi}{2}} < x < 10, \\ &y_2'' = -4x^2 y_2 - \frac{2}{\sqrt{y_1^2 + y_2^2}} y_1, \\ &y_1(x_0) = 0, \\ &y_1'(x_0) = -\sqrt{2\pi}, \\ &y_2(x_0) = 1, \\ &y_2'(x_0) = 0, \\ &\qquad x_0 = \sqrt{\frac{\pi}{2}}, \\ &\text{Exact: } y_1(x) = \cos\left(x^2\right), \\ &y_2(x) = \sin\left(x^2\right). \end{aligned} \tag{46}$$

Example 3. We consider the nonlinear BVP with mixed boundary conditions given in [17]:

$$\begin{aligned} &y'' = \frac{\left(y'\right)^2 + y^2}{2e^x}, \quad 0 < x < 1, \\ &y(0) - y'(0) = 0, \\ &y(1) + y'(1) = 2e, \\ &\text{Exact: } y(x) = e^x. \end{aligned} \tag{47}$$

Example 4. We consider the nonlinear BVP given in [18]:

$$\begin{aligned} &\frac{d^2 y_1}{dx^2} + 20 y_1' + 4\cos(x) y_1 + \sin\left(y_1 y_2\right) = f_1(x), \\ &\qquad 0 < x < 1, \\ &\frac{d^2 y_2}{dx^2} + 5e^x y_2' + 6\sinh(x) y_2 + \cos\left(y_2\right) = f_2(x), \\ &y_1(0) = 1, \\ &y_2(0) = 0, \\ &y_1(1) = e, \\ &y_2(1) = \sinh(1), \end{aligned} \tag{48}$$

where

$$\begin{aligned} &f_1(x) = 21e^x + 4e^x \cos(x) \\ &\qquad + \sin\left(e^x \sinh(x)\right), \\ &f_2(x) = x\cos(\sinh(x)) + 5e^x \cosh(x) \\ &\qquad + \sinh(x) + 6\sinh^2(x), \\ &\text{Exact: } y_1(x) = e^x, \\ &y_2(x) = \sinh(x). \end{aligned} \tag{49}$$

Table 1: Computational results for $\nu = 2$ for Example 5.1.

N	BVM		BUM		ETRs	
	l_∞ error	CPU Time	l_∞ error	CPU Time	l_∞ error	CPU Time
4	$2.184e-02$	0.281	$6.182e-05$	0.297	$3.641e-05$	0.296
8	$3.357e-06$	0.267	$1.225e-05$	0.328	$2.193e-06$	0.312
16	$1.867e-08$	0.329	$2.656e-07$	0.328	$8.754e-08$	0.365
32	$6.634e-10$	0.389	$4.351e-09$	0.359	$2.249e-09$	0.389
64	$2.884e-12$	0.422	$6.837e-11$	0.391	$4.564e-11$	0.441
128	$6.456e-14$	0.626	$1.070e-12$	0.546	$8.156e-13$	0.625

Table 2: Computational results for $\nu = 3$ for Example 5.1.

N	BVM		BUM		ETR_2s	
	l_∞ error	CPU Time	l_∞ error	CPU Time	l_∞ error	CPU Time
6	$9.982e-01$	0.281	$3.228e-06$	0.234	$1.837e-06$	0.328
12	$4.437e-08$	0.312	$3.948e-07$	0.313	$2.245e-08$	0.342
24	$1.913e-10$	0.358	$2.627e-09$	0.344	$1.353e-10$	0.359
48	$3.812e-13$	0.436	$1.142e-11$	0.390	$4.333e-13$	0.391
96	$4.442e-16$	0.577	$4.573e-14$	0.499	$1.030e-15$	0.514
192	$5.660e-16$	0.843	$6.145e-16$	0.689	$6.161e-15$	0.811

Table 3: Computational results for $\nu = 3$ for Example 5.2.

N	BVM		BUM		ETR_2s	
	l_∞ error	CPU Time	l_∞ error	CPU Time	l_∞ error	CPU Time
150	$1.326e-02$	1.266	$6.027e-02$	1.078	$1.631e-02$	1.178
300	$4.268e-05$	2.391	$1.133e-03$	2.000	$6.889e-05$	2.196
600	$1.511e-07$	4.470	$4.756e-05$	3.749	$2.714e-07$	4.461
1200	$5.520e-10$	9.171	$1.968e-08$	7.343	$1.067e-09$	8.786
2400	$3.132e-12$	19.125	$7.834e-11$	15.578	$4.204e-12$	17.468
4800	$3.677e-12$	42.936	$2.549e-12$	35.236	$8.294e-14$	40.312

Table 4: Computational results for $\nu = 4$ for Example 5.2.

N	BVM		BUM		TOMs	
	l_∞ error	CPU Time	l_∞ error	CPU Time	l_∞ error	CPU Time
150	$5.048e-04$	2.142	$2.261e-03$	1.671	$2.993e-03$	1.985
300	$2.979e-07$	3.952	$1.343e-05$	3.156	$4.373e-06$	3.766
600	$2.560e-10$	7.704	$1.447e-08$	5.890	$4.752e-09$	3.735
1200	$2.201e-13$	16.577	$1.499e-11$	11.860	$4.791e-12$	15.017
2400	$5.283e-13$	33.172	$6.877e-13$	25.750	$4.292e-14$	28.250
4800	$1.500e-12$	77.532	$1.505e-12$	59.281	$4.447e-15$	76.687

Example 5. Lastly, we consider the following BVP for $x, y \in [-1, 1]$ given in [19]:

$$u_{xx} + u_{yy} = -32\pi^2 \sin(4\pi x),$$
$$u(\pm 1, y) = u(x, \pm 1) = 0, \tag{50}$$
$$\text{Exact: } \sin(4\pi x)\sin(4\pi y).$$

5.1. Numerical Results and Discussion. Example 1 is a variable coefficient fourth-order BVP. The fourth-order BVP is transformed to a system of second-order BVP. We solved the system using the BVM and BUM of orders 6 and 8. The problem is also solved using the ETRs and ETR_2s of orders 6 and 8, respectively. Tables 1 and 2 show the computational results for this example. While the BUM produces solutions of approximate accuracy with the BVM, it uses shorter CPU Time. Example 2 is the well-known nonlinear Fehlberg problem. It was solved for $\nu = 3, 4$ and the maximum of the Euclidean norm of the errors in y_1 and y_2 was obtained in the interval of integration. Example 2 was also solved using the ETR_2s and the TOMs of orders 8 and 10, respectively. Tables 3 and 4 show that both methods produce solutions of approximate accuracy with the BUM using shorter CPU

Table 5: Computational results for $\nu = 2$ for Example 5.3.

N	BVM		BUM		ETRs	
	l_∞ error	CPU Time	l_∞ error	CPU Time	l_∞ error	CPU Time
4	$4.495e-01$	0.312	$2.536e-06$	0.266	$5.631e-06$	0.297
8	$4.144e-09$	0.297	$3.737e-08$	0.328	$6.782e-08$	0.343
16	$4.680e-11$	0.313	$5.753e-10$	0.328	$9.270e-10$	0.344
32	$5.347e-13$	0.375	$8.955e-12$	0.328	$1.518e-11$	0.366
64	$7.105e-15$	0.734	$1.399e-13$	0.500	$2.430e-13$	0.546
128	$4.441e-16$	2.031	$2.442e-15$	1.188	$3.553e-15$	1.156

Table 6: Computational results for $\nu = 4$ for Example 5.3.

N	BVM		BUM		TOMs	
	l_∞ error	CPU Time	l_∞ error	CPU Time	l_∞ error	CPU Time
8	$2.399e-00$	0.297	$4.823e-12$	0.234	$1.031e-11$	0.273
16	$2.220e-14$	0.297	$5.773e-15$	0.276	$7.994e-15$	0.297
32	$5.940e-14$	0.313	$1.332e-14$	0.281	$6.661e-16$	0.343
64	$3.055e-13$	0.453	$2.398e-14$	0.374	$1.332e-15$	0.374
128	$9.859e-14$	0.532	$5.063e-14$	0.423	$1.110e-15$	0.453
256	$4.596e-13$	0.797	$6.672e-14$	0.625	$8.882e-15$	0.671

Table 7: Computational results for $\nu = 2$ for Example 5.4.

N	BVM		BUM		TOMs	
	l_∞ error	CPU Time	l_∞ error	CPU Time	l_∞ error	CPU Time
4	$6.230e-02$	0.282	$6.379e-07$	0.273	$5.431e-06$	0.297
8	$2.632e-09$	0.359	$1.303e-08$	0.297	$8.024e-08$	0.327
16	$6.021e-11$	0.390	$1.865e-10$	0.405	$1.307e-09$	0.468
32	$5.291e-12$	0.703	$2.668e-12$	0.563	$2.078e-11$	0.828
64	$5.991e-13$	1.782	$4.157e-14$	1.281	$3.269e-13$	2.624
128	$5.361e-12$	9.156	$1.633e-15$	5.280	$5.336e-15$	10.532

Table 8: Computational results for $\nu = 3$ for Example 5.4.

N	BVM		BUM		ETR_2s	
	l_∞ error	CPU Time	l_∞ error	CPU Time	l_∞ error	CPU Time
6	$2.582e-00$	0.343	$1.044e-09$	0.298	$3.834e-10$	0.375
12	$8.390e-13$	0.375	$3.563e-12$	0.267	$1.449e-12$	0.453
24	$1.242e-13$	0.375	$1.460e-14$	0.328	$4.158e-15$	0.656
48	$3.999e-11$	0.455	$1.629e-14$	0.421	$4.578e-16$	0.861
96	$1.035e-11$	0.703	$2.517e-14$	0.593	$4.578e-16$	1.040
192	$1.775e-13$	1.077	$6.846e-14$	0.842	$4.965e-16$	1.235

Time. Example 3 was chosen to demonstrate the use of the methods on a nonlinear BVP with mixed boundary conditions. The computational results were given in Tables 5 and 6. Example 4 was chosen to show the performance of the schemes on systems of nonlinear BVPs. The maximum of the Euclidean norm of the errors in y_1 and y_2 is given in Tables 7 and 8. Lastly, we show the performance of the methods on a Poisson equation with boundary conditions. The partial differential equation is transformed into a system of second-order ordinary differential equations with boundary conditions using the method of lines. Table 9 shows the computational result for this example. Also, Figures 1–4 show the efficiency curves of these methods for the different examples where we have denoted BVM and BUM with $\nu = 2$ as BVM2 and BUM2, respectively.

From the foregoing, it can be concluded that the BUM and the BVM produce solutions of approximate accuracy with the BUM using shorter CPU Time. However, a 2ν-step

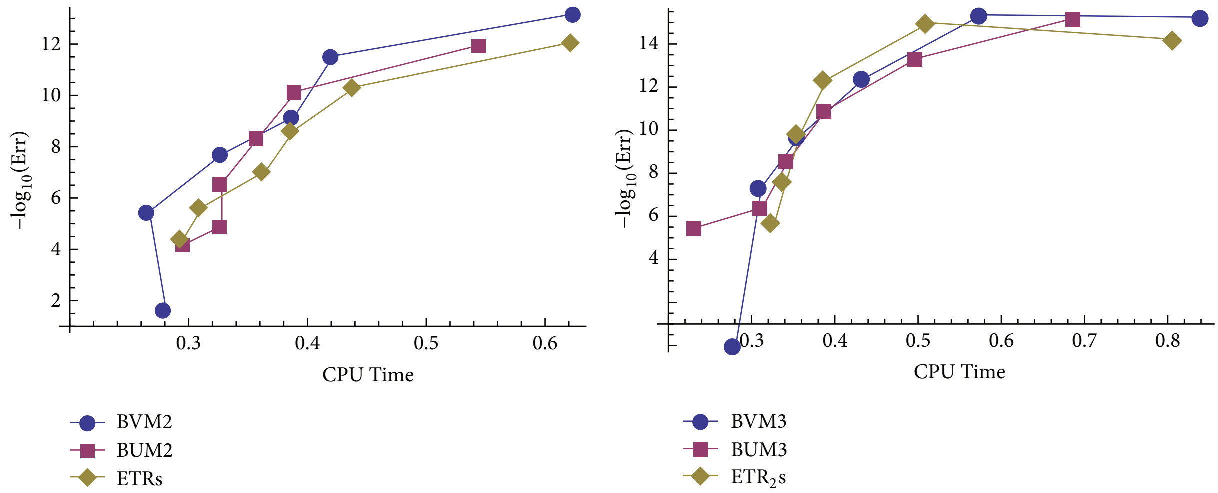

FIGURE 1: Efficiency curves for Example 1.

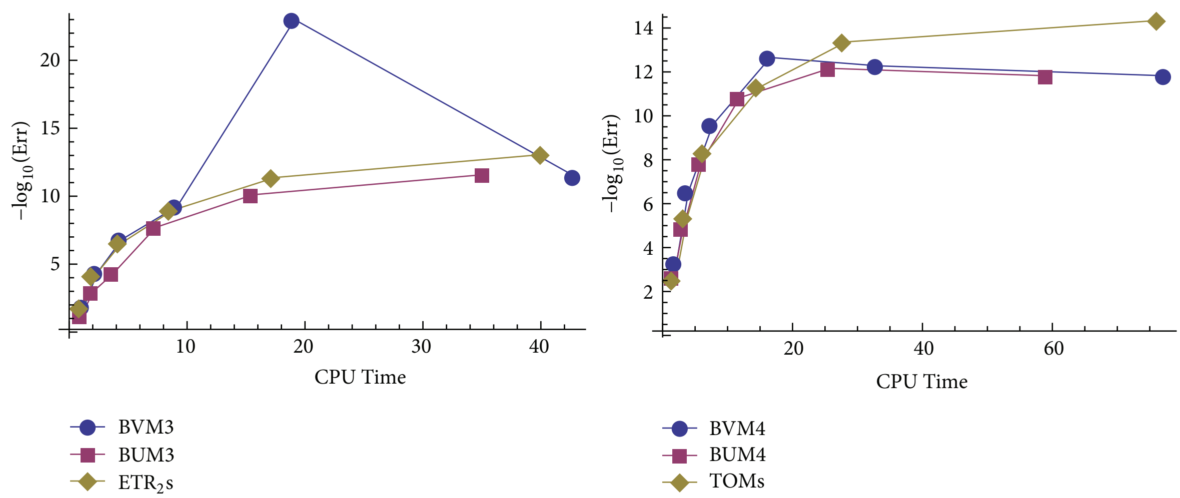

FIGURE 2: Efficiency curves for Example 2.

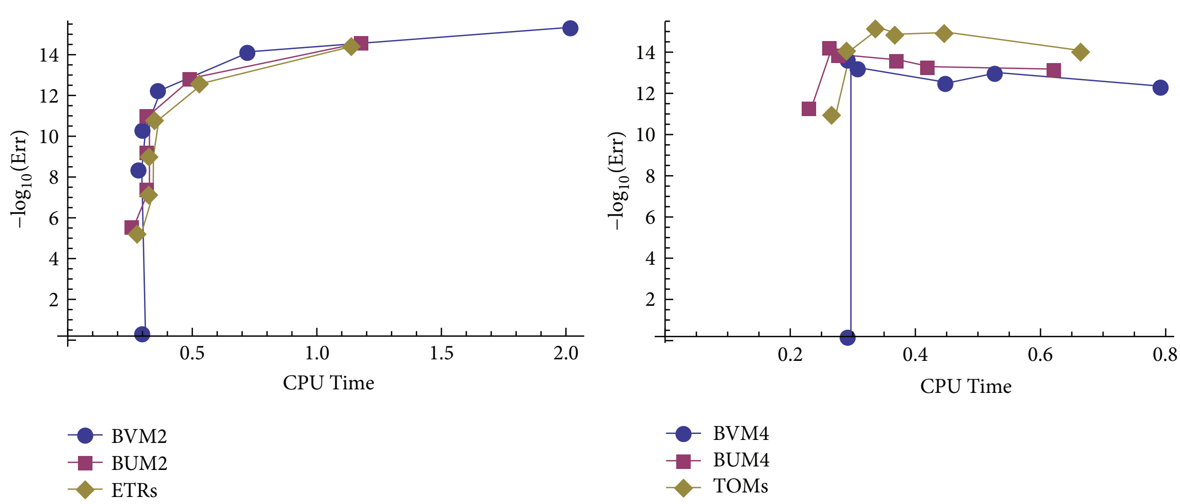

FIGURE 3: Efficiency curves for Example 3.

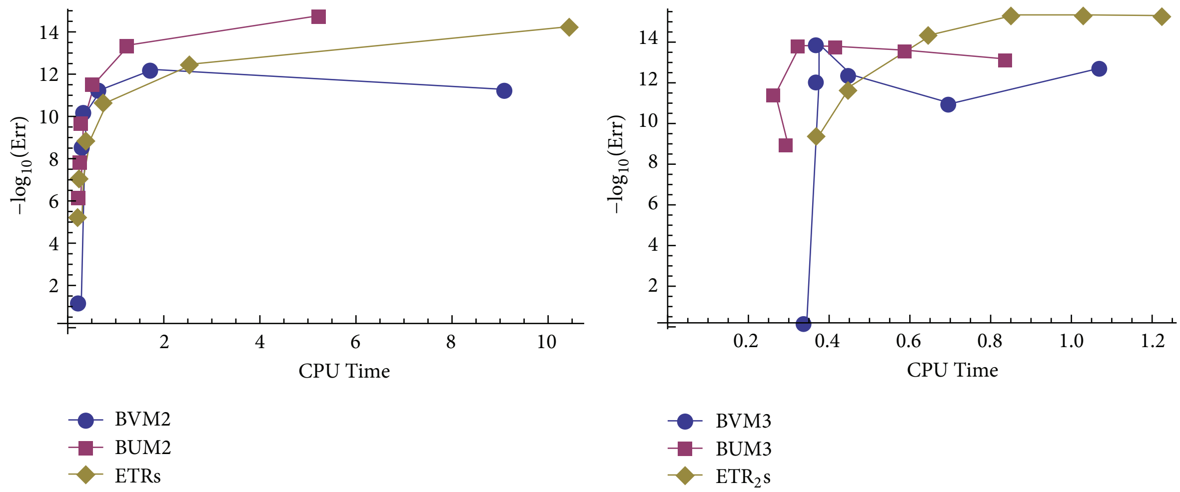

Figure 4: Efficiency curves for Example 4.

Table 9: Computational results for $v = 2$ for Example 5.5.

N	BVM		BUM	
	l_∞ error	CPU Time	l_∞ error	CPU Time
16	$9.662e-00$	0.483	$1.251e-01$	0.531
32	$2.582e-02$	1.235	$2.578e-02$	1.031
64	$6.433e-03$	5.358	$6.459e-03$	5.516
128	$1.607e-03$	43.641	$1.607e-03$	46.923
256	$2.00e-00$	512.843	$4.016e-04$	532.657

BVM performs poorly when the number of steps, N, is $2v$. This is because the main method together with the initial and final methods does not form a good discrete analog or approximation of the continuous boundary value problem. Also, the BUM has the drawback that it is only implemented for any N which is a multiple of $2v$.

6. Conclusion

In this paper, we have developed a new class of LMMs and implemented the LMMs via two approaches, the boundary value approach and the block unification strategy, which are used to solve initial and boundary value problems. The comparison of the two approaches was carried out in terms of accuracy and computational efficiency. The results given in Section 5 show that both approaches perform very well with the BUM using shorter CPU Time. Our future research will be to develop a variable stepsize version of the BVM and the BUM and a study of the conditioning of the matrices arising from the discretization of the continuous second-order problems.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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