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Research Article

Uniqueness of Solutions to a Nonlinear Elliptic Hessian Equation

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Through an Alexandrov-Fenchel inequality, we establish the general Brunn-Minkowski inequality. Then we obtain the uniqueness of solutions to a nonlinear elliptic Hessian equation on $\mathbb{S}^n$.

1. Introduction

According to a general Brunn-Minkowski inequality, we obtain a proof of the uniqueness of solutions to the following fully nonlinear elliptic Hessian equation:

$$\sigma_k\left(u_{ij} + u\delta_{ij}\right) = fu^{p-1} \quad \text{on } \mathbb{S}^n, \tag{1}$$

where u is the support function of convex bodies, u_{ij} are the second-order covariant derivations of u with respect to any orthonormal frame $\{e_1, e_2, \ldots, e_n\}$ on $\mathbb{S}^n$, δ_{ij} is the standard Kronecker symbol, $\mathbb{S}^n$ is the unit sphere of n-dimension, f is a positive function defined on $\mathbb{S}^n$, $k \in \{1, 2, \ldots, n\}$, $p > 1$, and σ_k is the *kth elementary symmetric function* defined as follows: for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n$,

$$\sigma_k(\lambda) = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}. \tag{2}$$

The definition can be extended to any symmetric matrix $W \in \mathbb{R}^{n\times n}$ by $\sigma_k(W) = \sigma_k(\lambda(W))$, where $\lambda(W) = (\lambda_1(W), \lambda_2(W), \ldots, \lambda_n(W))$ is the eigenvalue vector of W.

Equation (1) arrives from the geometry of convex bodies. A compact convex subset of Euclidean $(n+1)$-space $\mathbb{R}^{n+1}$ with nonempty interiors is called a *convex body*. An important concept related to a convex body Q is its support function.

Definition 1. Let M (the boundary of a convex body Q) be a smooth, closed, uniformly convex hypersurface enclosing the origin in $\mathbb{R}^{n+1}$. Assume that M is parameterized by its inverse Gauss map $X : \mathbb{S}^n \to M \subset \mathbb{R}^{n+1}$; the *support function* u of M (or Q) is defined by

$$u(x) = \langle x, X(x)\rangle, \quad \forall x \in \mathbb{S}^n, \tag{3}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard inner product in $\mathbb{R}^{n+1}$.

u is convex after being extended as a function of homogeneous degree 1 in $\mathbb{R}^{n+1}$. Conversely, any continuous convex function u of homogeneous degree 1 determines a convex body as follows:

$$Q = \left\{y \in \mathbb{R}^{n+1} : y \cdot x \leqslant u(x),\ \forall x \in \mathbb{S}^n\right\}. \tag{4}$$

From some basic concepts to support function, Minkowski sum [see Definition 4], and mixed volumes [see Definition 5], Minkowski developed a set of theories related to convex bodies. If $k = n$ and $p = 1$, (1) is the Monge-Ampère equation corresponding to the classical Minkowski problem

$$\det\left(u_{ij} + u\delta_{ij}\right) = f \quad \text{on } \mathbb{S}^n, \tag{5}$$

which has been solved by Nirenberg [1], Pogorelov [2, 3], Cheng and Yau [4], and many others. When $p = 1$, (1) is the classical Christoffel-Minkowski problem:

$$\sigma_k\left(u_{ij} + u\delta_{ij}\right) = f \quad \text{on } \mathbb{S}^n. \tag{6}$$

A necessary condition [3] for (6) to have a solution is

$$\int_{\mathbb{S}^n} x_i f(x)\, ds = 0, \quad \forall i = 1, 2, \ldots, n+1, \tag{7}$$

where ds is the standard area form on $\mathbb{S}^n$. Guan et al. [5] obtained that (7) is sufficient for (6) to have an admissible solution [see Definition 6].

Firey [6] generalized the Minkowski sum to p-sum [see Definition 4] from $p = 1$ to $p \geqslant 1$ in 1962. Later, Lutwak [7] extended the classical surface area measure to the p-sum cases. Also in [7], Lutwak first introduced the general Minkowski problem, which is called L_p-Minkowski problem thereafter. In the smooth category, L_p-Minkowski problem is equivalent to considering the following Monge-Ampère equation:

$$\det\left(u_{ij} + u\delta_{ij}\right) = f u^{p-1} \quad \text{on } \mathbb{S}^n. \tag{8}$$

The uniqueness of L_p-Minkowski problem for $p > 1$ and $p \neq n+1$ (the uniqueness holds up to a dilation if $p = n+1$) has been solved in [7]. However, the uniqueness for $p < 1$ is difficult and still open. In [8], Jian et al. obtained that, for any $-n-1 < p < 0$, there exists a positive function $f \in C^{\infty}(\mathbb{S}^n)$ to guarantee that (8) has two different solutions, which means that we need more conditions to consider the uniqueness.

When considering cases $1 \leqslant k < n$, attention is paid to the generalized Christoffel-Minkowski problem. In the smooth category, we need to study the k-Hessian equation (1).

For (1), Hu et al. [9] got the existence and uniqueness of solutions to (1) when $1 \leqslant k < n$ and $p > k+1$ under appropriate conditions. However, the uniqueness of (1) when $p < 1$ has not been solved well. In this paper, we study the uniqueness of (1) for $p > 1$.

Our main result is the following.

Theorem 2. *Suppose u is a positive admissible solution of*

$$\sigma_k\left(u_{ij} + u\delta_{ij}\right) = f u^{p_0} \quad \text{on } \mathbb{S}^n, \tag{9}$$

where $1 \leqslant k < n$, $k \in \mathbb{Z}$, $p_0 \in \mathbb{R}^+ \setminus \{k\}$, and f is a positive function defined on the unit sphere $\mathbb{S}^n$ and then the uniqueness holds. If $p_0 = k$, the uniqueness holds up to a dilation, which means that if u solves (9), then $\{au : \forall a \in \mathbb{R}^+\}$ are the whole solutions of (9).

Remark 3. Here, we rewrite (1) by (9), where $p_0 = p - 1$.

The organization of this paper is as follows. In Section 2, we show some basic concepts and lemmas which have been obtained by Guan et al. in [10]. In Section 3, we prove two useful propositions according to the methods in [11]. In the last section, we prove the main theorem.

2. Preliminaries

Definition 4. Given two convex bodies Q_1 and Q_2 in $\mathbb{R}^{n+1}$ with respective support functions u_1, u_2, and $\lambda, \mu \geqslant 0$ $(\lambda + \mu > 0)$, the *Minkowski sum* $\lambda Q_1 + \mu Q_2 \subset \mathbb{R}^{n+1}$ is defined by the convex body whose support function is $\lambda u_1 + \mu u_2$.

For $p \geqslant 1$, let Q_1 and Q_2 be two convex bodies containing the origin in $\mathbb{R}^{n+1}$ in their interiors, and $\lambda, \mu \geqslant 0$ $(\lambda + \mu > 0)$. The convex body $\lambda \circ Q_1 +_p \mu \circ Q_2$, whose support function is given by $(\lambda u_1^p + \mu u_2^p)^{1/p}$, is called *Firey's p-sum* of Q_1 and Q_2, where "$+_p$" means the p-summation and "$\circ$" means Firey's multiplication.

Definition 5. Let $Q_1, Q_2, \ldots, Q_r$ be convex bodies in $\mathbb{R}^{n+1}$ and the volume of their Minkowski sum

$$Q = \lambda_1 Q_1 + \lambda_2 Q_2 + \cdots + \lambda_r Q_r, \quad \lambda_i \geqslant 0, \tag{10}$$

is an $(n+1)$th degree homogeneous polynomial of the family $\lambda_1, \lambda_2, \ldots, \lambda_r$. Specially, the volume of Q is

$$\begin{aligned} \operatorname{Vol}(Q) &= \operatorname{Vol}\left(\lambda_1 Q_1 + \lambda_2 Q_2 + \cdots + \lambda_r Q_r\right) \\ &= \sum_{i_1, i_2, \ldots, i_{n+1}=1}^{r} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_{n+1}} V\left(Q_{i_1}, Q_{i_2}, \ldots, Q_{i_{n+1}}\right), \end{aligned} \tag{11}$$

where the functions V are symmetric. Then $V(Q_1, Q_2, \ldots, Q_{n+1})$ is called the *Minkowski mixed volume* of $Q_1, Q_2, \ldots, Q_{n+1}$.

Definition 6. For $k \in \{1, 2, \ldots, n\}$, let Γ_k be the *convex cone* in $\mathbb{R}^n$ which is determined by

$$\Gamma_k = \left\{\lambda \in \mathbb{R}^n : \sigma_1(\lambda) > 0, \sigma_2(\lambda) > 0, \ldots, \sigma_k(\lambda) > 0\right\}. \tag{12}$$

A function $u \in C^2(\mathbb{S}^n)$ is called *k-convex* if

$$W(x) = \left\{u_{ij}(x) + u(x)\delta_{ij}\right\} \in \Gamma_k, \quad \forall x \in \mathbb{S}^n, \tag{13}$$

and u is called an *admissible solution* to (1) if u is k-convex and satisfies (1).

Definition 7. Let $A_1, A_2, \ldots, A_m$ be symmetric real $k \times k$ matrices, $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{R}$; the determinant of $\lambda_1 A_1 + \cdots + \lambda_m A_m$ is a homogeneous polynomial of degree k in $\lambda_1, \lambda_2, \ldots, \lambda_m$. Namely,

$$\begin{aligned} &\det\left(\lambda_1 A_1 + \cdots + \lambda_m A_m\right) \\ &\quad = \sum_{i_1, \ldots, i_k = 1}^{m} \lambda_{i_1}\cdots\lambda_{i_k} D_k\left(A_{i_1}, \ldots, A_{i_k}\right). \end{aligned} \tag{14}$$

In fact, the coefficient $\lambda_{i_1}\cdots\lambda_{i_k}$ depends only on $A_{i_1}, \ldots, A_{i_k}$; then they are uniquely determined. $D_k(A_1, \ldots, A_k)$ is called the *mixed discriminant* of $A_1, \ldots, A_k$.

For later applications, we collect some results here which have been proved in [10].

Lemma 8. *Let $u_1, u_2, \ldots, u_{n+1}$ be the support function of convex bodies $Q_1, Q_2, \ldots, Q_{n+1}$, respectively. Denoting Minkowski mixed volume $V(Q_1, Q_2, \ldots, Q_{n+1})$ by $V(u_1, u_2, \ldots, u_{n+1})$ and*

$$W_m = \left\{(u_m)_{ij} + u_m\delta_{ij}\right\}, \quad m = 1, 2, \ldots, n+1, \tag{15}$$

then

$$V(u_1, u_2, \ldots, u_{n+1}) = \int_{\mathbb{S}^n} u_1 D_n(W_2, W_3, \ldots, W_{n+1})\, ds, \tag{16}$$

where $D_n(W_2, W_3, \ldots, W_{n+1})$ is the mixed discriminant [see Definition 7] of $W_2, W_3, \ldots, W_{n+1}$.

Remark 9. For all $1 \leqslant k \leqslant n$, setting $u_{k+2} = \cdots = u_{n+1} = 1$, then

$$V(u_1, \ldots, u_{k+1}, 1, \ldots, 1) \coloneqq V_{k+1}(u_1, u_2, \ldots, u_{k+1}) = \int_{\mathbb{S}^n} u_1 D_k(W_2, W_3, \ldots, W_{k+1})\, ds, \tag{17}$$

where $D_k(W_2, W_3, \ldots, W_{k+1})$ is the mixed discriminant of $W_2, W_3, \ldots, W_{k+1}$. Furthermore, if $u_1 = u_2 = \cdots = u_{n+1} = u$, denote $V(u_1, u_2, \ldots, u_{n+1}) \coloneqq V(u)$ and $V_{k+1}(u_1, u_2, \ldots, u_{k+1}) \coloneqq V_{k+1}(u)$; then

$$V(u) = \int_{\mathbb{S}^n} u \det(u_{ij} + u\delta_{ij})\, ds,$$
$$V_{k+1}(u) = \int_{\mathbb{S}^n} u\sigma_k(u_{ij} + u\delta_{ij})\, ds. \tag{18}$$

Lemma 10. *V is a symmetric multilinear form on $(C^2(\mathbb{S}^n))^{n+1}$.*

Lemma 11. *For any function $u \in C^2(\mathbb{S}^n)$, $W = \{u_{ij} + u\delta_{ij}\}$, $1 \leqslant k < n$, we have the Minkowski type integral formula,*

$$\int_{\mathbb{S}^n} u\sigma_k(W)\, ds = \int_{\mathbb{S}^n} \sigma_{k+1}(W)\, ds, \tag{19}$$

where ds is the standard area element on $\mathbb{S}^n$.

The following is a form of Alexandrov-Fenchel inequality for positive k-convex functions which comes from [10].

Lemma 12 (Alexandrov-Fenchel inequality). *If $u_1, u_2, \ldots, u_k$ are k-convex, u_1 is positive, and there exists $l \in \{2, 3, \ldots, k\}$ such that $u_l \geqslant 0$ on $\mathbb{S}^n$, then, for any $v \in C^2(\mathbb{S}^n)$,*

$$V_{k+1}^2(v, u_1, u_2, \ldots, u_k) \geqslant V_{k+1}(u_1, u_1, u_2, \ldots, u_k)\, V_{k+1}(v, v, u_2, \ldots, u_k), \tag{20}$$

with equality if and only if $v = au_1 + \sum_{i=1}^{n+1} a_i x_i$ for some constants $a, a_1, \ldots, a_{n+1}$.

3. Two Important Propositions

Now we prove two important propositions. The methods we use are from [11].

Proposition 13. *Suppose $u_0, u_1 > 0$ are k-convex; then*

$$V_{k+1}^{1/(k+1)}((1-t)u_0 + tu_1) \geqslant (1-t)V_{k+1}^{1/(k+1)}(u_0) + tV_{k+1}^{1/(k+1)}(u_1), \quad \forall t \in [0,1], \tag{21}$$

with equality if and only if $u_0 = au_1 + \sum_{i=1}^{n+1} a_i x_i$ for some constants $a, a_1, \ldots, a_{n+1}$.

Proof. We only need to prove that

$$F(t) = V_{k+1}^{1/(k+1)}((1-t)u_0 + tu_1) \tag{22}$$

is concave on $[0,1]$. Setting $u_t = (1-t)u_0 + tu_1$, $t \in [0,1]$, we have

$$F(t) = V_{k+1}^{1/(k+1)}\Big(\overbrace{u_t, u_t, \ldots, u_t}^{k+1}\Big). \tag{23}$$

By the symmetric multilinear property of V, it is obvious that

$$F'(t) = V_{k+1}^{1/(k+1)-1}\Big(\overbrace{u_t, \ldots, u_t}^{k+1}\Big) V_{k+1}\Big(-u_0 + u_1, \overbrace{u_t, \ldots, u_t}^{k}\Big), \tag{24}$$

$$\begin{aligned} F''(t) = {} & kV_{k+1}^{1/(k+1)-2}\Big(\overbrace{u_t, \ldots, u_t}^{k+1}\Big) \\ & \cdot \Big[V_{k+1}\Big(\overbrace{u_t, \ldots, u_t}^{k+1}\Big) \\ & \cdot V_{k+1}\Big(-u_0 + u_1, -u_0 + u_1, \overbrace{u_t, \ldots, u_t}^{k-1}\Big) \\ & - V_{k+1}^2\Big(-u_0 + u_1, \overbrace{u_t, \ldots, u_t}^{k}\Big)\Big] \leqslant 0, \end{aligned} \tag{25}$$

where the last inequality uses (20); thus F is a concave function on $[0,1]$. The equality condition is checked easily. ☐

Proposition 14 (general Brunn-Minkowski inequality). *Supposing $u_0, u_1 > 0$ are k-convex, then*

$$\int_{\mathbb{S}^n} u_1 \sigma_k((u_0)_{ij} + u_0\delta_{ij})\, ds \geqslant V_{k+1}^{1/(k+1)}(u_1)\, V_{k+1}^{1-1/(k+1)}(u_0), \tag{26}$$

with equality if and only if $u_0 = au_1 + \sum_{i=1}^{n+1} a_i x_i$ for some constants $a, a_1, \ldots, a_{n+1}$.

Proof. Setting

$$F(t) = V_{k+1}^{1/(k+1)}((1-t)u_0 + tu_1) - (1-t)V_{k+1}^{1/(k+1)}(u_0) - tV_{k+1}^{1/(k+1)}(u_1), \tag{27}$$

then $F(0) = F(1) = 0$. By (21), $F(t) \geqslant 0$; thus $F'(0) \geqslant 0$; namely,

$$V_{k+1}^{1/(k+1)-1}(u_0)\, V_{k+1}\Big(-u_0 + u_1, \overbrace{u_0, \ldots, u_0}^{k}\Big) + V_{k+1}^{1/(k+1)}(u_0) - V_{k+1}^{1/(k+1)}(u_1) \geqslant 0. \tag{28}$$

Then

$$V_{k+1}^{1/(k+1)-1}(u_0)\int_{\mathbb{S}^n}(-u_0+u_1)\,\sigma_k\left((u_0)_{ij}+u_0\delta_{ij}\right)ds + V_{k+1}^{1/(k+1)}(u_0) \geqslant V_{k+1}^{1/(k+1)}(u_1). \quad (29)$$

By (19),

$$V_{k+1}^{1/(k+1)-1}(u_0)\int_{\mathbb{S}^n}u_1\sigma_k\left((u_0)_{ij}+u_0\delta_{ij}\right)ds \geqslant V_{k+1}^{1/(k+1)}(u_1), \quad (30)$$

and then

$$\int_{\mathbb{S}^n}u_1\sigma_k\left((u_0)_{ij}+u_0\delta_{ij}\right)ds \geqslant V_{k+1}^{1/(k+1)}(u_1)\,V_{k+1}^{1-1/(k+1)}(u_0). \quad (31)$$

□

4. Proof of Theorem 2

Now we prove Theorem 2. The main methods are from [7, 12].

Proof. Assuming that (9) has two solutions u and v, then we consider the equation in the following three cases.

Case 1 ($p_0 > k$). Supposing x_0 is the maximum value point of $G = u/v$, then at x_0, we have

$$\begin{aligned} 0 &= \nabla \ln G = \frac{\nabla u}{u} - \frac{\nabla v}{v}, \\ 0 &\geqslant \nabla^2 \ln G = \left(\frac{\nabla^2 u}{u} - \frac{(\nabla u)^2}{u^2}\right) - \left(\frac{\nabla^2 v}{v} - \frac{(\nabla v)^2}{v^2}\right) \\ &= \frac{\nabla^2 u}{u} - \frac{\nabla^2 v}{v}; \end{aligned} \quad (32)$$

that is,

$$\frac{\nabla^2 u}{u} \leqslant \frac{\nabla^2 v}{v}. \quad (33)$$

Hence

$$\begin{aligned} fu^{p_0}(x_0) &= u^k(x_0)\,\sigma_k\left(\frac{u_{ij}}{u}+\delta_{ij}\right)(x_0) \\ &\leqslant u^k(x_0)\,\sigma_k\left(\frac{v_{ij}}{v}+\delta_{ij}\right)(x_0) \\ &= \frac{u^k(x_0)}{v^k(x_0)} f v^{p_0}(x_0); \end{aligned} \quad (34)$$

therefore

$$u^{p_0-k}(x_0) \leqslant v^{p_0-k}(x_0) \Longrightarrow G(x_0) = \frac{u(x_0)}{v(x_0)} \leqslant 1; \quad (35)$$

then

$$\frac{u}{v} \leqslant 1. \quad (36)$$

Similarly, we have $v/u \leqslant 1$. Thus $u \equiv v$.

Case 2 ($0 < p_0 < k$). We have

$$u^{-p_0}\sigma_k\left(u_{ij}+u\delta_{ij}\right) = v^{-p_0}\sigma_k\left(v_{ij}+v\delta_{ij}\right); \quad (37)$$

then

$$\begin{aligned} V_{k+1}(u) &= \int_{\mathbb{S}^n} u\sigma_k\left(u_{ij}+u\delta_{ij}\right)ds \\ &= \int_{\mathbb{S}^n}\left(\frac{u}{v}\right)^{p_0+1} v\sigma_k\left(v_{ij}+v\delta_{ij}\right)ds \\ &\geqslant \left[\int_{\mathbb{S}^n} u\sigma_k\left(v_{ij}+v\delta_{ij}\right)ds\right]^{p_0+1} \\ &\quad\cdot\left[\int_{\mathbb{S}^n} v\sigma_k\left(v_{ij}+v\delta_{ij}\right)ds\right]^{-p_0} \geqslant V_{k+1}^{(p_0+1)/(k+1)}(u) \\ &\quad\cdot V_{k+1}^{(kp_0+k)/(k+1)}(v)\,V_{k+1}^{-p_0}(v) = V_{k+1}^{(p_0+1)/(k+1)}(u) \\ &\quad\cdot V_{k+1}^{1-(p_0+1)/(k+1)}(v), \end{aligned} \quad (38)$$

where we have used Hölder inequality in the first inequality and used (26) in the second one. Hence $V_{k+1}(u) = V_{k+1}(v)$, which forces both the equalities to hold. By the equality condition, there exists a constant $a \in \mathbb{R}$ such that $v = au$. By (9), we know $a = 1$. Therefore, $u \equiv v$.

Case 3 ($p_0 = k$). According to Case 2, when $p_0 = k$, we have

$$\begin{aligned} V_{k+1}(u) &= \int_{\mathbb{S}^n} u\sigma_k\left(u_{ij}+u\delta_{ij}\right)ds = \int_{\mathbb{S}^n}\left(\frac{u}{v}\right)^{k+1} \\ &\quad\cdot v\sigma_k\left(v_{ij}+v\delta_{ij}\right)ds \geqslant \left[\int_{\mathbb{S}^n} u\sigma_k\left(v_{ij}+v\delta_{ij}\right)ds\right]^{k+1} \\ &\quad\cdot\left[\int_{\mathbb{S}^n} v\sigma_k\left(v_{ij}+v\delta_{ij}\right)ds\right]^{-k} \geqslant V_{k+1}(u)\,V_{k+1}^{k}(v) \\ &\quad\cdot V_{k+1}^{-k}(v) = V_{k+1}(u); \end{aligned} \quad (39)$$

then all the equalities hold. Thus there exists $a \in \mathbb{R}$, such that $v = au$. Therefore $\{au : \forall a \in \mathbb{R}^+\}$ are the whole solutions of (9).

Now we complete the proof of Theorem 2. □

Competing Interests

The author declares no competing interests.

References

[1] L. Nirenberg, "The Weyl and Minkowski problems in differential geometry in the large," *Communications on Pure and Applied Mathematics*, vol. 6, no. 3, pp. 337–394, 1953.

[2] A. V. Pogorelov, "On existence of a convex surface with a given sum principal radii of curvature," *Uspekhi Matematicheskikh Nauk*, vol. 8, no. 3, pp. 127–130, 1953 (Russian).

[3] A. V. Pogorelov, *The Multidimensional Minkowski Problem*, Winston, Washington, DC, USA, 1978.

[4] S. Y. Cheng and S. T. Yau, "On the regularity of the solution of the n-dimensional Minkowski problem," *Communications on Pure and Applied Mathematics*, vol. 29, no. 5, pp. 495–516, 1976.

[5] P. Guan, X.-N. Ma, and F. Zhou, "The Christofel-Minkowski problem. III. Existence and convexity of admissible solutions," *Communications on Pure and Applied Mathematics*, vol. 59, no. 9, pp. 1352–1376, 2006.

[6] W. J. Firey, "p-means of convex bodies," *Mathematica Scandinavica*, vol. 10, pp. 17–24, 1962.

[7] E. Lutwak, "The Brunn-Minkowski-Firey theory I: mixed volumes and the Minkowski problem," *Journal of Differential Geometry*, vol. 38, no. 1, pp. 131–150, 1993.

[8] H. Jian, J. Lu, and X.-J. Wang, "Nonuniqueness of solutions to the L_p-Minkowski problem," *Advances in Mathematics*, vol. 281, pp. 845–856, 2015.

[9] C. Q. Hu, X.-N. Ma, and C. Shen, "On the Christoffel-Minkowski problem of Firey's p-sum," *Calculus of Variations and Partial Differential Equations*, vol. 21, no. 2, pp. 137–155, 2004.

[10] P. F. Guan, X.-N. Ma, N. Trudinger, and X. Zhu, "A form of Alexandrov-Fenchel inequality," *Pure and Applied Mathematics Quarterly*, vol. 6, no. 4, pp. 999–1012, 2010.

[11] R. Schneider, *Convex Bodies: the Brunn-Minkowski Theory*, Cambridge University Press, Cambridge, UK, 2013.

[12] K.-S. Chou and X.-J. Wang, "The L_p-Minkowski problem and the Minkowski problem in centroaffine geometry," *Advances in Mathematics*, vol. 205, no. 1, pp. 33–83, 2006.