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Research Article

Density by Moduli and Statistical Boundedness

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We have generalized the notion of statistical boundedness by introducing the concept of f-statistical boundedness for scalar sequences where f is an unbounded modulus. It is shown that bounded sequences are precisely those sequences which are f-statistically bounded for every unbounded modulus f. A decomposition theorem for f-statistical convergence for vector valued sequences and a structure theorem for f-statistical boundedness have also been established.

1. Introduction and Background

The idea of statistical convergence was given in the first edition (published in Warsaw in 1935) of the monograph of Zygmund [1]. Formally the concept of statistical convergence was introduced by Steinhaus [2] and Fast [3] and later reintroduced by Schoenberg [4]. Although statistical convergence was introduced over nearly last 60 years, it has become an active area of research in recent years. Statistical convergence has been studied most recently by several authors [5–15].

The standard definition of "(x_k) is convergent to L" requires that the set $\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ should be finite for every $\varepsilon > 0$, where $\mathbb{N}$ is the set of natural numbers.

The number sequence (x_k) is said to be statistically convergent to the number L provided that the set $\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$, instead of being finite, has natural density 0, where the natural density of a subset $K \subset \mathbb{N}$ (see [16], chapter 11) is defined by

$$d(K) = \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in K\}|, \qquad (1)$$

where $|\{k \leq n : k \in K\}|$ denotes the number of elements of K not exceeding n. A set K is said to be statistically dense [17] if $d(K) = 1$. A subsequence of a sequence is said to be statistically dense if the set of all indices of its elements is statistically dense. In other words, a subsequence of a sequence is said to be statistically dense if the complement of the set of all indices of its elements has natural density 0. Obviously, we have $d(K) = 0$ provided that K is a finite set of positive integers.

We will be particularly concerned with those subsets of $\mathbb{N}$ which have natural density zero. To facilitate this, Fridy [18] introduced the following notation: if $x = (x_k)$ is a sequence such that x_k satisfies property P for all k except a set of natural density zero, then we say that $x = (x_k)$ satisfies P for "almost all k" and we abbreviate this by "a.a. k."

Using this notation, we have the following.

Definition 1. The number sequence $x = (x_k)$ is said to be statistically convergent to the number L if, for each $\varepsilon > 0$,

$$\begin{gathered} d(A_\varepsilon) = 0, \\ \text{where } A_\varepsilon = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}, \text{ that is,} \\ \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0, \text{ that is,} \\ |x_k - L| < \varepsilon \quad \text{a.a. } k. \end{gathered} \qquad (2)$$

Definition 2. A number sequence $x = (x_k)$ is said to be statistically Cauchy if, for each $\varepsilon > 0$, there exists a positive integer $p = p(\varepsilon)$ such that

$$d\left\{k \in \mathbb{N} : \left|x_k - x_p\right| \geq \varepsilon\right\} = 0, \text{ that is,} \quad \left|x_k - x_p\right| < \varepsilon \quad \text{a.a. } k. \tag{3}$$

In 1997, Fridy and Orhan [19] introduced the concept of statistical boundedness as follows.

Definition 3. The number sequence $x = (x_k)$ is said to be statistically bounded if there is a number $M > 0$ such that $d(\{k \in \mathbb{N} : |x_k| > M\}) = 0$; that is,

$$\left|x_k\right| \leq M \quad \text{a.a. } k. \tag{4}$$

We denote the set of all statistically bounded sequences by $S(b)$.

In the same year, that is, 1997, Tripathy [20] proved a decomposition theorem for statistically bounded sequences and also established a necessary and sufficient condition for a sequence to be statistically bounded.

Quite recently, Bhardwaj and Gupta [10] have introduced and studied the concepts of statistical boundedness of order α, λ-statistical boundedness, and λ-statistical boundedness of order α for scalar sequences. Bhardwaj et al. [11] have also introduced and studied a new concept of lacunary statistical boundedness as a lacunary analog of the concept of statistical boundedness.

The idea of a modulus function was structured by Nakano [21] in 1953. Following Ruckle [22] and Maddox [23], we recall that a modulus f is a function from $[0,\infty)$ to $[0,\infty)$ such that (i) $f(x) = 0$ if and only if $x = 0$, (ii) $f(x+y) \leq f(x) + f(y)$ for $x \geq 0$, $y \geq 0$, (iii) f is increasing, and (iv) f is continuous from the right at 0. Hence f must be continuous everywhere on $[0,\infty)$. A modulus may be unbounded or bounded. For example, $f(x) = x^p$, where $0 < p < 1$, is unbounded, but $f(x) = x/(1+x)$ is bounded.

Connor [24], Pehlivan [25], Pehlivan and Fisher [26], Kolk [27], Ghosh and Srivastava [28], Bhardwaj and Singh [29, 30], Çolak [31], Altin and Et [32], and some others have used a modulus function to construct some sequence spaces.

In the year 2014, Aizpuru et al. [6] have defined a new concept of density with the help of modulus function and consequently obtained a new concept of f-statistical convergence, which is in fact a generalization of the notion of statistical convergence. They proved that the ordinary convergence is equivalent to the f-statistical convergence for every unbounded modulus function f.

Quite recently, Bhardwaj and Dhawan have introduced and studied the concepts of f-statistical convergence of order α [9] and f-lacunary statistical convergence [33] by using the approach of Aizpuru et al. [6].

Throughout the paper, unless otherwise specified, X will denote a real normed space.

First we recall some definitions from [6].

Definition 4. Let f be an unbounded modulus function. The f-density of a set $A \subset \mathbb{N}$ is defined by

$$d_f(A) = \lim_n \frac{f(|\{k \leq n : k \in A\}|)}{f(n)}, \tag{5}$$

in case this limit exists. Clearly, finite sets have zero f-density and $d_f(\mathbb{N} - A) = 1 - d_f(A)$ does not hold, in general.

Remark 5. For any unbounded modulus f and $A \subset \mathbb{N}$, $d_f(A) = 0$ implies $d(A) = 0$ (see [6]). But the converse need not be true in the sense that a set having natural density zero may not have f-density zero with respect to some unbounded modulus f.

Definition 6. A sequence (x_k) in X is said to be f-statistically convergent to $L \in X$, if, for each $\varepsilon > 0$, $d_f(\{k \in \mathbb{N} : \|x_k - L\| \geq \varepsilon\}) = 0$ and one writes it as $f\text{-st}\lim x_k = L$.

In view of Definition 6 and Remark 5, it follows that every f-statistically convergent sequence is statistically convergent but a statistically convergent sequence need not be f-statistically convergent for every unbounded modulus f.

Definition 7. A sequence (x_k) in X is said to be f-statistically Cauchy if, for each $\varepsilon > 0$, there exists a positive integer $p = p(\varepsilon)$ such that

$$d_f\left(\left\{k \in \mathbb{N} : \left\|x_k - x_p\right\| \geq \varepsilon\right\}\right) = 0. \tag{6}$$

We now introduce the following notation: if $x = (x_k)$ is a sequence such that x_k satisfies property P for all k except a set of f-density zero, then we say that $x = (x_k)$ satisfies P for "almost all k with respect to f," where f is any unbounded modulus and we abbreviate this by "a.a. k w.r.t. f."

Using this notation, the definitions of f-statistical convergence and f-statistically Cauchy can be reformulated as follows.

Definition 8. An X-valued sequence (x_k) is said to be f-statistically convergent to $L \in X$, if, for each $\varepsilon > 0$,

$$\left\|x_k - L\right\| < \varepsilon \quad \text{a.a. } k \text{ w.r.t. } f. \tag{7}$$

Definition 9. An X-valued sequence (x_k) is said to be f-statistically Cauchy if, for each $\varepsilon > 0$, there exists a positive integer $p = p(\varepsilon)$ such that

$$\left\|x_k - x_p\right\| < \varepsilon \quad \text{a.a. } k \text{ w.r.t. } f. \tag{8}$$

The main object of this paper is to introduce and study a new concept of f-statistical boundedness for scalar sequences defined as follows.

Definition 10. A number sequence (x_k) is said to be f-statistically bounded if there exists $M > 0$ such that $d_f(\{k \in \mathbb{N} : |x_k| > M\}) = 0$; that is, $|x_k| \leq M$ a.a. k w.r.t. f. By $s_f(b)$, one will denote the space of all f-statistically bounded scalar sequences.

In this paper, we establish a relation between statistical boundedness and f-statistical boundedness. It is shown that the concept of f-statistical boundedness is intermediate between the ordinary boundedness and the statistical boundedness. We also prove that bounded sequences are precisely those sequences which are f-statistically bounded for every unbounded modulus f. Apart from studying f-statistical boundedness we also propose to derive further properties concerning f-statistical convergence. We establish a decomposition theorem for f-statistical convergence. It is proved that the terms of an f-statistically convergent sequence (x_k) are coincident with those of a convergent sequence for almost all k with respect to f. We also show that ℓ_∞^f, the set of all bounded f-statistically convergent sequences of scalars, is a closed linear subspace of the normed linear space ℓ_∞ of all bounded sequences of scalars and hence is a nowhere dense set in ℓ_∞.

2. Some More Results on f-Statistical Convergence

Throughout this section, unless otherwise specified, we deal with X-valued sequences. By $s_f(X)$, we will denote the space of all X-valued f-statistically convergent sequences.

We begin this section by establishing a decomposition theorem for f-statistical convergence. In fact, the decomposition theorem for statistically convergent sequences of scalars was given by Connor [34]. The following theorem extends the decomposition theorem of Connor [34] to f-statistical convergence.

Theorem 11 (decomposition theorem). *If $x = (x_k)$ is f-statistically convergent to L, then there is a sequence y converging to L and an f-statistically null sequence z such that $x = y + z$. Moreover, if x is bounded then z is also bounded and $\|z\|_\infty \le \|x\|_\infty + \|L\|$.*

Proof. As $x = (x_k)$ is f-statistically convergent to L, there exists $A \subset \mathbb{N}$ with $d_f(A) = 0$ such that $\lim_{k\in\mathbb{N}-A} x_k = L$. For $k \in \mathbb{N}$, let

$$y_k = \begin{cases} x_k, & \text{if } k \in \mathbb{N} - A; \\ L, & \text{if } k \in A, \end{cases} \qquad z_k = \begin{cases} 0, & \text{if } k \in \mathbb{N} - A; \\ x_k - L, & \text{if } k \in A. \end{cases} \tag{9}$$

Clearly $x = y + z$. As $\{k \in \mathbb{N} : \|z_k - 0\| > \epsilon\} \subset A$ for every $\epsilon > 0$, we have $d_f(\{k \in \mathbb{N} : \|z_k - 0\| > \epsilon\}) = 0$ for every $\epsilon > 0$. Thus $z = (z_k)$ is an f-statistically null sequence and $\|z\|_\infty \le \|x\|_\infty + \|L\|$, if x is bounded.

For $k \in \mathbb{N}$, we have

$$\|y_k - L\| = \begin{cases} \|x_k - L\|, & \text{if } k \in \mathbb{N} - A; \\ 0, & \text{if } k \in A; \end{cases} \tag{10}$$

and so

$$\{k \in \mathbb{N} : \|y_k - L\| > \epsilon\} \subset \{k \in \mathbb{N} : \|x_k - L\| > \epsilon\} \cap (\mathbb{N} - A). \tag{11}$$

As $\lim_{k\in\mathbb{N}-A} x_k = L$, the set on right hand side of (11) is finite for each $\varepsilon > 0$ and hence $\lim_k y_k = L$. □

Remark 12. If we denote the space of all X-valued convergent and f-statistically null sequences by $c(X)$ and $s_f^0(X)$, respectively, then in view of Theorem 11 we have $s_f(X) = c(X) + s_f^0(X)$. Moreover $s_f(X) \ne c(X) \oplus s_f^0(X)$ as $c(X) \cap s_f^0(X) \supset c_0(X)$, the space of X-valued null sequences.

Fridy [18] proved that, in case of scalar sequences, every statistically convergent sequence has entries coincident with those of a convergent sequence for almost all k. We establish a similar result for f-statistical convergence of X-valued sequences, which includes the above stated result of Fridy [18].

Theorem 13. *A sequence $x = (x_k)$ is f-statistically convergent if and only if there exists a convergent sequence $y = (y_k)$ such that $x_k = y_k$ a.a. k w.r.t. f.*

Proof. First suppose $x = (x_k)$ is an f-statistically convergent sequence. Proceeding on the same lines as in Theorem 11, we get a convergent sequence $y = (y_k)$ with $d_f(\{k \in \mathbb{N} : x_k \ne y_k\}) \le d_f(A) = 0$; that is, $x_k = y_k$ a.a. k w.r.t. f. Conversely, it is given that there exists a convergent sequence $y = (y_k)$ such that $x_k = y_k$ a.a. k w.r.t. f. Now

$$\{k \in \mathbb{N} : \|x_k - L\| > \epsilon\} \subset \{k \in \mathbb{N} : x_k \ne y_k\} \cup \{k \in \mathbb{N} : \|y_k - L\| > \epsilon\}. \tag{12}$$

Since $\lim_k y_k = L$, the latter set on the right hand side of (12) is finite. Therefore $d_f(\{k \in \mathbb{N} : \|x_k - L\| > \epsilon\}) = 0$ for every $\epsilon > 0$. Thus $x = (x_k)$ is an f-statistically convergent sequence. □

As an immediate consequence of Theorem 13, we have the following.

Corollary 14. *If $x = (x_k)$ is a sequence such that f-st $\lim x_k = L$, then x has a subsequence $y = (y_k)$ such that $\lim_k y_k = L$.*

Aizpuru et al. [6] proved that, in a Banach space X, (x_n) is f-statistically Cauchy if and only if (x_n) is f-statistically convergent. Combining this result with Theorem 13, we get the following result which includes Theorem 7 of Fridy [18].

Theorem 15. *Let X be a Banach space and f an unbounded modulus. Then the following are equivalent:*

(a) *(x_k) is f-statistically convergent.*

(b) *(x_k) is f-statistically Cauchy.*

(c) *There exists a convergent sequence $y = (y_k)$ such that $x_k = y_k$ a.a. k w.r.t. f.*

Remark 16. We know that every subsequence of a convergent sequence is convergent but this is no longer true in case of f-statistical convergence; that is, an f-statistically convergent sequence may have a subsequence which is not f-statistically convergent. This can be verified by the following example.

Example 17. Consider $X = \mathbb{C}$, the space of complex numbers, and $f(x) = x^p$ with $0 < p \leq 1$. Let $(x_n) = (1, 0, 0, 4, 0, 0, 0, 0, 9, \ldots)$. Now $d_f(\{i \in \mathbb{N} : |x_i - 0| > \epsilon\}) = d_f(A)$ for every $\epsilon > 0$ where $A = \{1, 4, 9, \ldots\}$. Then $|A(n)| = |\{k \leq n : k \in A\}| \leq \sqrt{n}$ for every $n \in \mathbb{N}$ and so

$$\frac{f(|A(n)|)}{f(n)} \leq \frac{(n^{1/2})^p}{n^p} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty; \tag{13}$$

that is, $d_f(A) = 0$. Thus (x_n) is f-statistically convergent, whereas $(1, 4, 9, \ldots)$ is a subsequence of (x_n) which is not f-statistically convergent.

Definition 18. A subsequence of a sequence (x_k) is said to be f-statistically dense if the complement of the set of all indices of its elements has f-density zero.

Remark 19. Every f-statistically dense subsequence of a sequence (x_k) is statistically dense.

Burgin and Duman [17] proved that a number sequence (x_k) is statistically convergent if and only if every statistically dense subsequence of it is statistically convergent. We extend this result to the following.

Theorem 20. *A sequence (x_k) is f-statistically convergent if and only if every f-statistically dense subsequence of it is f-statistically convergent.*

The proof is similar to Theorem 2.1 of Burgin and Duman [17] and hence is omitted.

Corollary 21. *f-statistically dense subsequences of an f-statistically convergent sequence are f-statistically convergent.*

The following theorem shows that continuous functions preserve the f-statistical convergence of sequences.

Theorem 22. *If f-st $\lim x_k = L$ and $g(x)$ defined for all $x \in X$ is continuous at L, then f-st $\lim g(x_k) = g(L)$.*

Proof. As f-st $\lim x_k = L$, it follows by Theorem 3.1 of Aizpuru et al. [6] that there exists a set $A \subset \mathbb{N}$ such that $d_f(A) = 0$ and $\lim_{k \in \mathbb{N}-A} x_k = L$. As g is continuous at L, $\lim_{k \in \mathbb{N}-A} g(x_k) = g(L)$. Using Theorem 3.1 of Aizpuru et al. [6], the result follows. □

Šalát [35] proved that the set m_0 of all bounded statistically convergent sequences of real numbers is a closed linear subspace of the normed linear space m of all bounded sequences of real numbers. We establish a similar result for ℓ_∞^f, the set of all bounded f-statistically convergent sequences of scalars.

Theorem 23. *The set ℓ_∞^f is a closed linear subspace of the normed linear space ℓ_∞.*

The proof is similar to Theorem 2.1 of Šalát [35] and hence is omitted.

The above theorem provides us with the following information related to the structure of the set ℓ_∞^f.

Theorem 24. *The set ℓ_∞^f is a nowhere dense set in ℓ_∞.*

Since the sequence $((-1)^k) \in \ell_\infty$ does not belong to ℓ_∞^f, the proof follows from the fact that every proper closed linear subspace of an arbitrary normed linear space E is a nowhere dense set in E.

3. f-Statistical Boundedness

In this section we show that the concept of f-statistical boundedness is intermediate between the ordinary boundedness and the statistical boundedness. Some of the results of this section include the corresponding earlier results of Bhardwaj and Gupta [10] on statistical boundedness. f-statistical analog of monotone convergence theorem is established in Theorem 38. Theorem 40 shows that a sequence which is f-statistically bounded for each modulus f is also bounded in the ordinary sense.

Throughout this section, we deal with the sequences of scalars.

Theorem 25. *Every bounded sequence is f-statistically bounded; however, the converse need not be true.*

Proof. The result follows in view of the fact that empty set has zero f-density for every unbounded modulus f. For the converse part, the sequence $x = (x_k) = (1, 0, 0, 4, 0, 0, 0, 0, 9, \ldots)$ of Example 17 serves the purpose. □

Theorem 26. *Every f-statistically bounded sequence is statistically bounded.*

The proof follows in view of the fact that $A \subset \mathbb{N}$, $d_f(A) = 0$ implies $d(A) = 0$.

Remark 27. The converse of the above theorem need not be true which can be verified by the following example.

Example 28. Let $f(x) = \log(x + 1)$ and $(x_k) = (1, 0, 0, 4, 0, 0, 0, 0, 9, \ldots)$. Let $A = \{1, 4, 9, \ldots\}$ = the set of squares of natural numbers. For any $M > 0$,

$$\{k \in \mathbb{N} : |x_k| > M\} = A - \text{a finite subset of } \mathbb{N}. \tag{14}$$

Since $d_f(A) = 1/2 \neq 0$ and $d(A) = 0$, $(x_k) \notin s_f(b)$ and $(x_k) \in s(b)$. Consequently, $s_f(b) \subsetneq s(b)$.

Remark 29. From Theorems 25 and 26 and Example 28, we get $\ell_\infty \subsetneq s_f(b) \subsetneq s(b)$; that is, f-statistical boundedness is intermediate between the ordinary boundedness and the statistical boundedness and agrees with the statistical boundedness when $f = I$, the identity mapping.

Aizpuru et al. [6] proved that f-st $\lim x_k = L$ if and only if there exists $A \subset \mathbb{N}$ with $d_f(A) = 0$ and $\lim_{k\in\mathbb{N}-A} x_k = L$. We also establish a similar structure theorem for f-statistically bounded sequences; however, our proof is quite different from that of [6].

Theorem 30 (structure theorem). *A sequence $x = (x_k)$ is f-statistically bounded if and only if there exists $A \subset \mathbb{N}$ such that $d_f(A) = 0$ and $(x_k)_{k\in\mathbb{N}-A} \in \ell_\infty$.*

Proof. First, we suppose (x_k) is f-statistically bounded. So there exists $M > 0$ such that $d_f(\{k \in \mathbb{N} : |x_k| > M\}) = 0$. Take $A = \{k \in \mathbb{N} : |x_k| > M\}$. Then $d_f(A) = 0$ and for $k \in \mathbb{N} - A$, we have $|x_k| \le M$; that is, $(x_k)_{k\in\mathbb{N}-A} \in \ell_\infty$. Conversely, it is given that there exists $A \subset \mathbb{N}$ such that $d_f(A) = 0$ and $(x_k)_{k\in\mathbb{N}-A} \in \ell_\infty$. As $(x_k)_{k\in\mathbb{N}-A} \in \ell_\infty$ there exists $M > 0$ such that $|x_k| \le M$ for all $k \in \mathbb{N} - A$. This implies that $\{k \in \mathbb{N} : |x_k| > M\} \subset A$ and so $d_f(\{k \in \mathbb{N} : |x_k| > M\}) = 0$. Hence $(x_k)_{k\in\mathbb{N}}$ is f-statistically bounded. □

Remark 31. Let $(x_k) \in s_f(b)$. Then there exists $M > 0$ such that $d_f(\{k \in \mathbb{N} : |x_k| > M\}) = 0$. Take $A = \{k \in \mathbb{N} : |x_k| > M\}$. For $k \in \mathbb{N}$, let

$$y_k = \begin{cases} x_k, & \text{if } k \in \mathbb{N} - A; \\ 0, & \text{if } k \in A, \end{cases}$$
$$z_k = \begin{cases} 0, & \text{if } k \in \mathbb{N} - A; \\ x_k, & \text{if } k \in A. \end{cases} \tag{15}$$

Clearly $y = (y_k) \in \ell_\infty$ and $z = (z_k) \in s_f^0$, the space of f-statistically null sequences. Here $x_k = y_k + z_k$ for $k \in \mathbb{N}$. Thus $s_f(b) \subset \ell_\infty + s_f^0$. As $\ell_\infty, s_f^0 \subset s_f(b)$ and $s_f(b)$ is a linear space, we have $s_f(b) = \ell_\infty + s_f^0$.

It is easy to note that $s_f(b) \ne \ell_\infty \oplus s_f^0$ as $\ell_\infty \cap s_f^0 \ne \{0\}$. In fact $\ell_\infty \cap s_f^0 \supset c_0$, the space of null scalar sequences.

Remark 32. A subsequence of an f-statistically bounded sequence need not be f-statistically bounded. The sequence $x = (x_k) = (1, 0, 0, 4, 0, 0, 0, 0, 9, \ldots)$ of Example 17 is f-statistically bounded whereas $(1, 4, 9, \ldots)$ is a subsequence of it which is not f-statistically bounded.

We now characterize f-statistically bounded sequences in terms of their subsequences.

Theorem 33. *A sequence is f-statistically bounded if and only if every f-statistically dense subsequence of it is f-statistically bounded.*

The proof is easy in view of Theorem 2.1 of Burgin and Duman [17] and hence is omitted.

Theorem 34. *Every f-statistically convergent sequence is f-statistically bounded; however, the converse need not be true.*

Proof. The proof follows from the fact that $\{k \in \mathbb{N} : |x_k| > |L| + \epsilon\} \subset \{k \in \mathbb{N} : |x_k - L| > \epsilon\}$. For the converse part, taking $f = I$, identity map, and $x_k = (-1)^k$, we get $(x_k) \in s_f(b)$; however $(x_k) \notin s_f$, the space of f-statistically convergent sequences of scalars. □

The following theorem shows that every f-statistically bounded sequence has entries coincident with those of a bounded sequence for almost all k with respect to f.

Theorem 35. *A sequence $x = (x_k)$ is f-statistically bounded if and only if there exists a bounded sequence $y = (y_k)$ such that $x_k = y_k$ a.a. k w.r.t. f.*

The proof is easy and hence omitted.

Theorem 36. *Every f-statistically Cauchy sequence is f-statistically bounded; however, the converse need not be true.*

Theorem 37. *(a) $s_f(b)$ is normal and hence monotone.*
(b) $s_f(b)$ is a sequence algebra.
(c) $s_f(b)$ is not symmetric, in general.

Theorem 38. *Every monotone and f-statistically bounded sequence is f-statistically convergent.*

Proof. Let (x_k) be a monotone and f-statistically bounded sequence. By Theorem 30, there exists $A \subset \mathbb{N}$ with $d_f(A) = 0$ such that $(x_k)_{k\in\mathbb{N}-A} \in \ell_\infty$. So there exists $L \in \mathbb{C}$ such that $\lim_{k\in\mathbb{N}-A} x_k = L$. Using Theorem 3.1 of [6], we have $(x_k) \in s_f$. □

Lemma 39 (see [6]). *If $A \subset \mathbb{N}$ is infinite, then there exists an unbounded modulus f such that $d_f(A) = 1$.*

The following theorem shows that a sequence which is f-statistically bounded for each modulus f is also bounded in ordinary sense.

Theorem 40. *If, for every unbounded modulus f, $(x_k) \in s_f(b)$, then $(x_k) \in \ell_\infty$.*

Proof. Suppose, if possible, $(x_k) \notin \ell_\infty$. Then for every $M > 0$, we have that $A = \{k \in \mathbb{N} : |x_k| > M\}$ is an infinite set and so by Lemma 39, there exists an unbounded modulus f such that $d_f(A) = 1$ which contradicts the assumption that $(x_k) \in s_f(b)$ for every modulus f. □

Remark 41. From Theorem 25, we have $\ell_\infty \subset s_f(b)$ for every unbounded modulus f. Using this and Theorem 40, we can say that bounded sequences are precisely those sequences which are f-statistically bounded for every unbounded modulus f.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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