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Research Article

On the Property N^{-1}

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We construct a continuous function $f : [0,1] \to \mathbb{R}$ such that f possesses N^{-1}-property, but f does not have approximate derivative on a set of full Lebesgue measure. This shows that Banach's Theorem concerning differentiability of continuous functions with Lusin's property (N) does not hold for N^{-1}-property. Some relevant properties are presented.

1. Introduction

First we will specify some basic notations. By $|E|$ we denote the Lebesgue measure of $E \subset \mathbb{R}$. For any $f : I \to \mathbb{R}$, where I is an interval, by $f \restriction E$ we denote the restriction of f to $E \subset I$ and the symbol $f'_{\mathrm{ap}}(x)$ stands for approximate derivative of f at x.

Definition 1 (see [1]). Let $D \subset \mathbb{R}$ be measurable. We say that $f : D \to \mathbb{R}$ has Lusin's property (N), if the image $f(E)$ of every set $E \subset D$ of Lebesgue measure 0 has Lebesgue measure 0.

This condition was studied exhaustively; some of results can be found in [1]. For the present paper the most important is the following.

Theorem 2 (Third Banach Theorem, [1] Theorem 7.3). *If $f : [0,1] \to \mathbb{R}$ is continuous and has Lusin's property (N), then f is differentiable on a set of positive Lebesgue measure.*

In the present paper we will study a similar property.

Definition 3 (see [2, 3]). We say that $f : D \to \mathbb{R}$, defined on a measurable set $D \subset \mathbb{R}$, has N^{-1}-property, if the inverse image $f^{-1}(E)$ of every set $E \subset \mathbb{R}$ of Lebesgue measure 0 has Lebesgue measure 0.

Some of results concerning N^{-1}-property are presented in [2, 3]. In [2] a systematic study of N^{-1}-property for smooth and almost everywhere differentiable functions can be found. Some applications of N^{-1}-property in functional equation and geometric function theory can be found in [4–6].

2. Main Results

Our goal is to construct a continuous function $f : [0,1] \to [0,1]$ with N^{-1}-property which is not approximately differentiable on a set of full measure. We start with the basic theorem.

Theorem 4. *Let $B_1 = \{(2k-1)/2^n : k \in \{1,2,\dots,2^{n-1}\}, n \in \mathbb{N}\}$, $B_2 = \{(2k-1)/2^n + 1/(3\cdot 2^n) : k \in \{1,2,\dots,2^{n-1}\}, n \in \mathbb{N}\}$, and $A = (0,1) \setminus (B_1 \cup B_2)$. There exists a homeomorphism $f : A \to A$ such that*

(a1) *$f = f^{-1}$,*

(a2) *f has Lusin's property (N) and N^{-1}-property,*

(a3) *f has no approximate derivative (finite or not) at any $x \in A$.*

Proof. Let $x = 0.\overline{i_1 i_2 \cdots i_n \cdots}$ denote a binary decomposition of $x \in (0,1)$. It is easily seen that $(2k-1)/2^n + 1/(3\cdot 2^n) = 0.\overline{i_1 i_2 \cdots i_{n-1} 101010 \cdots}$. Therefore, $x \in (0,1)$ and $x = 0.\overline{i_1 i_2 \cdots i_n \cdots}$ belongs to A if and only if it has a binary decomposition $x = 0.\overline{i_1 i_2 \cdots i_n \cdots}$ such that

$$\begin{gathered} \overline{\overline{\{n \in \mathbb{N} : i_{2n} = 0\}}} = \aleph_0 = \overline{\overline{\{n \in \mathbb{N} : i_{2n} = 1\}}} \text{ or} \\ \overline{\overline{\{n \in \mathbb{N} : i_{2n-1} = 0\}}} = \aleph_0 = \overline{\overline{\{n \in \mathbb{N} : i_{2n-1} = 1\}}} \end{gathered} \tag{1}$$

(in other words x has infinitely many 0s and infinitely many 1s at even places or infinitely many 0s and infinitely many 1s at odd places). Let $\Delta_n^k = (k/2^n, (k+1)/2^n)$ for $k \in \{0,1,\ldots,2^n-1\}$ and $n \in \mathbb{N}$. Obviously, $A \subset \bigcup_{k=0}^{2^n-1} \Delta_n^k$ for every $n \in \mathbb{N}$. Moreover,

$$A \cap \Delta_n^k = \left\{ x \in A : x = 0.\overline{i_1 i_2 \cdots i_n \cdots} \text{ where } \sum_{j=1}^{n} 2^{n-j} i_j = k \right\}. \tag{2}$$

Define $f : A \to A$ by

$$f(x) = 0.\overline{i_1 i_2' i_3 i_4' \cdots i_{2n-1} i_{2n}' \cdots}, \tag{3}$$

where $x = 0.\overline{i_1 i_2 \cdots i_n \cdots}$ and $i_j' = 1 - i_j$. In other words, $f(x) = 0.\overline{m_1 m_2 \cdots m_n \cdots}$, where $m_j = i_j$ for odd j and $m_j = 1 - i_j$ for even j. By (1), $f(x) \in A$ for $x \in A$ and f is well-defined. Moreover, directly from the definition of f, it follows that f is a bijection and the composition $f \circ f$ is the identity function, whence $f^{-1} = f$. Moreover, by (2), for each $n \in \mathbb{N}$ and $k \in \{0,1,\ldots,2^n-1\}$, $k = \sum_{j=1}^{n} 2^{n-j} i_j$, $i_j \in \{0,1\}$, we have

$$f\left(A \cap \Delta_n^k\right) = A \cap \Delta_n^{k'}, \tag{4}$$

where $k' = \sum_{j=1}^{n} 2^{n-j} m_j$.

We claim that f is continuous. Fix $x_0 \in A$ and $\varepsilon > 0$. Choose $n_0 \in \mathbb{N}$ such that $1/2^{n_0} < \varepsilon$. There exists $k_0 \le 2^{n_0} - 1$ for which $x_0 \in \Delta_{n_0}^{k_0}$. By (4), $f(A \cap \Delta_{n_0}^{k_0}) = A \cap \Delta_{n_0}^{k_0'}$. Since $A \cap \Delta_{n_0}^{k_0}$ is a neighborhood of x_0 and $|y_1 - y_2| \le 1/2^{n_0} < \varepsilon$ for all $y_1, y_2 \in \Delta_{n_0}^{k_0'}$, we conclude that f is continuous at x_0. Thus, f is continuous, because x_0 was arbitrary. By the equality $f = f^{-1}$, f is a homeomorphism.

Now we will show condition (a2). Let $H \subset A$ be any set of Lebesgue measure zero. Fix any $\varepsilon > 0$. There exists an open in A set $U \subset A$ such that $H \subset U$ and $|U| < \varepsilon$. Let

$$\mathscr{B} = \left\{ \Delta_n^k \cap A : k \in \{0,1,\ldots,2^n-1\},\ n \in \mathbb{N} \right\}. \tag{5}$$

Clearly, $\mathscr{B}$ is a base of the natural topology in A. Since either any two sets from $\mathscr{B}$ are disjoint or one of them is contained in the other, it is easy to see that any open subset of A can be represented as a union of some subfamily of pairwise disjoint sets from $\mathscr{B}$. Thus, $U = \bigcup_{j\in J} (\Delta_{n_j}^{k_j} \cap A)$, where J is at most countable and $\Delta_{n_{j_1}}^{k_{j_1}} \cap \Delta_{n_{j_2}}^{k_{j_1}} = \emptyset$ for $j_1, j_2 \in J$, $j_1 \ne j_2$. Then, by (4), $\bigcup_{j\in J} f(\Delta_{n_j}^{k_j} \cap A) = \bigcup_{j\in J} (\Delta_{n_j}^{k_j'} \cap A)$ is an open in A set containing $f(H)$ and

$$\sum_{j\in J} \left| \Delta_{n_j}^{k_j'} \cap A \right| = \sum_{j\in J} \left| \Delta_{n_j}^{k_j} \cap A \right| = |U| < \varepsilon. \tag{6}$$

Since $\varepsilon > 0$ was arbitrary, $|f(H)| = 0$ and f has Lusin's property (N). Since $f = f^{-1}$, f has also N^{-1}-property.

Finally, we will show that f has no approximate derivative at any $x \in A$. Fix $x \in A$ and an even $n \in \mathbb{N}$. Then $x \in \Delta_n^k$ for some $k \le 2^n - 1$. Let $i_1, i_2, \ldots, i_n \in \{0,1\}$ be such that $k = \sum_{j=1}^{n} 2^{n-j} i_j$. Moreover, let k' be understood as before. It is clear that

$$\begin{aligned} A \cap \Delta_n^k &= \left(A \cap \Delta_{n+2}^{4k}\right) \cup \left(A \cap \Delta_{n+2}^{4k+1}\right) \\ &\quad \cup \left(A \cap \Delta_{n+2}^{4k+2}\right) \cup \left(A \cap \Delta_{n+2}^{4k+3}\right), \\ f\left(A \cap \Delta_{n+2}^{4k}\right) &= A \cap \Delta_{n+2}^{4k'+1}, \\ f\left(A \cap \Delta_{n+2}^{4k+1}\right) &= A \cap \Delta_{n+2}^{4k'}, \\ f\left(A \cap \Delta_{n+2}^{4k+2}\right) &= A \cap \Delta_{n+2}^{4k'+3}, \\ f\left(A \cap \Delta_{n+2}^{4k+3}\right) &= A \cap \Delta_{n+2}^{4k'+2} \end{aligned} \tag{7}$$

(remember that n is even).

Note that

$$\frac{f(y) - f(x)}{y - x} < 0 \tag{8}$$

if $x \in \Delta_{n+2}^{4k}$ and $y \in A \cap \Delta_{n+2}^{4k+1}$ or $x \in \Delta_{n+2}^{4k+2}$ and $y \in A \cap \Delta_{n+2}^{4k+3}$. Moreover,

$$\frac{f(y) - f(x)}{y - x} > \frac{2^n}{3 \cdot 2^n} = \frac{1}{3} \tag{9}$$

if $x \in \Delta_{n+2}^{4k}$ and $y \in A \cap \Delta_{n+2}^{4k+2}$ or $x \in \Delta_{n+2}^{4k+1}$ and $y \in A \cap \Delta_{n+2}^{4k+3}$.

Thus, if $n \in \mathbb{N}$ is even and $x \in A \cap \Delta_n^k$, we can find $B, C \subset \Delta_n^k \cap A$ such that

$$|B| = |C| = \frac{1}{4} \left| \Delta_n^k \right|, \tag{10}$$

$$\frac{f(x) - f(y)}{x - y} < 0 \quad \forall y \in B, \tag{11}$$

$$\frac{f(x) - f(y)}{x - y} > \frac{1}{3} \quad \forall y \in C. \tag{12}$$

Since this is true for every even n and $|\Delta_n^k| = 1/2^n$, we conclude that f has no approximate derivative (finite or not) at x. The proof is completed. □

From Banach's Theorem 2, we easily get the following.

Corollary 5. *Any function f, defined on an interval, which possesses Lusin's condition (N) such that the set of discontinuity points of f is finite, is derivable at every point of some set of positive Lebesgue measure.*

Meanwhile, by Theorem 4, we have the following.

Theorem 6. *There exists a bijection $g : [0,1] \to [0,1]$ such that*

(b1) *g has Lusin's property (N) and N^{-1}-property,*

(b2) *the set of discontinuity points of g is countable,*

(b3) *g has no approximate derivative at any point.*

Proof. Let B_1, B_2, A, and f be the same as in Theorem 4. It is easily seen that every member of B_2 is of the form $0.\overline{i_1 \cdots i_n 001010\cdots}$ or $0.\overline{i_1 \cdots i_n 1101010\cdots}$ for some $n \in \mathbb{N}$, except 1/6, 1/3, 2/3, 5/6. Define $\varphi : \{0,1\} \cup B_1 \cup B_2 \to \{0,1\} \cup B_1 \cup B_2$ by

$$\varphi(x) = \begin{cases} x & \text{for } x \in \left\{0, 1, \frac{1}{6}, \frac{1}{3}, \frac{2}{3}, \frac{5}{6}\right\}, \\ 0.\overline{j_1 j_2 \cdots j_n 001010\cdots} & \text{for } x = 0.\overline{i_1 \cdots i_n 001010\cdots}, \\ 0.\overline{j_1 j_2 \cdots j_n 110101\cdots} & \text{for } x = 0.\overline{i_1 \cdots i_n 110101\cdots}, \\ 0.\overline{j_1 j_2 \cdots j_{n-1} 1} & \text{for } x = 0.\overline{i_1 \cdots i_n 1}, \end{cases} \tag{13}$$

where $j_{2m-1} = i_{2m-1}$ and $j_{2m} = 1 - i_{2m}$. It is easy to see that φ is a bijection. Let $g : [0,1] \to [0,1]$ be defined by

$$g(x) = \begin{cases} f(x) & \text{for } x \in A, \\ \varphi(x) & \text{for } x \in \{0,1\} \cup B_1 \cup B_2. \end{cases} \tag{14}$$

Fix $x \in (0,1) \setminus (B_1 \cup B_2)$, $x = 0.\overline{i_1 i_2 \cdots i_n \cdots}$, and $\varepsilon > 0$. Let m be a positive integer such that $1/2^m < \varepsilon$. The set

$$\begin{aligned} C = &\left\{0, 1, \frac{1}{6}, \frac{1}{3}, \frac{2}{3}, \frac{5}{6}\right\} \\ &\cup \left\{\frac{2k-1}{2^n} : k \in \{1, 2, \ldots, 2^{n-1}\},\ n \le m\right\} \\ &\cup \left\{0.\overline{l_1 \cdots l_n 001010\cdots} : n \le m\right\} \\ &\cup \left\{0.\overline{l_1 \cdots l_n 1101010\cdots} : n \le m\right\} \end{aligned} \tag{15}$$

is finite and $C \subset \{0,1\} \cup B_1 \cup B_2$. Hence, we can find $\delta \in (0, 1/2^{m+3})$ for which $(x - \delta, x + \delta) \cap C = \emptyset$. Take any $y \in (x - \delta, x + \delta) \cap (B_1 \cup B_2)$. Since $|x - y| < 1/2^{m+3}$ and $y \in B_1 \cup B_2$, we conclude $y = 0.\overline{i_1 i_2 \cdots i_m \cdots 001010\cdots}$ or $y = 0.\overline{i_1 i_2 \cdots i_m \cdots 110101\cdots}$ or $y = 0.\overline{i_1 i_2 \cdots i_m \cdots 1}$. Hence, $g(x) = f(x) = 0.\overline{j_1 j_2 \cdots j_m \cdots}$ and $g(y) = \varphi(y) = 0.\overline{j_1 j_2 \cdots j_m \cdots}$ or $g(y) = \varphi(y) = 0.\overline{j_1 j_2 \cdots j_m \cdots 1}$. Therefore, $|g(x) - g(y)| < 1/2^m < \varepsilon$. Since f is continuous, g is continuous at x. Thus, we have proved that the set of all discontinuity points of g is contained in $\{0,1\} \cup B_1 \cup B_2$. Therefore, g satisfies (b1), (b2), and (b3). □

Theorem 7. *For each $\varepsilon \in (0,1)$ there exist a closed nowhere dense set $F \subset (0,1)$ and a homeomorphism $h : F \to F$ such that*

- (c1) $|F| > 1 - \varepsilon$,
- (c2) $h = h^{-1}$,
- (c3) *h has Lusin's property (N) and N^{-1}-property,*
- (c4) *h has no approximate derivative (finite or not) at any $x \in F$ (more precisely, if $\widetilde{h} : [0,1] \to [0,1]$ is any extension of h then $\widetilde{h}$ has no approximate derivative (finite or not) at any $x \in F$).*

Proof. Let B_1, B_2, A, and f be the same as in Theorem 4. Let $\{x_n\}_{n=1}^{\infty} = B_1 \cup B_2$. Fix $\varepsilon > 0$ and choose a sequence $(m_n)_{n \ge 0}$ of even natural numbers satisfying

$$4 \sum_{n=0}^{\infty} \frac{1}{2^{m_n}} < \varepsilon, \tag{16}$$

$$4 \sum_{j=n+1}^{\infty} \frac{1}{2^{m_j}} < \frac{1}{8} \cdot \frac{1}{2^{m_n}} \quad \forall n \ge 0. \tag{17}$$

For each $n \ge 1$ there exists $k_n \in \{1, \ldots, 2^{m_n} - 1\}$ such that $x_n \in ((k_n - 1)/2^{m_n}, (k_n + 1)/2^{m_n})$. Let

$$\begin{aligned} B = &\left(\Delta_{m_0}^{0} \cup \{0\} \cup \Delta_{m_0}^{2^{m_0}-1} \cup \{1\}\right) \\ &\cup \bigcup_{n=1}^{\infty} \left(\Delta_{m_n}^{k_n - 1} \cup \left\{\frac{k_n}{2^{m_n}}\right\} \cup \Delta_{m_n}^{k_n}\right). \end{aligned} \tag{18}$$

Since

$$\Delta_{m_n}^{k_n - 1} \cup \left\{\frac{k_n}{2^{m_n}}\right\} \cup \Delta_{m_n}^{k_n} = \left(\frac{k_n - 1}{2^{m_n}}, \frac{k_n + 1}{2^{m_n}}\right), \tag{19}$$

B is an open subset of $[0,1]$. Moreover, $B_1 \cup B_2 \cup \{0,1\} \subset B$ and, by (16),

$$|B| \le \frac{2}{2^{m_0}} + \sum_{n=1}^{\infty} \frac{2}{2^{m_n}} < \frac{\varepsilon}{2}. \tag{20}$$

By (4), in the proof of Theorem 4, for each $n \ge 1$ there exist $u_n, v_n \in \{1, \ldots, 2^{m_n}\}$ such that

$$\begin{aligned} f\left(\Delta_{m_n}^{k_n - 1} \cap A\right) &= \Delta_{m_n}^{u_n} \cap A, \\ f\left(\Delta_{m_n}^{k_n} \cap A\right) &= \Delta_{m_n}^{v_n} \cap A. \end{aligned} \tag{21}$$

Moreover,

$$\begin{aligned} f\left(\Delta_{m_0}^{0} \cap A\right) &= \Delta_{m_0}^{i_0} \cap A, \\ f\left(\Delta_{m_0}^{2^{m_0}-1} \cap A\right) &= \Delta_{m_0}^{i_1} \cap A \end{aligned} \tag{22}$$

for some $i_0, i_1 \in \{0, 1, \ldots, 2^{m_0}\}$. Hence,

$$\begin{aligned} C = f(B \cap A) = f\Bigg(&\left(\left(\Delta_{m_0}^{0} \cap A\right) \cup \left(\Delta_{m_0}^{2^{m_0}-1} \cap A\right)\right) \\ &\cup \bigcup_{n=1}^{\infty} \left(\left(\Delta_{m_n}^{k_n - 1} \cap A\right) \cup \left(\Delta_{m_n}^{k_n} \cap A\right)\right)\Bigg) = \left(\Delta_{m_0}^{i_0} \cap A\right) \\ &\cup \left(\Delta_{m_0}^{i_1} \cap A\right) \cup \bigcup_{n=1}^{\infty} \left(\left(\Delta_{m_n}^{u_n} \cap A\right) \cup \left(\Delta_{m_n}^{v_n} \cap A\right)\right) \\ &= \left(\Delta_{m_0}^{i_0} \cup \Delta_{m_0}^{i_1} \cup \bigcup_{n=1}^{\infty} \left(\Delta_{m_n}^{u_n} \cup \Delta_{m_n}^{v_n}\right)\right) \cap A. \end{aligned} \tag{23}$$

Again, applying (16), we have $|C| = |B| < \varepsilon/2$. Moreover, since $[0,1] \setminus A \subset \operatorname{Int} B$, the set

$$\begin{aligned} B \cup C = &([0,1] \setminus A) \cup B \\ &\cup \left(\Delta_{m_0}^{i_0} \cup \Delta_{m_0}^{i_1} \cup \bigcup_{n=1}^{\infty} \left(\Delta_{m_n}^{u_n} \cup \Delta_{m_n}^{v_n}\right)\right) \end{aligned} \tag{24}$$

is open in $[0,1]$.

Finally, put $H = [0,1] \setminus (B \cup C)$. It is clear that $H \subset A$, H is a closed subset of $[0,1]$, and $|H| > 1 - 2(\varepsilon/2) = 1 - \varepsilon$. Since f is a bijection and $f = f^{-1}$, we have

$$f((B \cup C) \cap A) = f(B \cap A) \cup f(C \cap A) = C \cup (B \cap A) = (B \cup C) \cap A. \tag{25}$$

It follows that $f(H) = H$ and $h = f \restriction H$ is a homeomorphism.

Fix $x_0 \in H$ and $n \in \mathbb{N}$. There exists $k \in \{0, 1, \ldots, 2^n - 1\}$ such that $x_0 \in \Delta^k_{m_n}$. Certainly, $\Delta^k_{m_n} \not\subset B \cup C$. Therefore, by (17), $|(B \cup C) \cap \Delta^k_{m_n}| < (1/8)|\Delta^k_{m_n}|$. By (10), (11), and (12),

$$\left|\left\{x \in \Delta^k_{m_n} : \frac{f(x) - f(x_0)}{x - x_0} < 0\right\}\right| > \frac{1}{4} \cdot \frac{1}{8}\left|\Delta^k_{m_n}\right|,$$
$$\left|\left\{x \in \Delta^k_{m_n} : \frac{f(x) - f(x_0)}{x - x_0} > \frac{1}{3}\right\}\right| > \frac{1}{4} \cdot \frac{1}{8}\left|\Delta^k_{m_n}\right|. \tag{26}$$

Therefore, any extension $\tilde{h} : [0,1] \to [0,1]$ of h has no approximate derivative at x_0. □

Lemma 8. *Let $a, b, c, d \in \mathbb{R}$, $a < b$, and $c < d$. For every $\varepsilon \in (0,1)$ there exist a closed nowhere dense set $H \subset (a,b)$ and a continuous injection $g : H \to [c,d]$ such that*

(d1) $|H| > (1/2)(b - a)$,

(d2) *$g^{-1} : g(H) \to H$ is continuous,*

(d3) *g has Lusin's property (N) and N^{-1}-property,*

(d4) *if $\tilde{g} : [a,b] \to [c,d]$ is any extension of g, then $\tilde{g}$ has no approximate derivative (finite or not) at any $x \in H$,*

(d5) *$|g(\min H) - c| < \varepsilon$, $|d - g(\max H)| < \varepsilon$, and $|g(b_n) - g(a_n)| < \varepsilon$ for all $n \in \mathbb{N}$, where $\{(a_n, b_n) : n \in \mathbb{N}\}$ is the set of all connected components of $(a,b) \setminus H$.*

Proof. Fix $\varepsilon > 0$ and choose $n \in \mathbb{N}$ such that $1/(n+1) < \varepsilon/2(d-c)$. Let $a = y_0 < x_1 < y_1 < x_2 < \cdots < x_n < y_n < x_{n+1} = b$ be a partition of $[a,b]$ such that $y_i - x_i = (1/(n+1))(b-a)$ for $i \in \{1, \ldots, n\}$ and $x_j - y_{j-1} = (1/(n+1)^2)(b-a)$ for $j \in \{1, \ldots, n, n+1\}$. Let $\psi : [a,b] \to [c,d]$ be a linear homeomorphism, $\psi(x) = ((d-c)/(b-a))(x-a) + c$. By Theorem 7, there exist a closed nowhere dense set $F \subset (0,1)$ and a homeomorphism $h : F \to F$ satisfying conditions (c2)–(c4) such that $|F| > (n+1)/2n$. For each $k \in \{1, \ldots, n\}$ define linear homeomorphisms $\psi_k : [x_k, y_k] \to [0,1]$,

$$\psi_k(x) = \frac{1}{y_k - x_k}(x - x_k), \tag{27}$$

and $\phi_k : [0,1] \to [\psi(x_k), \psi(y_k)]$,

$$\phi_k(x) = (\psi(y_k) - \psi(x_k))x + \psi(x_k). \tag{28}$$

Moreover, let $F_k = \psi_k^{-1}(F)$ for $k \le n$. Obviously, each F_k is a closed nowhere dense subset of (x_k, y_k). Besides, $|F_k| = |F| \cdot (y_k - x_k) = |F| \cdot (b-a)/(n+1)$. For each $k \in \{1, \ldots, n\}$ define $h_k : F_k \to [\psi(x_k), \psi(y_k)]$ by $h_k = \phi_k \circ h \circ \psi_k$. It is easy to see that each h_k is a continuous injection, h_k has Lusin's property (N) and N^{-1}-property, and, moreover, any extension of h_k to $[x_k, y_k]$ is not approximately differentiable at any point $x \in F_k$. Finally, let $H = \bigcup_{k=1}^n F_k$ and define $g : H \to [c,d]$ by $g(x) = h_k(x)$ for $x \in F_k$, $k \in \{1, \ldots, n\}$.

It is clear that H and g satisfy conditions (d1)–(d4). Let (α, β) be any connected component of $(a,b) \setminus H$. If $(\alpha, \beta) \subset [x_k, y_k]$ for some $k \in \{1, \ldots, n\}$ then

$$|g(\beta) - g(\alpha)| \le \psi(y_k) - \psi(x_k) = \frac{b-a}{n+1} \cdot \frac{d-c}{b-a} = \frac{d-c}{n+1} < \varepsilon. \tag{29}$$

If $(\alpha, \beta) \supset [y_{k-1}, x_k]$ for some $k \in \{2, \ldots, n\}$ then

$$|g(\beta) - g(\alpha)| \le \psi(y_k) - \psi(x_{k-1}) = 2\frac{d-c}{n+1} < \varepsilon. \tag{30}$$

Similarly,

$$|g(\min F) - c| \le \psi(y_1) - c = \frac{d-c}{n+1} + \frac{d-c}{(n+1)^2} = \frac{n+2}{(n+1)^2}(d-c) < \frac{2(d-c)}{n+1} < \varepsilon. \tag{31}$$

Analogously, $|d - g(\max F)| < \varepsilon$. This completes the proof. □

Now, we can prove the main theorem of the present paper.

Theorem 9. *There exists a continuous function $f : [0,1] \to [0,1]$ such that f has N^{-1}-property, but f'_{ap} exists almost nowhere.*

Proof. We will construct inductively a sequence $(F_n)_{n\in\mathbb{N}}$ of closed subsets of $[0,1]$ and a sequence $(f_n)_{n\in\mathbb{N}}$ of continuous functions $f_n : [0,1] \to [0,1]$ such that

(1) $F_n \subset F_{n+1}$ and $|F_n| > 1 - 1/2^n$ for all $n \ge 1$,

(2) $f_n \restriction F_k = f_k \restriction F_k$ for all $n > k$,

(3) $|f_n(x) - f_{n+1}(x)| < 1/2^n$ for $n \in \{1, 2, \ldots\}$ and $x \in [0,1]$,

(4) every f_n restricted to F_n has N^{-1}-property,

(5) every extension of $f_n \restriction F_n$ has no approximate derivative at any $x \in F_n$.

First, we give a useful definition. If $E \subset (0,1)$ is closed and $\varphi : E \to (0,1)$, then by the linear extension of f we mean $\psi : [0,1] \to [0,1]$ such that $\psi \restriction E = \varphi$, $\psi(0) = 0$, $\psi(1) = 1$, and ψ is linear on every closed interval contiguous to $E \cup \{0,1\}$. It is clear that ψ is continuous if and only if φ is continuous.

By Theorem 7, there exist a closed set $F \subset (0,1)$, $|F| > 1/2$, and a bijection $g_1 : F \to F$ satisfying conditions (c1)–(c4). Let $F_1 = F$ and $f_1 : [0,1] \to [0,1]$ be the linear extension of g_1. Then f_1 is continuous, f_1 has N^{-1}-property, and every extension of $f_1 \restriction F_1 = g_1$ has no approximate derivative at any $x \in F_1$.

Let $((a_k^1, b_k^1))_{k\geq 1}$ be the family of all connected components of $[0,1] \setminus (F_1 \cup \{0,1\})$. Moreover, for every $k \in \mathbb{N}$, let J_k^1 be an open interval with endpoints $f_1(a_k^1)$ and $f_1(b_k^1)$. By Lemma 8, for each $k \in \mathbb{N}$ there exist closed $F_k^1 \subset (a_k^1, b_k^1)$ and $g_k^1 : F_k^1 \to J_k^1$ satisfying conditions (d1)–(d5) with $\varepsilon = 1/2$. Let $F_2 = F_1 \cup \bigcup_{k=1}^{\infty} F_k^1$ and let $g_2 : F_2 \to F_1 \cup \bigcup_{k=1}^{\infty} J_k^1$ be defined by $g_2(x) = f_1(x)$ for $x \in F_1$ and $g_2(x) = g_k^1(x)$ for $x \in F_k^1$, $k \in \{1, 2, \ldots\}$. We claim that g_2 is continuous. The continuity of g_2 at each point of $\bigcup_{k=1}^{\infty} F_k^1$ is obvious. Fix $x_0 \in F_1$ and $\varepsilon > 0$. If x_0 is not isolated from the right in F_2, then there exist $\delta > 0$ such that $|g_2(x) - g_2(x_0)| = |g_1(x) - g_1(x_0)| < \varepsilon$ for $x \in F_1 \cap (x_0, x_0 + \delta]$ and $F_2 \cap (x_0, x_0 + \delta) = (F_1 \cap (x_0, x_0 + \delta)) \cup \bigcup_{k \in K} F_k^1$ for some $K \subset \mathbb{N}$. Since

$$|g_2(x) - g_2(x_0)| < \max\left\{\left|g_2\left(a_k^1\right) - g_2(x_0)\right|, \left|g_2\left(b_k^1\right) - g_2(x_0)\right|\right\} \quad (32)$$

for $x \in J_k^1$, we have $|g_2(x) - g_2(x_0)| < \varepsilon$ for $x \in F_2 \cap (x_0, x_0 + \delta)$. Hence, g_2 is continuous from the right at x_0. Similarly, we can show that g_2 is continuous from the left at x_0. Since x_0 was arbitrary, g_2 is continuous.

Let f_2 be the linear extension of g_2. It is clear that $F_1 \subset F_2$, $|F_2| > 1 - 1/4$, $f_2 \restriction F_1 = f_1 \restriction F_1$, f_2 restricted to F_2 has N^{-1}-property, and every extension of $f_2 \restriction F_2 = g_2$ has no approximate derivative at any $x \in F_2$. Moreover, $|f_2(x) - f_1(x)| < 1/2$ for $x \in [0,1]$.

Assume that closed sets $F_1, \ldots, F_n \subset (0,1)$, $F_1 \subset \cdots \subset F_n$, and continuous functions $f_r : [0,1] \to [0,1]$, $r \in \{1, \ldots, n\}$, are chosen. Moreover, assume that for every $r \in \{2, \ldots, n\}$ we have $|F_r| > 1 - 1/2^r$, f_r restricted to F_r has N^{-1}-property, every extension of $f_r \restriction F_r$ has no approximate derivative at any $x \in F_r$, $|f_r(x) - f_{r-1}(x)| < 1/2^{r-1}$ for each $x \in [0,1]$, and $f_r \restriction F_s = f_s \restriction F_s$ for every $s \in \{1, \ldots, r-1\}$.

Let $((a_k^n, b_k^n))_{k\geq 1}$ be the family of all connected components of $[0,1] \setminus (F_n \cup \{0,1\})$. Moreover, for every $k \in \mathbb{N}$ let J_k^n be an open interval with endpoints $f_n(a_k^n)$ and $f_n(b_k^n)$. By Lemma 8, for each $k \in \mathbb{N}$ there exist closed $F_k^n \subset (a_k^n, b_k^n)$ and $g_k^n : F_k^n \to J_k^n$ satisfying conditions (d1)–(d5) with $\varepsilon = 1/2^n$. Let $F_{n+1} = F_n \cup \bigcup_{k=1}^{\infty} F_k^n$ and let $g_{n+1} : F_{n+1} \to F_1 \cup \bigcup_{k=1}^{\infty} J_k^n$ be defined by $g_{n+1}(x) = f_n(x)$ for $x \in F_n$ and $g_{n+1}(x) = g_k^n(x)$ for $x \in F_k^n$, $k \in \{1, 2, \ldots\}$. Similarly, as in the case of g_2, we can check that g_{n+1} is continuous.

Let f_{n+1} be the linear extension of g_{n+1}. It is clear that $F_n \subset F_{n+1}$, $|F_{n+1}| > 1 - 1/2^{n+1}$, $f_{n+1} \restriction F_n = f_n \restriction F_n$, f_{n+1} restricted to F_{n+1} has N^{-1}-property, and every extension of $f_{n+1} \restriction F_{n+1}$ has no approximate derivative at any $x \in F_{n+1}$. Moreover, $|f_{n+1}(x) - f_n(x)| < 1/2^n$ for $x \in [0,1]$. Thus, we have proved inductively that there exist a sequence $(F_n)_{n\in\mathbb{N}}$ of closed subsets of $[0,1]$ and a sequence $(f_n)_{n\in\mathbb{N}}$ of continuous functions $f_n : [0,1] \to [0,1]$ satisfying conditions (1)–(5).

Since $|f_{n+1}(x) - f_n(x)| < 1/2^n$ for $x \in [0,1]$ and $n \in \mathbb{N}$, the sequence $(f_n)_{n\in\mathbb{N}}$ is uniformly convergent to some continuous function $f : [0,1] \to [0,1]$. Moreover, $f \restriction F_n = f_n \restriction F_n$ for all $n \in \mathbb{N}$. Therefore, by (5), f has no approximate derivative at any point from $\bigcup_{n=1}^{\infty} F_n$. Since, by (1), $|\bigcup_{n=1}^{\infty} F_n| = 1$, f'_{ap} exists almost nowhere.

It remains to prove that f has N^{-1}-property. Take any $E \subset [0,1]$ of the Lebesgue measure zero. Then, by (2),

$$\begin{aligned} f^{-1}(E) &\subset \bigcup_{n=1}^{\infty}\left(F_n \cap f^{-1}(E)\right) \cup \left([0,1] \setminus \bigcup_{n=1}^{\infty} F_n\right) \\ &= \bigcup_{n=1}^{\infty}\left(F_n \cap f_n^{-1}(E)\right) \cup \left([0,1] \setminus \bigcup_{n=1}^{\infty} F_n\right). \end{aligned} \quad (33)$$

Applying (1) and (4), we conclude that $|f^{-1}(E)| = 0$. Thus, f has N^{-1}-property. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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