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Research Article

φ-Multipliers on Banach Algebras and Topological Modules

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We prove some results concerning Arens regularity and amenability of the Banach algebra $M_\varphi(A)$ of all φ-multipliers on a given Banach algebra A. We also consider φ-multipliers in the general topological module setting and investigate some of their properties. We discuss the φ-strict and φ-uniform topologies on $M_\varphi(A)$. A characterization of φ-multipliers on $L_1(G)$-module $L_p(G)$, where G is a compact group, is given.

1. Introduction

The concept of a multiplier was introduced by Helgason [1] as follows. Let A be a commutative and semisimple Banach algebra and let $\Delta(A)$ be its maximal ideal space. Let $\widehat{A}$ denote the Gelfand representation of A as a subalgebra of the algebra of continuous functions on $\Delta(A)$. A bounded continuous function g on $\Delta(A)$ is a multiplier on A if $g\widehat{A} \subseteq \widehat{A}$. The general theory of multipliers on faithful Banach algebras was developed by Wang [2] and Birtel [3].

Recall that a mapping $T : A \to A$ is called a left (resp., right) multiplier on A if

$$T(xy) = T(x)\,y \quad (\text{resp., } T(xy) = xT(y)) \tag{1}$$

for all $x, y \in A$. We say T is a multiplier on A if it is both a left multiplier and a right multiplier on A.

We denote with $M(A)$ the algebra of all multipliers on A.

A Banach algebra A is called left (resp., right) faithful if, for all $x \in A$, $xA = \{0\}$ (resp., $Ax = \{0\}$) implies that $x = 0$; A is called faithful if it is both left and right faithful.

In [4] we generalized the concept of multipliers on faithful Banach algebras to φ-multipliers as follows. Let A be a Banach algebra and let $\varphi : A \to A$ be an algebra homomorphism. A linear continuous mapping $T : A \to A$ is called a left (resp., right) φ-multiplier on A if

$$T(xy) = T(x)\,\varphi(y) \quad (\text{resp., } T(xy) = \varphi(x)\,T(y)) \tag{2}$$

for all $x, y \in A$. We say T is a φ-multiplier on A if it is both a left φ-multiplier and a right φ-multiplier on A. We denote by $M_\varphi(A)$ (resp., $M_\varphi^l(A)$, $M_\varphi^r(A)$) the collection of all φ-multipliers (resp., left φ-multipliers, right φ-multipliers) on A.

It turns out that this concept is considerably more general than the concept of multipliers on Banach algebras. Also by using some well-known homomorphisms like Jordan homomorphism, spectrum preserving homomorphism, and idempotent preserving homomorphism, we can transfer these useful properties from homomorphism φ to the algebra of φ-multipliers.

In [4], we studied various properties of φ-multipliers, for instance, the faithfulness of the Banach algebra $M_\varphi(A)$ and the existence of a bounded approximate identity in the range of a φ-multiplier. Finally, as an example, we have characterized φ-multipliers on $L_1(G)$.

In Section 2 we are concerned by Arens regularity and amenability of the Banach algebra $M_\varphi(A)$ under some suitable conditions. We introduce the notion of Jordan φ-multiplier and prove that every Jordan φ-multiplier is a φ-multiplier whenever the range of φ is dense in the algebra.

In Section 3 we extend the notion of φ-multipliers on Banach algebras to topological modules and investigate some of their properties. We discuss the φ-strict and φ-uniform topologies on $M_\varphi(A)$ and apply our results to $L_1(G)$-module $L_p(G)$.

Let X be a topological vector space and let A be a topological algebra, both over the same field $\mathbb{K}$ (= $\mathbb{R}$ or $\mathbb{C}$). Then X is called a topological left A-module if it is a left A-module and the module multiplication $(a, x) \to a \cdot x$ from $A \times X$ into X is separately continuous. If $b(A)$ denotes the collection of all bounded sets in A, then module multiplication $(a, x) \to a\cdot x$ is called $b(A)$-hypocontinuous [5] if, given any neighborhood G of 0 in X and any $D \in b(A)$, there exists a neighborhood H of 0 in X such that $D \cdot H \subseteq G$. Clearly, joint continuity $\Rightarrow$ hypocontinuity $\Rightarrow$ separate continuity. A mapping ψ from a left A-module X into another left A-module Y is called an A-module homomorphism if $\psi(a \cdot x) = a \cdot \psi(x)$ for all $a \in A$ and $x \in X$.

2. Some Properties of φ-Multipliers on Banach Algebras

Let us start with the following result proved in [4].

Theorem 1 (see [4, Theorem 2.2]). *Let A be a faithful commutative Banach algebra and let φ be an idempotent homomorphism on A. Then $M_\varphi(A)$ is a Banach algebra. Moreover, if $A^2 = A$ and $\varphi \circ T = T \circ \varphi$ for all $T \in M_\varphi(A)$, then $M_\varphi(A)$ is a faithful commutative Banach algebra.*

Definition 2. Let A be a Banach algebra and let φ be a homomorphism from A to A. The mapping $T : A \to A$ is called a left (resp., right) Jordan φ-multiplier on A if for all $x \in A$

$$T\left(x^2\right) = T(x)\varphi(x) \quad \left(\text{resp., } T\left(x^2\right) = \varphi(x)T(x)\right). \tag{3}$$

T is called a Jordan φ-multiplier on A if it is both a left Jordan φ-multiplier and a right Jordan φ-multiplier on A.

Theorem 3. *Let A be a faithful commutative Banach algebra and let φ be a homomorphism from A to A with dense range. Then T is a φ-multiplier if and only if T is a Jordan φ-multiplier.*

Proof. It is clear that every φ-multiplier is a Jordan φ-multiplier. Conversely, suppose T is a Jordan φ-multiplier. Then

$$\begin{aligned} T\left((x+y)^2\right) &= \varphi(x+y)T(x+y) \\ &= \varphi(x)T(x) + \varphi(x)T(y) \\ &\quad + \varphi(y)T(x) + \varphi(y)T(y) \end{aligned} \tag{4}$$

for all $x, y \in A$.

On the other hand, we have

$$\begin{aligned} T\left((x+y)^2\right) &= T\left(x^2 + 2xy + y^2\right) \\ &= \varphi(x)T(x) + 2T(xy) + \varphi(y)T(y). \end{aligned} \tag{5}$$

Comparing (4), (5) we obtain

$$2T(xy) = \varphi(x)T(y) + \varphi(y)T(x). \tag{6}$$

From (6) and using commutativity of A, for each sequence $\{z_n\}_{n=1}^{\infty} \subset A$ we have

$$\begin{aligned} 2T(xyz_n) &= \varphi(y)T(xz_n) + \varphi(xz_n)T(y) \\ &= \frac{\varphi(y)\left[\varphi(x)T(z_n) + \varphi(z_n)T(x)\right]}{2} \\ &\quad + \varphi(xz_n)T(y), \end{aligned} \tag{7}$$

so we have

$$\begin{aligned} 2T(xyz_n) = {} & [\varphi(y)\varphi(x)T(z_n) + \varphi(y)\varphi(z_n)T(x) \\ & + 2\varphi(x)\varphi(z_n)T(y)] \cdot 2^{-1}; \end{aligned} \tag{8}$$

similarly by using (6) we have

$$\begin{aligned} 2T(xyz_n) = {} & [\varphi(x)\varphi(y)T(z_n) + \varphi(x)\varphi(z_n)T(y) \\ & + 2\varphi(y)\varphi(z_n)T(x)] \cdot 2^{-1}; \end{aligned} \tag{9}$$

comparing (8), (9) we obtain

$$\lim_{n\to\infty}\varphi(x)\varphi(z_n)T(y) = \lim_{n\to\infty}\varphi(y)\varphi(z_n)T(x) \tag{10}$$

for all $x, y, z_n \in A$. Since φ has dense range and A is a faithful commutative Banach algebra, we have

$$\varphi(x)T(y) = T(x)\varphi(y); \tag{11}$$

hence T is a φ-multiplier. □

We mention that Theorem 3 holds for certain noncommutative cases, but not in general. For instance, Zalar has proved in [6] that any left (right) Jordan multiplier on a 2-torsion free semiprime ring is a left (right) multiplier. Vukman [7] has shown that an additive map $\varphi : R \to R$, where R is a 2-torsion free semiprime ring, with the property that $2\varphi(a^2) = a\varphi(a) + \varphi(a)a$ for all $a \in A$, is a multiplier.

The following example shows that, in general, the above theorem need not hold for noncommutative Banach algebras.

Example 4. Consider the subalgebra

$$\mathscr{A} = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in \mathbb{C} \right\} \tag{12}$$

of the algebra of all 3×3 matrices. It is obvious that $\mathscr{A}$ is a Banach algebra with respect to the norm given by

$$\left\| \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \right\| = |a| + |b| + |c| + |d|. \tag{13}$$

Let

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{14}$$

and define a continuous linear map $\varphi : \mathscr{A} \to \mathscr{A}$ by $\varphi(A) = AX + XA$. By a straightforward calculation one can prove that

$$BAX + XAB = BXA + AXB, \quad A, B \in \mathscr{A}. \tag{15}$$

If $'\circ'$ denotes the Jordan product $A \circ B = AB + BA$, then we have $\varphi(A \circ B) = A \circ \varphi(B)$ for each $A, B \in \mathscr{A}$ and hence φ is a right Jordan multiplier. Also $A \circ \varphi(B) = \varphi(1)$ for all $A, B \in \mathscr{A}$ with $A \circ B = 1$. If we consider

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{16}$$

then $\varphi(A) \neq 0$ and $A\varphi(1) = 0$, where $0, 1$ are the zero matrix and the identity matrix, respectively. Thus φ is not a right multiplier.

Lemma 5 (see [8]). *Let A be an amenable Banach algebra and let ψ be a continuous homomorphism of A onto a dense subalgebra of a Banach algebra B. Then B is amenable.*

Theorem 6. *(a) Let A be a unital commutative Banach algebra and let φ be an idempotent homomorphism on A such that φ commutes with each $S \in M_\varphi(A)$. If A is Arens regular then $M_\varphi(A)$ is Arens regular.*

(b) Let A be a commutative Banach algebra and let φ be as in part (a). If A is amenable then $M_\varphi(A)$ is amenable.

Proof. (a) Define $\mu : A \to M_\varphi(A)$ by $\mu(a) = {}_a\varphi$, where ${}_a\varphi(b) = \varphi(ab)$. The homomorphism μ is onto. Namely, if $S \in M_\varphi(A)$, then $S = {}_{S(e)}\varphi$. Of course, $\mu^{**} : A^{**} \to (M_\varphi(A))^{**}$ has the same property, as well. Let $F', G' \in (M_\varphi(A))^{**}$. Then there exist $F, G \in A^{**}$ such that $\mu^{**}(F) = F'$, $\mu^{**}(G) = G'$. Let $\circ$ and $\circ'$ be two Arens multiplications on the second dual A^{**}. Thus,

$$\begin{aligned} F' \circ G' &= \mu^{**}(F) \circ \mu^{**}(G) = \mu^{**}(F \circ G) = \mu^{**}\left(F \circ' G\right) \\ &= \mu^{**}(F) \circ' \mu^{**}(G) = F' \circ' G'. \end{aligned} \tag{17}$$

(b) Let $T \in M_\varphi(A)$ and $\{e_\alpha : \alpha \in I\}$ be a bounded approximate identity in A (see [9]). A simple computation shows that $T = \lim_\alpha {}_{T(e_\alpha)}\varphi$, which means $\overline{\mu(A)} = M_\varphi(A)$. So by Lemma 5 we conclude that $M_\varphi(A)$ is amenable. □

Theorem 7. *Let A be a unital Banach algebra and let $\varphi : A \to A$ be a spectrum preserving homomorphism with dense range. Then each $T \in M_\varphi(A)$ is spectrum preserving.*

Proof. Let $a \in A$ and $\lambda \notin \sigma(T(a))$. Since φ has dense range, there exists a sequence $\{c_n\}_n \in A$ such that $(T(a) - \lambda)\lim_n \varphi(c_n) = 1$. Thus

$$\begin{aligned} &(\varphi(a) - \lambda) \lim_n T(c_n) \\ &\quad = \lim_n (\varphi(a) - \lambda) T(c_n) = \lim_n \varphi(a - \lambda) T(c_n) \\ &\quad = \lim_n T(a - \lambda) \varphi(c_n) = (T(a) - \lambda) \lim_n \varphi(c_n) \\ &\quad = 1. \end{aligned} \tag{18}$$

Similarly $\lim_n T(c_n)(\varphi(a) - \lambda) = 1$. Thus $\lambda \notin \sigma(\varphi(a))$. Since $\sigma(\varphi(a)) = \sigma(a)$, we have $\lambda \notin \sigma(a)$.

Now, let $\lambda \notin \sigma(a)$. Then there exists $b \in A$ such that $(a - \lambda)b = 1$. Thus

$$(T(a) - \lambda)\varphi(b) = T(a - \lambda)\varphi(b) = T((a - \lambda)b) = 1. \tag{19}$$

Similarly $\varphi(b)(T(a) - \lambda) = 1$. Hence $\lambda \notin \sigma(T(a))$, which means $\sigma(T(a)) = \sigma(a)$. □

3. φ-Multipliers on Topological Modules and Their Properties

Now, we consider φ-multipliers in the general topological module setting and investigate some of their properties.

Definition 8. Let A be a topological algebra and let X, Y be two topological A-bimodules and let φ be a nonzero and continuous idempotent A-module homomorphism on X. A linear and bounded mapping $T : X \to Y$ is called a left (resp., right) φ-multiplier if $T(a \cdot x) = T(\varphi(x)) \cdot a$ (resp., $T(x \cdot a) = a \cdot T(\varphi(x))$) for all $a \in A, x \in X$. We say T is a φ-multiplier if it is both a left φ-multiplier and a right φ-multiplier.

We denote by $M_\varphi(X, Y)$ (resp., $M_\varphi^l(X, Y), M_\varphi^r(X, Y)$) the collection of all φ-multipliers (resp., left φ-multipliers, right φ-multipliers).

It is easy to check that $\varphi \in M_\varphi(X, X)$. So $M_\varphi(X, X) \neq \{0\}$.

Example 9. Let A be a topological algebra, X an A-bimodule, and φ an idempotent A-module homomorphism on X. For each $a \in A$ the mapping ${}_a\varphi : X \to X$ defined by ${}_a\varphi(x) = a \cdot \varphi(x)$ is a left φ-multiplier on X.

Proof. Let $a, b \in A$ and $x \in X$,

$$\begin{aligned} {}_a\varphi(b \cdot x) &= a \cdot \varphi(b \cdot x) = a \cdot b \cdot \varphi(x) = b \cdot a \cdot \varphi(\varphi(x)) \\ &= b \cdot {}_a\varphi(\varphi(x)), \\ {}_a\varphi(x \cdot b) &= a \cdot \varphi(x \cdot b) = a \cdot \varphi(x) \cdot b = a \cdot \varphi(\varphi(x)) \cdot b \\ &= {}_a\varphi(\varphi(x)) \cdot b. \end{aligned} \tag{20}$$

Hence ${}_a\varphi \in M_\varphi(X, X)$. □

In the sequel, A denotes a topological algebra and X, Y are two topological A-bimodules. In general, φ is an A-module homomorphism on X such that it is also idempotent, linear, and continuous. Sometimes φ is on A; it will be mentioned when this happens.

Lemma 10. *$M_\varphi^l(X, Y)$ is a left A-module.*

Proof. $M_\varphi^l(X, Y)$ denotes the vector space of all left φ-multipliers from X to Y. Let $T \in M_\varphi^l(X, Y)$ and $a \in A$ be arbitrary. Define $a * T$ as $(a * T)(x) := T(x \cdot a)$ where $x \in X$ is arbitrary. Since the equalities

$$\begin{aligned} (a * T)(b \cdot x) &= T(b \cdot x \cdot a) = b \cdot T(\varphi(x \cdot a)) \\ &= b \cdot T(\varphi(x) \cdot a) = b \cdot (a * T)(\varphi(x)) \end{aligned} \tag{21}$$

hold for all $b \in A$ and $x \in X$, we conclude that $a * T$ is a left φ-multiplier. Then, since X is an A-bimodule and T is linear, $M^l_\varphi(X, Y)$ is a left A-module. □

Definition 11. An A-bimodule X is said to be commutative if $a \cdot x = x \cdot a$ holds for all $a \in A$ and $x \in X$.

Definition 12 (see [10, 11]). Let X be a left (resp., right) A-module. A is said to be left (resp., right) faithful in X if, for any $x \in A$, $a \cdot x = 0$ (resp., $x \cdot a = 0$) for all $a \in A$ implies that $x = 0$. If X is an A-bimodule then A is said to be faithful in X if it is both left and right faithful in X.

The following definition generalizes Definition 12.

Definition 13. Let X be a left (resp., right) A-module. A is said to be left (resp., right) φ-faithful in X if, for any $x \in A$, $\varphi(a) \cdot x = 0$ (resp., $x \cdot \varphi(a) = 0$) for all $a \in A$ implies that $x = 0$. If X is an A-bimodule then A is said to be φ-faithful in X if it is both left and right φ-faithful in X.

Definition 14. Let A be a topological algebra, X a commutative A-bimodule, and φ an idempotent A-module homomorphism on A. For any $x \in X$, define

$$_x\varphi : A \longrightarrow X \quad \text{by} \quad {}_x\varphi(a) = x \cdot \varphi(a), \ a \in A. \qquad (22)$$

It is easy to see that $_x\varphi \in M^l_\varphi(A, X)$. Now, we define $\psi : X \to M^l_\varphi(A, X)$, by $\psi(x) = {}_x\varphi$.

Lemma 15. *Let A be a topological algebra with an approximate identity $\{e_\lambda : \lambda \in I\}$ and let X be a topological A-bimodule. If A is φ-faithful in X and $\psi : X \to M^l_\varphi(A, X)$ is onto, then A is left faithful in $M^l_\varphi(A, X)$.*

Proof. Let $T \in M^l_\varphi(A, X)$. In view of Lemma 10, it is enough to show that $T = 0$ if $a * T = 0$ for all $a \in A$. Since ψ is onto, there exist $x \in X$ such that $T = {}_x\varphi$. Therefore for any $b \in A$ and $\lambda \in I$,

$$x \cdot \varphi(b \cdot e_\lambda) = {}_x\varphi(b \cdot e_\lambda) = (e_\lambda * {}_x\varphi)(b) = 0. \qquad (23)$$

The continuity of φ implies that $x \cdot \varphi(b) = 0$. Now, since A is φ-faithful in X, we conclude that $x = 0$. Hence $T = {}_x\varphi = 0$. □

Definition 16. Let A be a Hausdorff topological algebra and (X, τ) a Hausdorff topological A-bimodule. Let φ be an A-module homomorphism on A. The φ-uniform operator topology u_φ (resp., φ-strong operator topology s_φ) on $M^l_\varphi(A, X)$ is defined as the linear topology which has a base of neighborhoods of 0 consisting of all sets of the form

$$N(\varphi(B), V) = \{T \in M^l_\varphi(A, X) : T(\varphi(B)) \subseteq V\}, \qquad (24)$$

where B is a bounded (resp., finite) subset of A and V is a neighborhood of 0 in X. Clearly $s_\varphi \le u_\varphi$.

Theorem 17. *Let φ be an A-module homomorphism on A and let (X, τ) be a topological A-bimodule with $b(A)$-hypocontinuous module multiplication.*

Then $(M^l_\varphi(A, X), s_\varphi)$ and $(M^l_\varphi(A, X), u_\varphi)$ are topological left A-modules.

Proof. By Lemma 10, $M^l_\varphi(A, X)$ is a left A-module. Now, let us prove that the module multiplication $(a, T) \to a * T$ from $A \times M^l_\varphi(A, X)$ into $M^l_\varphi(A, X)$ is separately continuous in u_φ-topology. Let $T \in M^l_\varphi(A, X)$ and $\{a_\alpha : \alpha \in I\}$ be a net in A with $a_\alpha \to a \in A$ and let D be a bounded subset of A and let V be a neighborhood of 0 in X. By $b(A)$-hypocontinuity, there exists a balanced neighborhood H of 0 in X such that $\varphi(D) \cdot H \subset V$. Since T and φ are continuous, there exist $\alpha_0 \in I$ such that

$$\begin{aligned}(a_\alpha * T)(\varphi(b)) - (a * T)(\varphi(b)) &= T(\varphi(b) \cdot a_\alpha) - T(\varphi(b) \cdot a) \\ &= \varphi(b) \cdot [T(\varphi(a_\alpha)) - T(\varphi(a))] \\ &\in \varphi(D) \cdot H \subset V\end{aligned} \qquad (25)$$

for all $b \in D$ and $\alpha \ge \alpha_0$. Hence $a_\alpha * T \to_{u_\varphi} a * T$.

Next, let $a \in A$ and $\{T_\alpha : \alpha \in I\}$ be a net in $M^l_\varphi(A, X)$ such that $T_\alpha \to_{u_\varphi} T \in M^l_\varphi(A, X)$ and let D be a bounded subset of A and let V be a neighborhood of 0 in X. Since the mappings φ and $R_a(x) = xa$ are continuous, it follows that $\varphi(D) \cdot a$ is a bounded subset in A. So there exist $\alpha_0 \in I$ such that

$$\begin{aligned}(a * T_\alpha - a * T)(\varphi(b)) &= T_\alpha(\varphi(b) \cdot a) - T(\varphi(b) \cdot a) \\ &= (T_\alpha - T)(\varphi(D) \cdot a) \subseteq V\end{aligned} \qquad (26)$$

for all $\alpha \ge \alpha_0$ and $b \in D$. Hence $a * T_\alpha \to_{u_\varphi} a * T$. That means $(M^l_\varphi(A, X), u_\varphi)$ is a left topological module. A similar computation shows that $(M^l_\varphi(A, X), s_\varphi)$ is a left topological module. □

Lemma 18. *Let A be a topological algebra with an approximate identity $\{e_\lambda : \lambda \in I\}$ and let (X, τ) be a commutative A-bimodule and φ an idempotent A-module homomorphism on A. Then $\overline{\psi(X)}^{s_\varphi} = M^l_\varphi(A, X)$.*

Proof. Let $T \in M^l_\varphi(A, X)$, and let B be a finite subset of A and let V be a neighborhood of 0 in X. For each $a \in A$ we have $e_\lambda\varphi(a) \to \varphi(a)$. Then, since T is continuous and B is finite, there exist $\lambda_0 \in I$ such that $T(e_\lambda\varphi(a)) - T(\varphi(a)) \in V$, for all $a \in B$ and $\lambda \ge \lambda_0$. Then, for any $a \in B$ and $\lambda \ge \lambda_0$

$$\begin{aligned}{}_{\{T(\varphi(e_\lambda))\}_\lambda}\varphi(\varphi(a)) - T(\varphi(a)) &= T(\varphi(e_\lambda)) \cdot \varphi(\varphi(a)) - T(\varphi(a)) \\ &= T(e_\lambda\varphi(a)) - T(\varphi(a)) \in V.\end{aligned} \qquad (27)$$

Therefore $_{\{T(\varphi(e_\lambda))\}_\lambda}\varphi \to_{s_\varphi} T$. □

Theorem 19. *Let A be a topological algebra with an approximate identity $\{e_\lambda : \lambda \in I\}$ and let X be a commutative* *topological A-bimodule such that (X,τ) is complete. Suppose φ is an idempotent A-module homomorphism on A and A is φ-faithful in X. Then (X,τ) is isomorphic to $(M^l_\varphi(A,X),s_\varphi)$.*

Proof. Let ψ be as in Definition 14. It is obvious that ψ is a continuous module homomorphism. We first show that ψ is onto. In view of Lemma 18, it is enough to prove that $\psi(X)$ is s_φ-closed. Let $T\in\overline{\psi(X)}^{s_\varphi}$. Then there exists a net $\{x_\alpha\}_\alpha\subseteq X$ such that ${}_{x_\alpha}\varphi\rightarrow_{s_\varphi}T$. It follows that the net $\{x_\alpha\cdot\varphi(a)\}_\alpha=\{{}_{x_\alpha}\varphi(\varphi(a))\}_\alpha$ is τ-Cauchy in X for every $a\in A$. Now, since A is φ-faithful in X, the net $\{x_\alpha\}$ is τ-Cauchy in X. By completeness of (X,τ), there exist $x\in X$ such that $x_\alpha\rightarrow x$. Hence ${}_{x_\alpha}\varphi\rightarrow_{s_\varphi}{}_x\varphi$. By uniqueness of limit in Hausdorff space $T={}_x\varphi$. Therefore $\psi(X)$ is s_φ-closed.

To show that ψ is one-to-one, let $x,y\in X$ such that ${}_x\varphi={}_y\varphi$. Then for any $a\in A$, $(x-y)\cdot\varphi(a)=0$. Since A is φ-faithful in X, this implies that $x=y$. Thus ψ is one-to-one. □

Definition 20. Let A be a topological algebra and let X be a topological A-bimodule. The uniform topology γ_φ (strict topology β_φ) on $M^l_\varphi(A,X)$ is defined as the linear topology which has a base of neighborhoods of 0 consisting of all sets

$$N'(\varphi(D),G)=\left\{T\in M^l_\varphi(A,X):\varphi(D)*T\subset G\right\},\qquad(28)$$

where D is a bounded (finite) subset of A and G is a neighborhood of 0 in $(M^l_\varphi(A,X),u_\varphi)[(M^l_\varphi(A,X),s_\varphi)]$.

Lemma 21. *Let A be a topological algebra with a bounded approximate identity $\{e_\lambda:\lambda\in I\}$ and let X be a topological A-bimodule. Then $u_\varphi=\gamma_\varphi$ and $s_\varphi=\beta_\varphi$.*

Proof. Let $\{T_\alpha\}_\alpha$ be a net in $M^l_\varphi(A,X)$ with $T_\alpha\rightarrow_{u_\varphi}T$. Let $G=N(\varphi(C),V)$ be a neighborhood of 0 in u_φ-topology. Since φ is continuous, $\varphi(C)\varphi(D)$ is a bounded subset of A for each bounded subset D of A. Then there exist α_0 such that

$$(\varphi(D)*(T_\alpha-T))(\varphi(C))=(T_\alpha-T)(\varphi(C)\varphi(D))\in V\qquad(29)$$

for all $\alpha\geq\alpha_0$. That means $\varphi(D)*(T_\alpha-T)\in G$. Hence $T_\alpha\rightarrow_{\gamma_\varphi}T$.

Conversely, let $\{T_\alpha\}_\alpha$ be a net in $M^l_\varphi(A,X)$ with $T_\alpha\rightarrow_{\gamma_\varphi}T$. Let D be a bounded subset of A and let V be a closed neighborhood of 0 in X. Choose $C=\{e_\lambda\}_\lambda$. Then there exist α_0 such that

$$\begin{aligned}(T_\alpha-T)(\varphi(D))&=\lim_\lambda(T_\alpha-T)(\varphi(C)\varphi(D))\\&=\lim_\lambda(\varphi(D)*(T_\alpha-T))(\varphi(C))\in V\end{aligned}\qquad(30)$$

for all $\alpha\geq\alpha_0$. That means $T_\alpha-T\in N(\varphi(D),V)$. Hence $T_\alpha\rightarrow_{u_\varphi}T$. □

At the end we characterize the φ-multipliers on $L_p(G)$, where G is a compact Abelian group. Of course, $L_p(G)$ is a Banach algebra and several authors studied its multipliers. For instance, Larsen [5] showed that a linear transformation $T:L_p(G)\rightarrow L_p(G)$, where G is a locally compact Abelian group, is a multiplier if and only if there exists a unique $\varphi\in L_\infty(\widehat{G})$ such that $\widehat{Tf}=\varphi\widehat{f}$ for each $f\in L_p(G)$.

However, we now consider $L_p(G)$ as a left Banach module over the group algebra $L_1(G)$. Namely, the algebra $L_1(G)$ acts on $L_p(G)$ through the convolution $L_1(G)*L_p(G)=L_p(G)$.

Example 22. Let G be a compact Abelian group and let $\varphi:L_1(G)\rightarrow L_p(G)$ be an idempotent $L_1(G)$-module homomorphism with dense range. If $T:L_p(G)\rightarrow L_p(G)$ is a φ-multiplier then there exists a unique function $H_T\in L_p(G)$ such that

$$T(\varphi(f))=\varphi(f)*H_T,\quad(f\in L_1(G)).\qquad(31)$$

Proof. Let $\{e_\beta\}_\beta$ be a bounded approximate identity in $L_1(G)$. Then $\{T(\varphi(e_\beta))\}_\beta\subseteq\text{Ball}(L_q(G))^*$. By the Alaoglu theorem, there exists a function $H_T\in L_p(G)$ such that $T(\varphi(e_\beta))\rightarrow_{\text{weak}*}H_T$. Then for each $f\in L_1(G)$

$$\varphi(f)*T(\varphi(e_\beta))\longrightarrow_{\text{weak}*}\varphi(f)*H_T.\qquad(32)$$

On the other hand, since T is a φ-multiplier,

$$\varphi(f)*T(\varphi(e_\beta))=T(\varphi(f*e_\beta))\longrightarrow T(\varphi(f))\qquad(33)$$

for each $f\in L_1(G)$. By uniqueness of limit, $T(\varphi(f))=\varphi(f)*H_T$.

To show that H_T is unique, let ψ be a second function in $L_p(G)$ such that $T(\varphi(f))=\varphi(f)*\psi$ for each $f\in L_1(G)$. Since φ has dense range, $T(\varphi(f))=f*\psi$ for each $f\in L_1(G)$. Therefore

$$\widehat{f}(\gamma)\,(\widehat{\psi-H_T})(\gamma)=0\qquad(34)$$

for each $f\in L_1(G)$ and $\gamma\in\widehat{G}$. By compactness of G, for each $\gamma\in\widehat{G}$ there exist $f\in L_1(G)$ such that $\widehat{f}(\gamma)\neq0$. Hence the semisimplicity of $L_p(G)$ implies that $H_T=\psi$. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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