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Research Article

Generalized Solutions for Nonlocal Elliptic Equations and Systems with Nonlinear Singularities

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We use the topological degree method to study the existence of solutions for nonlocal elliptic equations (systems) with a strong singular nonlinearity.

1. Introduction and Main Results

Given $s \in (0,1)$, an integer $n > 2s$, and a bounded open set Ω of $\mathbb{R}^n$ with Lipschitz boundary, let $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ be a function satisfying the following properties:

(i) $\gamma K \in L^1(\mathbb{R}^n)$ with $\gamma(x) = \min\{|x|^2, 1\}$.

(ii) There exists $\theta > 0$ such that $K(x) \geq \theta |x|^{-(n+2s)}$ for any $x \in \mathbb{R}^n \setminus \{0\}$.

(iii) $K(x) = K(-x)$ for any $x \in \mathbb{R}^n \setminus \{0\}$.

The so-called nonlocal elliptic operator $\mathfrak{L}_K$ is defined by

$$\mathfrak{L}_K u(x) = \frac{1}{2}\int_{\mathbb{R}^n} \left(u(x+y) + u(x-y) - 2u(x)\right) K(y)\, dy, \quad x \in \mathbb{R}^n. \tag{1}$$

In particular, when $K(x) = |x|^{-(n+2s)}$, $\mathfrak{L}_K$ is equal to the fractional Laplace operator $-(-\Delta)^s$ (up to normalization factors).

For a Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$, the following problem

$$\begin{gathered} \mathfrak{L}_K u + f(x,u) = 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega \end{gathered} \tag{2}$$

and its special case

$$\begin{gathered} (-\Delta)^s u = f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega \end{gathered} \tag{3}$$

have been widely studied under various contexts; see a recent survey [1] for details.

1.1. Previous Work. Motivated by the work of Caffarelli and Silvestre [2], several authors have considered an equivalent problem of (3) by means of an auxiliary variable; see [2–6]. Precisely, let (x,y) denote the points in $\mathscr{C}_\Omega := \Omega \times (0,\infty) \subset \mathbb{R}^{n+1}_+$ and $\partial_L \mathscr{C}_\Omega = \partial\Omega \times (0,\infty)$. Take $\alpha = 2s$ and $X_0^\alpha(\mathscr{C}_\Omega)$ as the completion of $C_0^\infty(\Omega \times (0,\infty))$ with respect to the norm

$$\|z\|_{X_0^\alpha(\mathscr{C}_\Omega)} = \left(\kappa_\alpha \int_{\mathscr{C}_\Omega} y^{1-\alpha} |\nabla z|^2\, dx\, dy\right)^{1/2}, \tag{4}$$

where κ is a normalization constant. For $w \in X_0^\alpha(\mathscr{C}_\Omega)$, let

$$\begin{gathered} L_\alpha w := -\operatorname{div}\left(y^{1-\alpha}\nabla w\right), \\ \frac{\partial w}{\partial \nu^\alpha} := \kappa_\alpha \lim_{y\to 0^+} y^{1-\alpha}\frac{\partial w}{\partial y} \end{gathered} \tag{5}$$

and consider the problem

$$\begin{aligned} L_\alpha w &= 0 \quad \text{in } \mathscr{C}_\Omega, \\ w &= 0 \quad \text{in } \partial_L \mathscr{C}_\Omega, \\ \frac{\partial w}{\partial \nu^\alpha} &= f(x, w) \quad \text{in } \Omega \times \{y = 0\}. \end{aligned} \tag{3$'$}$$

An energy solution to this problem is a function $w \in X_0^\alpha(\mathscr{C}_\Omega)$ such that

$$\kappa_\alpha \int_{\mathscr{C}_\Omega} y^{1-\alpha} \langle \nabla w, \nabla \varphi \rangle \, dx\, dy = \int_\Omega f(x, w) \varphi \, dx \qquad \forall \varphi \in X_0^\alpha(\mathscr{C}_\Omega). \tag{6}$$

Such an energy solution w yields a function $u = w(\cdot, 0)$ in the sense of traces, which belongs to the space $H_0^{\alpha/2}(\Omega)$ and is a weak solution of (3). The converse is also true. The reader may refer to [2–6] for dealing with (3) with this method. In particular, Stinga and Torrea [6] generalized the arguments and results in [2] to the fractional powers L^σ, $0 < \sigma < 1$, of a linear second order partial differential operator L that is nonnegative, densely defined, and self-adjoint in $L^2(\Omega, d\eta)$ with a positive measure $d\eta$ on Ω.

Servadei and Valdinoci developed a variational framework to study the problem (2) in a series of papers [7–11]. They introduced the following Hilbert space $(X_0(\Omega, K), \langle \cdot, \cdot \rangle_{0,\Omega,K})$ in [7, 8]. Let $Q := \mathbb{R}^{2n} \setminus (\mathscr{C}\Omega \times \mathscr{C}\Omega)$, where $\mathscr{C}\Omega = \mathbb{R}^n \setminus \Omega$, and let $X(\Omega, K)$ be the space of all Lebesgue measurable functions $u : \mathbb{R}^n \to \mathbb{R}$ such that $u|_\Omega \in L^2(\Omega)$ and that the map

$$Q \ni (x, y) \longmapsto (u(x) - u(y)) \sqrt{K(x - y)} \text{ is in } L^2(Q). \tag{7}$$

$X(\Omega, K)$ is a Banach space endowed with the so-called *Gagliardo norm*

$$\|u\|_{\Omega,K} = \left(\int_\Omega |u(x)|^2 \, dx + \int_Q |u(x) - u(y)|^2 K(x - y) \, dx\, dy \right)^{1/2} \tag{8}$$

and is contained in $H^s(\mathbb{R}^n)$. Consider the subspace of $X(\Omega, K)$:

$$X_0(\Omega, K) = \{u \in X(\Omega, K) \mid u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}. \tag{9}$$

It was proved in [12, Theorem 6] that this space is the closure of $C_0^\infty(\Omega)$ in $X(\Omega, K)$. Clearly, the space $X_0(\Omega, K)$ depends on K. In fact, when $K(x) = |x|^{-(n+2s)}$, $X_0(\Omega, K) = \{u \in H^s(\mathbb{R}^n) \mid u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}$ ([11, Lemma 7-b]). $X_0(\Omega, K)$ can be endowed with a Hilbert space structure given by the inner product

$$\langle u, v \rangle_{0,\Omega,K} = \int_Q (u(x) - u(y))(v(x) - v(y)) K(x - y) \, dx\, dy \tag{10}$$

([8, Lemma 7]) and contains $C_0^2(\Omega)$ ([7, Lemma 11]). Denote by $\|\cdot\|_{0,\Omega,K}$ the induced norm of the inner product in (10). This norm is equivalent to the restriction of $\|\cdot\|_{\Omega,K}$ to $X_0(\Omega, K)$. Call $u \in X_0(\Omega, K)$ a *weak solution* of the problem (2) if u satisfies

$$\int_{\mathbb{R}^{2n}} (u(x) - u(y))(\phi(x) - \phi(y)) K(x - y) \, dx\, dy = \int_\Omega f(x, u(x)) \phi(x) \, dx \tag{11}$$

for all $\phi \in X_0(\Omega, K)$. Define $F(x, t) = \int_0^t f(x, \tau) d\tau$ for the above Carathéodory function f. Suppose that there exist $a_1 > 0$, $a_2 > 0$ and $q \in (2, 2_s^*)$, $2_s^* = 2n/(n - 2s)$, such that

$$|f(x, t)| \le a_1 + a_2 |t|^{q-1} \quad \text{a.e. } x \in \Omega,\ t \in \mathbb{R}. \tag{12}$$

Then, the functional $\mathscr{J} : X_0(\Omega, K) \to \mathbb{R}$ defined by

$$\mathscr{J}(u) = \frac{1}{2} \int_Q |u(x) - u(y)|^2 K(x - y) \, dx\, dy - \int_\Omega F(x, u(x)) \, dx \tag{13}$$

is of class C^1 and the critical points of $\mathscr{J}$ are exactly the weak solutions of (2). Condition (12) is always assumed in the proofs of present several existence results on (2) via variational methods [7–11, 13–15]. Except for [13] $u = 0$ is also assumed to be a solution in all other works. For studies of (3), there is a great deal of literature; see [2, 4, 16] and references therein.

However, all previous results cannot include the following case: $\Omega = B^n(0, 1) = \{x \in \mathbb{R}^n \mid |x| < 1\}$ and $f(x, t) = 1/(|x| - 1) + h(t)$, where $h \in C^1(\mathbb{R})$ such that $h(t) = \cos t$ for $|t| \ll 1$ and that $h(t) = \sin t$ for $|t| \gg 1$.

1.2. Main Results of This Paper. We will use the topological degree theory developed by [17] to study generalized solutions of problem (2).

Our result can apply to the example just mentioned. Without special statements, we write

$$r = \frac{2n}{(n + 2s)}, \qquad 2_s^* = \frac{2n}{(n - 2s)} \tag{14}$$

(the latter plays the role of a critical Sobolev exponent).

Theorem 1. *For an integer $n \ge 2$ let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ satisfy (i)–(iii) and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function verifying the following conditions: there exist positive numbers α, β, $q \in (2, 2_s^*)$, $p \in [1, 2)$ and functions $a \in L^r_{\mathrm{loc}}(\Omega)$ and $c \in L^1(\Omega)$ such that*

$$|f(x, t)| \le a(x) + \alpha |t|^{q-1} \quad \forall (x, t) \in \Omega \times \mathbb{R}, \tag{15}$$

$$\Omega \ni x \longmapsto b(x) := f(x, 0) \ \textit{belongs to } L^r(\Omega), \tag{16}$$

$$-f(x, t) t \ge -\beta |t|^p - c(x) \quad \forall (x, t) \in \Omega \times \mathbb{R}. \tag{17}$$

Then, problem (2) has at least one generalized solution u in $X_0(\Omega, K)$; that is, it satisfies

$$\int_{\mathbb{R}^{2n}} (u(x)-u(y))(\phi(x)-\phi(y)) K(x-y)\,dx\,dy = \int_\Omega f(x,u(x))\phi(x)\,dx \quad \forall \phi \in C_0^\infty(\Omega). \tag{18}$$

In particular, it must have a nontrivial generalized solution if b is not identically zero.

Corollary 2. *Under the assumptions of Theorem 1, let $1<\nu<2$, $l>2n/(2n-(n-2s)\nu)$ and let $G \in L^l_{\mathrm{loc}}(\Omega)$. If either $\lambda G \le 0$ or $(\lambda G)^+ := \max\{0, \lambda G\}$ belongs to $L^\kappa(\Omega)$ with $\kappa > 2/(2-\nu)$, then*

$$\begin{aligned} \mathfrak{L}_K u + \lambda G(x)|u|^{\nu-2}u + f(x,u) &= 0 \quad \text{in } \Omega,\\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega, \end{aligned} \tag{19}$$

has at least one nontrivial solution in $X_0(\Omega, K)$ provided b is not identically zero.

Corollary 3. *Under the assumptions of Theorem 1, let $2 \le \nu < 2_s^*$ and $G \in L^l_{\mathrm{loc}}(\Omega)$ with $l \ge r\vartheta$ and $\vartheta > (n+2s)/(n+2s-(\nu-1)(n-2s))$. If $\lambda \in \mathbb{R}$ is such that $\lambda G \le 0$, then*

$$\begin{aligned} \mathfrak{L}_K u + \lambda G(x)|u|^{\nu-2}u + f(x,u) &= 0 \quad \text{in } \Omega,\\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega, \end{aligned} \tag{20}$$

has at least one nontrivial solution in $X_0(\Omega, K)$ provided b is not identically zero.

In particular, this corollary includes the example at the end of Section 1.2. See Example 1 for more general cases.

Our methods can also be used to study the case of nonlocal elliptic operator systems. Let $K_1, K_2 : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ be functions satisfying conditions (i)–(iii). Given two Carathéodory functions $f_i : \Omega \times \mathbb{R} \to \mathbb{R}$, $i=1,2$, consider the following problem:

$$\begin{aligned} \mathfrak{L}_{K_1} u + f_1(x,v) &= 0 \quad \text{in } \Omega,\\ \mathfrak{L}_{K_2} v + f_2(x,u) &= 0 \quad \text{in } \Omega,\\ u = v &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega. \end{aligned} \tag{21}$$

Call $(u,v) \in X_0(\Omega, K_1) \times X_0(\Omega, K_2)$ a *generalized solution* of system (21) if

$$\begin{aligned} &\int_{\mathbb{R}^{2n}} (u(x)-u(y))(\phi(x)-\phi(y)) K_1(x-y)\,dx\,dy \\ &+ \int_{\mathbb{R}^{2n}} (v(x)-v(y))(\psi(x)-\psi(y)) \\ &\cdot K_2(x-y)\,dx\,dy - \int_\Omega f_1(x,v)\phi(x)\,dx \\ &- \int_\Omega f_2(x,u)\psi(x)\,dx = 0 \end{aligned} \tag{22}$$

for every $(\phi, \psi) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)$. Here is the second main result.

Theorem 4. *Under the above assumptions, suppose also that there exist positive numbers α_j, β_j, $q_j \in (2, 2_s^*)$, $1 \le p_j, \widehat{p}_j < 2$, and functions $a_j \in L^r_{\mathrm{loc}}(\Omega)$ and $c_j \in L^1(\Omega)$ such that*

$$\left|f_j(x,t)\right| \le a_j(x) + \alpha_i |t|^{q_j-1} \quad \forall (x,t) \in \Omega \times \mathbb{R}, \tag{23}$$

$$\begin{aligned} &\Omega \ni x \longmapsto b_j(x) \\ &\quad := f_j(x,0) \ \text{belongs to } L^r(\Omega), \end{aligned} \tag{24}$$

$$\begin{aligned} &- f_j(x,t)h \ge -\beta_j |t|^{p_j/2} |h|^{\widehat{p}_j/2} - c_j(x) \\ &\qquad \forall (x,t,h) \in \Omega \times \mathbb{R} \times \mathbb{R}. \end{aligned} \tag{25}$$

Then, problem (21) has at least one generalized solution in $X_0(\Omega, K_1) \times X_0(\Omega, K_2)$. In particular, it must have a nontrivial generalized solution if one of b_1 and b_2 is not identically zero.

Similarly, consider the following problem:

$$\begin{aligned} &-\Delta u(x) + \sum_{j=1}^n f_j(x, v(x)) \frac{\partial u}{\partial x_j}(x) + f_0(x, v(x)) \\ &+ a_1(x) = 0, \\ &-\Delta v(x) + \sum_{j=1}^n g_j(x, u(x)) \frac{\partial v}{\partial x_j}(x) + g_0(x, u(x)) \\ &+ a_2(x) = 0 \end{aligned} \tag{26}$$

or

$$\begin{aligned} &-\Delta u(x) + \sum_{j=1}^n f_j(x, v(x)) \frac{\partial v}{\partial x_j}(x) + f_0(x, v(x)) \\ &+ a_1(x) = 0, \\ &-\Delta v(x) + \sum_{j=1}^n g_j(x, u(x)) \frac{\partial u}{\partial x_j}(x) + g_0(x, u(x)) \\ &+ a_2(x) = 0. \end{aligned} \tag{27}$$

Assume

(A) $f_i, g_i : \Omega \times \mathbb{R}^n \to \mathbb{R}$ are Carathéodory functions, $i = 0, \ldots, n$;

(B) there exist constants

$$\begin{aligned} &r_0 \in \left(\frac{2n}{n+2}, \infty\right), \\ &r_i \in (n, \infty), \quad \frac{n-2}{2n} r_i < \frac{1}{s_i} < \infty, \ i = 1, \ldots, n, \end{aligned} \tag{28}$$

and measurable functions $b_i, d_i \in L^{r_i}_{\mathrm{loc}}(\Omega)$, $i = 0, 1, \ldots, n$, $a_1, a_2 \in L^{r_0}(\Omega)$, such that

$$\begin{aligned} & f_0(x,0) = g_0(x,0) = 0, \quad \forall x \in \Omega, \\ & |f_i(x,t)| \le b_i(x) + k_i |t|^{s_i}, \\ & \qquad \forall (x,t) \in \Omega \times \mathbb{R},\ i = 0, \ldots, n, \\ & |g_i(x,t)| \le d_i(x) + l_i |t|^{s_i}, \\ & \qquad \forall (x,t) \in \Omega \times \mathbb{R},\ i = 0, \ldots, n; \end{aligned} \tag{29}$$

(C) there exist measurable functions $c(x), \widehat{c}(x) \in L^1(\Omega)$ and constants $1 < q_1, q_2, \widehat{q}_1, \widehat{q}_2 < 2$ such that, for any $(x, t_1, t_2, z) \in \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$,

$$\begin{aligned} & -\frac{1}{2}|z|^2 - k|t_1|^{q_1/2}|t_2|^{\widehat{q}_1/2} c(x) \\ & \quad \le \left[\sum_{i=1}^n f_i(x,t_2) z_i + f_0(x,t_2) + a_1(x)\right] t_1, \\ & -\frac{1}{2}|z|^2 - \widehat{k}|t_1|^{q_2/2}|t_2|^{\widehat{q}_2/2} \widehat{c}(x) \\ & \quad \le \left[\sum_{i=1}^n g_i(x,t_2) z_i + g_0(x,t_2) + a_2(x)\right] t_1. \end{aligned} \tag{30}$$

Combing the proof of [17] and that of Theorem 4, we can prove the following.

Theorem 5. *Under the conditions (A), (B), and (C), if a_1 or a_2 is not zero, then the equation systems (26) and (27) have at least a nontrivial generalized solution $(u, v) \in W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$.*

Finally, let us point out that the corresponding results of Theorems 1 and 4 can be also proved if the operator $\mathfrak{L}_K$ is replaced by L^σ in [6, (1.10)]. They will be given in other places.

The arrangements of this paper are as follows. In Section 2, we give some necessary preliminaries. The proof of Theorem 1 will be completed in Section 3. In Section 4, we will prove Corollaries 2 and 3 and give an example. Theorem 4 will be proved in Section 5.

2. Preliminaries

Firstly, we review the topological degree theory for mappings of class (B_+) developed in [17]. Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let $\{E_n\}_n$ be a strictly increasing sequence of finite dimensional subspace of H such that $E = \cup_{n=1}^\infty E_n$ is dense in H. Denote by P_n the orthogonal projection from H onto E_n for every integer $n \in \mathbb{N}$. Let G be an open bounded set in E and let g be a mapping from $\overline{G}^E$, the closure of G in E, into H. Put

$$\begin{aligned} & G_n = G \cap E_n \quad \forall n \in \mathbb{N}, \\ & g_n(x) = P_n(g(x)) \quad \forall n \in \mathbb{N},\ x \in \overline{G_n}. \end{aligned} \tag{31}$$

Since H and E induce equivalent topologies on all finite dimensional spaces E_n the subset $G_n \subset E_n$ has the same closure in E_n, E, and H, denoted by $\overline{G_n}$.

Definition 6. Under the above assumptions, g is said to be of *class* (B_+) on $\overline{G}^E$ if and only if the following conditions are satisfied:

(a) $g_n : \overline{G_n} \to E_n$ is a continuous mapping for each $n \in \mathbb{N}$.

(b) There is not any sequence $\{x_{n_k}\}_k$ in E such that the sequence $\{x_{n_k}\}_k$ is weakly convergent in H, $x_{n_k} \in \partial_{E_{n_k}} G_{n_k}$, $\langle g(x_{n_{k+1}}), x_{n_{k+1}} \rangle \le 0$ and $\langle g(x_{n_{k+1}}), v \rangle = 0$ for all $k \in \mathbb{N}$ and v in E_{n_k}.

Lemma 7 (see [17, Lemma 2.3]). *Let $H, \{E_n\}_n, E, G, g$ and $\{g_n\}_n$ be as in Definition 6. Assume that g is of class (B_+) on $\overline{G}^E$. Then, there exists an integer n_0 such that the Leray-Schauder degree* $\deg(g_n, G_n, 0)$ *is defined and*

$$\deg(g_n, G_n, 0) = \deg(g_{n_0}, G_{n_0}, 0) \quad \forall n \ge n_0. \tag{32}$$

It follows that

$$\deg(g, G, 0) := \lim_{n \to \infty} \deg(g_n, G_n, 0) \tag{33}$$

is defined. It was the topological degree of g on G at 0 in [17]. The corresponding versions with usual properties of the Leray-Schauder degree were given in [17, Theorem 2.1]. In particular, the identity map Id is of class (B_+), and $\deg(\mathrm{Id}, G, 0) = 1$ if $0 \in G$. Moreover, the following proposition is key for the proof of our main results.

Proposition 8 (see [17, Corollary 2.1]). *Let $H, \{E_n\}_n, E$ and G be as in Definition 6. Let g be a mapping from $\overline{G}^E$ into H such that g_m is continuous on $\overline{G_m}^{E_m}$ for any $m \in \mathbb{N}$. Suppose that G contains 0 and*

$$\langle g(x), x \rangle > 0, \quad \forall x \in \partial_E G. \tag{34}$$

Then there is a weakly Cauchy sequence $\{x_n\}_n$ in G such that

$$\lim_{n \to \infty} \langle g(x_n), v \rangle = 0, \quad \forall v \in E. \tag{35}$$

Next, we need the following results on the space $X_0(\Omega, K)$.

Lemma 9. (*a*) *$X_0(\Omega, K)$ and $X(\Omega, K)$ are continuously embedded in $H^s(\mathbb{R}^n)$ and $H^s(\Omega)$, respectively ([8, Lemma 5]).*

(*b*) *If $\Omega \subset \mathbb{R}^n$ is a bounded open subset with continuous boundary, the embedding $X_0(\Omega, K) \hookrightarrow L^p(\mathbb{R}^n)$ is compact for any $p \in [1, 2_s^*)$ ([8, Lemma 8] and [11, Lemma 9-a]).*

(*c*) *The embedding $X_0(\Omega, K) \hookrightarrow L^p(\mathbb{R}^n)$ is continuous for $p = 2_s^*$ ([11, Lemma 9-b]).*

(*d*) *The embedding $H^s(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$ is continuous for any $p \in [1, 2_s^*]$ ([18, Theorem 6.5]).*

(*e*) *If Ω is an open set in $\mathbb{R}^n$ of class $C^{0,1}$ with bounded boundary, then there exist continuous embeddings $W^{1,p}(\Omega) \hookrightarrow W^{s,p}(\Omega)$ and $W_0^{1,p}(\Omega) \hookrightarrow W_0^{s,p}(\Omega)$ for any $p \in [1, \infty)$ and $s \in (0, 1)$ ([18, Proposition 2.2]).*

Lemma 10. *Let Ω be a bounded open set in $\mathbb{R}^n$ with boundary of class $C^{0,1}$. Then, the space $X_0(\Omega, K)$ is separable. Furthermore, there exists a sequence $\{v_m\}_m$ in $C_0^\infty(\Omega)$ such that $\{v_m\}_m$ is a maximal orthogonal set of $X_0(\Omega, K)$.*

Proof. By Proposition 9(f) of [9], there exists a Hilbert basis $\{e_k\}_{k\geq 1}$ of $X_0(\Omega, K)$, which implies separability of $X_0(\Omega, K)$. So $\cup_{m=1}^\infty \{r_1 e_1 + \cdots + r_m e_m \mid r_i \in \mathbb{Q}, i = 1, \ldots, m\}$ is a dense countable subset in $X_0(\Omega, K)$. Let $\{f_m\}_{m\geq 1}$ denote this countable set. Since $C_0^\infty(\Omega)$ is dense in $X_0(\Omega, K)$ by [12, Theorem 6], for each $m \in \mathbb{N}$ we can take $f_{m,k} \in C_0^\infty(\Omega)$ such that $\|f_m - f_{m,k}\|_{0,\Omega,K} < 1/k \ \forall k \in \mathbb{N}$. Then, $\{f_{m,k} \mid (m, k) \in \mathbb{N} \times \mathbb{N}\}$ is also dense in $X_0(\Omega, K)$. Let $\{h_l \mid l \in \mathbb{N}\}$ be a maximal subset of $\{f_{m,k} \mid (m, k) \in \mathbb{N} \times \mathbb{N}\}$ such that any finite elements in $\{h_l \mid l \in \mathbb{N}\}$ are linearly independent. Then, $\operatorname{Span}(\{h_l \mid l \in \mathbb{N}\}) = \operatorname{Span}(\{f_{m,k} \mid (m, k) \in \mathbb{N} \times \mathbb{N}\})$ is dense in $X_0(\Omega, K)$. Making the Hilbert-Schmidt orthogonalization procedure for $\{h_m \mid m \in \mathbb{N}\}$, we obtain an orthogonal set $\{e_m \mid m \in \mathbb{N}\}$, which is also a maximal in $X_0(\Omega, K)$. □

3. Proof of Theorem 1

Take an increasing sequence of open subsets of Ω, $\{\Omega_k\}_k$, such that each of them has $C^{0,1}$-boundary and that

$$\overline{\Omega_k} \subset \Omega_{k+1} \quad \forall k \in \mathbb{N}, \qquad \Omega = \bigcup_{k=1}^\infty \Omega_k. \tag{36}$$

By Lemma 10, we may choose a sequence $\{v_{1,m}\}_m$ in $C_0^\infty(\Omega_1)$ such that $\{v_{1,m}\}_m$ is a maximal orthogonal set of $X_0(\Omega_1, K)$. Then, we can find a sequence $\{v_{2,m}\}_m$ in $C_0^\infty(\Omega_2) \setminus C_0^\infty(\Omega_1, K)$ such that $\{v_{1,m}, v_{2,m} \mid m \in \mathbb{N}\}$ is a maximal orthogonal set of $X_0(\Omega_2, K)$. By the mathematical induction, it is easy to find the set $\{v_{k,m} \mid k, m \in \mathbb{N}\}$ in $C_0^\infty(\Omega)$ such that $\{v_{j,m} \mid m \in \mathbb{N}, j = 1, \ldots, k\}$ is a maximal orthogonal set of $X_0(\Omega_k, K)$ for every $k \in \mathbb{N}$. Let us rewrite the countable set $\{v_{k,m} \mid k, m \in \mathbb{N}\}$ as a sequence $\{e_k\}_k$. Let E_m be the vector subspace of $X_0(\Omega, K)$ spanned by $\{e_1, \ldots, e_m\}$, and $E = \cup_m E_m$. For conveniences we set $H = X_0(\Omega, K)$ and denote by P_m the orthogonal projection from H onto E_m.

Lemma 11. (*a*) *E is dense in $X_0(\Omega, K)$.*

(*b*) *For each $u \in C_0^\infty(\Omega)$, there are $k \in \mathbb{N}$ and a sequence $\{u_m\}_m$ in E such that the supports of all u_m are contained in Ω_k and that $u_m \to u$ in $X_0(\Omega, K)$ as $m \to \infty$.*

(*c*) *For every $m \in \mathbb{N}$ and for every given $u \in X_0(\Omega, K)$, there exists a unique $T_m(u)$ in $X_0(\Omega, K)$ such that*

$$\langle T_m(u), v\rangle_{0,\Omega,K} = -\int_{\Omega_m} f(x, u(x))\, v(x)\, dx \qquad \forall v \in X_0(\Omega, K). \tag{37}$$

Moreover, if $v \in C_0^\infty(\Omega_k)$, then

$$\langle T_m(u), v\rangle_{0,\Omega,K} = \langle T_k(u), v\rangle_{0,\Omega,K} \quad \forall m \geq k. \tag{38}$$

(*d*) *Suppose that a sequence $\{u_k\}_k \subset X_0(\Omega, K)$ weakly converges to u in $X_0(\Omega, K)$. Then, $\{T_m(u_k)\}_k$ weakly converges to $T_m(u)$ in $X_0(\Omega, K)$ for $m = 1, 2, \ldots$.*

(*e*) *For every given $u \in E$ (the support of u must be contained in some Ω_{m_0} by the construction of E), there exists a unique $T(u) \in X_0(\Omega, K)$ such that*

$$\begin{aligned}\langle T(u), v\rangle_{0,\Omega,K} &= -\int_\Omega f(x, u(x))\, v(x)\, dx \qquad \forall v \in X_0(\Omega, K),\\ \langle T(u), v\rangle_{0,\Omega,K} &= \langle T_m(u), v\rangle_{0,\Omega,K} \qquad \forall v \in X_0(\Omega, K),\ \forall m \geq m_0.\end{aligned} \tag{39}$$

(*f*) *$P_m \circ (T|_{E_m}) : E_m \to E_m$ is continuous for every $m \in \mathbb{N}$.*

(*g*) *There exists a constant $C > 0$ such that*

$$\langle u + T(u), u\rangle_{0,\Omega,K} \geq \|u\|_{0,\Omega,K}^2 \left(1 - C\|u\|_{0,\Omega,K}^{p-2} - \|c\|_{L^1(\Omega)}\|u\|_{0,\Omega,K}^{-2}\right) \qquad \forall u \in E \setminus \{0\}. \tag{40}$$

Proof. (a) Since $\{e_k\}_k$ is a maximal orthogonal set of $X_0(\Omega, K)$, E is dense in $X_0(\Omega, K)$.

(b) For a given $u \in C_0^\infty(\Omega)$, by the choices of $\{\Omega_m\}_m$, there exists $k \in \mathbb{N}$ such that the support of u is contained in Ω_k. Let $\{v_{j,m} \mid m \in \mathbb{N}, j = 1, \ldots, k\}$ be the maximal orthogonal set of $X_0(\Omega_k, K)$ as constructed above. Then, $\operatorname{Span}(\{v_{j,m} \mid m \in \mathbb{N}, j = 1, \ldots, k\})$ is dense in $X_0(\Omega_k, K)$. Hence, we can find a sequence $\{u_m\}_{m\geq 1}$ in $E \cap C_0^\infty(\Omega_k)$ such that $\|u_m - u\|_{0,\Omega,K} \to 0$ as $m \to \infty$.

(c) By (16), we can write $f(x,t) = b(x) + f_0(x,t)\ \forall (x,t) \in \Omega \times \mathbb{R}$. Then, $f_0(x, 0) = 0\ \forall x \in \Omega$, and (15) implies

$$|f_0(x,t)| \leq a(x) + b(x) + \alpha |t|^{q-1} \quad \forall (x,t) \in \Omega \times \mathbb{R}. \tag{41}$$

Note that $2_s^* < 2n/(n-2)$ and $1/r + 1/2_s^* = 1$. Moreover, since $q \in (2, 2_s^*)$ we get

$$\begin{aligned}\frac{2n}{n+2s} &= r < r(q-1) < r(2_s^* - 1)\\ &= \frac{2n}{n+2s}\left(\frac{2n}{n-2s} - 1\right) = \frac{2n}{n-2s} = 2_s^*.\end{aligned} \tag{42}$$

For given $m \in \mathbb{N}$, by [19, Theorem 3.2.4] (see also [20, page 30]) we have a continuous mapping $\mathfrak{F}_m$ from $L^{r(q-1)}(\Omega)$ into $L^r(\Omega_m)$, where

$$\mathfrak{F}_m(u)(x) = -f_0(x, u(x)) \quad \forall x \in \Omega_m. \tag{43}$$

For $u, v \in X_0(\Omega, K)$, we have $u \in L^{r(q-1)}(\Omega)$ and $v \in L^{2_s^*}(\Omega)$ by Lemma 9(b) and (c). Thus, $\mathfrak{F}_m(u) \in L^r(\Omega_m)$ and

$$\begin{aligned}&\left|\int_{\Omega_m} f_0(x, u(x))\, v(x)\, dx\right|\\ &\quad = \left|\int_{\Omega_m} \mathfrak{F}_m(u)\, v(x)\, dx\right| \leq \|\mathfrak{F}_m(u)\|_{L^r(\Omega_m)} \|v\|_{L^{2_s^*}}.\end{aligned} \tag{44}$$

Using Lemma 9(c) again, there exists a constant $C > 0$ such that $\|v\|_{L^{2_s^*}} \le C\|v\|_{0,\Omega,K}$ for all $v \in X_0(\Omega, K)$. It follows that

$$\left| - \int_{\Omega_m} f_0(x, u(x)) v(x)\, dx - \int_\Omega b(x) v(x)\, dx \right| \le C\left(\|\mathfrak{F}_m(u)\|_{L^r(\Omega_m)} + \|b\|_{L^r(\Omega)} \right) \|v\|_{0,\Omega,K}. \tag{45}$$

Hence, Riesz representative theorem yields a unique $T_m(u) \in X_0(\Omega, K)$ such that

$$\langle T_m(u), v\rangle_{0,\Omega,K} = \int_{\Omega_m} \mathfrak{F}_m(u) v\, dx - \int_\Omega b(x) v(x) \quad \forall v \in X_0(\Omega, K). \tag{46}$$

If $v \in C_0^\infty(\Omega_k)$, for each integer $m \ge k$ we deduce

$$\begin{aligned}\langle T_m(u), v\rangle_{0,\Omega,K} &= \int_{\Omega_m} \mathfrak{F}_m(u) v\, dx - \int_\Omega b(x) v(x) \\ &= \int_{\Omega_k} \mathfrak{F}_k(u) v\, dx - \int_\Omega b(x) v(x) \\ &= \langle T_k(u), v\rangle_{0,\Omega,K}.\end{aligned} \tag{47}$$

(d) Let $\{u_k\}_k$ be a sequence weakly converging to u in $X_0(\Omega, K)$. Since

$$1 < r(q-1) < 2_s^* < \frac{2n}{n-2}, \tag{48}$$

from Lemma 9(b), we deduce that $\{u_k\}_k$ converges to u in $L^{r(q-1)}(\Omega)$. Then, the continuity of the map $\mathfrak{F}_m : L^{r(q-1)}(\Omega) \to L^r(\Omega_m)$ implies that $\{\mathfrak{F}_m(u_k)\}_k$ converges to $\mathfrak{F}_m(u)$ in $L^r(\Omega_m)$. Moreover, for $v \in X_0(\Omega, K)$, we have $v \in L^{2_s^*}(\Omega)$. Recall that $r^{-1} + (2_s^*)^{-1} = 1$ and $L^{2_s^*}(\Omega) = (L^r(\Omega))^*$. We deduce that

$$\int_{\Omega_m} v(x) \mathfrak{F}_m(u_k)(x)\, dx \longrightarrow \int_{\Omega_m} v(x) \mathfrak{F}_m(u)(x)\, dx \quad \text{as } k \longrightarrow \infty, \tag{49}$$

and hence $\lim_{k\to\infty} \langle T_m(u_k), v\rangle_{0,\Omega,K} = \langle T_m(u), v\rangle_{0,\Omega,K}$ by (46).

(e) Since $f_0(x, 0) = 0\ \forall x \in \Omega$, and $\mathrm{Supp}(u) \subset \Omega_{m_0}$, by (46) we deduce that

$$\begin{aligned}&- \int_\Omega f(x, u(x)) v(x)\, dx \\ &= - \int_\Omega f_0(x, u(x)) v(x)\, dx - \int_\Omega b(x) v(x)\, dx \\ &= - \int_{\Omega_m} f_0(x, u(x)) v(x)\, dx - \int_\Omega b(x) v(x)\, dx \\ &= \langle T_m(u), v\rangle_{0,\Omega,K} \quad \forall v \in X_0,\ \forall m \ge m_0.\end{aligned} \tag{50}$$

This shows that $X_0(\Omega, K) \ni v \mapsto -\int_\Omega f(x, u(x)) v(x) dx$ is a continuous linear functional. Using the Riesz representative theorem again we obtain a unique $T(u) \in X_0(\Omega, K)$ such that

$$\langle T(u), v\rangle_{0,\Omega,K} = - \int_\Omega f(x, u(x)) v(x)\, dx \quad \forall v \in X_0. \tag{51}$$

Clearly, $\langle T(u), v\rangle_{0,\Omega,K} = \langle T_m(u), v\rangle_{0,\Omega,K}\ \forall v \in X_0$ for all $m \ge m_0$.

(f) By the construction of E_m, we have an integer $m_0 \in \mathbb{N}$ such that each $u \in E_m$ has a support contained in Ω_{m_0}. Let $\{u_k\}_k \subset E_m$ converge to $u \in E_m$. By (d), $\{T_n(u_k)\}_k$ weakly converges to $T_n(u)$ in $X_0(\Omega, K)$ for every $n \in \mathbb{N}$. Then, (e) implies that $\langle T(u_k), v\rangle_{0,\Omega,K} \to \langle T(u), v\rangle_{0,\Omega,K}\ \forall v \in X_0(\Omega, K)$ as $k \to \infty$. In particular, since $v \in E_m$ satisfies $P_m v = v$, we have

$$\begin{aligned}\langle P_m \circ T(u_k), v\rangle_{0,\Omega,K} &= \langle T(u_k), P_m v\rangle_{0,\Omega,K} \\ &\longrightarrow \langle T(u), P_m v\rangle_{0,\Omega,K} \\ &= \langle P_m \circ T(u), v\rangle_{X_{0,\Omega,K}}.\end{aligned} \tag{52}$$

This shows that $\{P_m \circ T(u_k)\}_k$ weakly converges to $P_m T(u)$ in E_m. However, the strong converge and the weak ones on finitely dimensional space E_m are equivalent. Hence, $P_m \circ T(u_k) \to P_m \circ T(u)$ as $m \to \infty$.

(g) Since $1 \le p < 2 < 2_s^*$, by Lemma 9(b) and (c) there is a constant $C > 0$ such that $\|u\|_{L^p(\Omega)} \le C\|u\|_{0,\Omega,K}\ \forall u \in X_0(\Omega, K)$. It follows from this and (17) that

$$\begin{aligned}\langle u + T(u), u\rangle_{0,\Omega,K} &= \|u\|_{0,\Omega,K}^2 - \int_\Omega f(x, u(x)) u(x)\, dx \\ &\ge \|u\|_{0,\Omega,K}^2 - \int_\Omega \left(\beta |u|^p + |c|\right) dx \\ &= \|u\|_{0,\Omega,K}^2 - \beta \int_\Omega |u|^p\, dx - \|c\|_{L^1} \\ &= \|u\|_{0,\Omega,K}^2 - \beta \|u\|_{L^p}^p - \|c\|_{L^1} \\ &\ge \|u\|_{0,\Omega,K}^2 - \beta C^p \|u\|_{0,\Omega,K}^p - \|c\|_{L^1}.\end{aligned} \tag{53}$$

This leads to (g). □

Proof of Theorem 1. Let β, C, and c be as above. Since $1 \le p < 2$, we have $R > 0$ such that

$$1 - \beta C^p R^{p-2} - \|c\|_{L^1} R^{-2} > \frac{1}{4}. \tag{54}$$

Let $G = \{u \in E : \|u\|_{0,\Omega,K} < R\}$. Define $g : \overline{G}^E \to X_0(\Omega, K)$ by

$$g(u) = u + T(u) \quad \forall u \in \overline{G}^E. \tag{55}$$

Let us prove that g is of class (B_+) on $\overline{G}^E$. Note that $G_n = G_n \cap E_n$ has the same closure in E_n, E, and H, denoted by $\overline{G_n}$. Let $g_n : \overline{G_n} \to E_n$ be defined by $g_n(u) = P_n(g(u)) = P_n u + P_n \circ T(u) = u + P_n \circ T(u)$ for each $n \in \mathbb{N}$. Suppose that a sequence $\{u_k\}_k \subset \overline{G_n}$ converges to $u \in \overline{G_n}$. Lemma 11(f) implies $P_n \circ T(u_k) \to P_n \circ T(u)$ in E_n. Hence, g_n is continuous. By the proof of Lemma 11(g) and (54) we deduce that

$$\langle g(u), u\rangle_{0,\Omega,K} = \langle u + T(u), u\rangle_{0,\Omega,K} \geq \frac{R^2}{4} \quad \forall u \in \partial_E G = \{u \in E : \|u\|_{0,\Omega,K} = R\}. \tag{56}$$

This implies that Definition 6(b) is satisfied. Hence, g is of class (B_+) on $\overline{G}^E$. Moreover, it also shows that g satisfies the conditions of Proposition 8. Thus, we have a weakly Cauchy sequence $\{u_n\}_n \subset G$ such that

$$\lim_{n\to\infty} \langle g(u_n), v\rangle_{0,\Omega,K} = 0 \quad \forall v \in E. \tag{57}$$

Let u be the weak limit of $\{u_n\}_n$ in $X_0(\Omega, K)$. For a given $v \in E$, the support of it is contained in some Ω_k, and thus

$$\begin{aligned}\langle g(u_n), v\rangle_{0,\Omega,K} &= \langle u_n + T(u_n), v\rangle_{0,\Omega,K}\\ &= \langle u_n + T_m(u_n), v\rangle_{0,\Omega,K}\\ &= \langle u_n + T_k(u_n), v\rangle_{0,\Omega,K}\\ &\qquad \forall m \geq k,\ \forall n \in \mathbb{N}\end{aligned} \tag{58}$$

by Lemma 11(c). For each fixed $n \in \mathbb{N}$, there exists $\tilde{n} \in \mathbb{N}$ such that $u_n \in C_0^\infty(\Omega_{\tilde{n}})$. Hence, Lemma 11(e) yields

$$\begin{aligned}\langle g(u_n), v\rangle_{0,\Omega,K} &= \langle u_n + T(u_n), v\rangle_{0,\Omega,K}\\ &= \langle u_n + T_m(u_n), v\rangle_{0,\Omega,K}\\ &= \langle u_n + T_k(u_n), v\rangle_{0,\Omega,K}\\ &\qquad \forall m \geq \max\{k, \tilde{n}\}.\end{aligned} \tag{59}$$

Taking $n \to \infty$ in both sides of $\langle g(u_n), v\rangle_{0,\Omega,K} = \langle u_n + T_k(u_n), v\rangle_{0,\Omega,K}$ and using (57) and Lemma 11(d), we deduce

$$\langle u + T_m(u), v\rangle_{0,\Omega,K} = 0, \quad \forall m \geq k. \tag{60}$$

For any given $v \in C_0^\infty(\Omega)$, by Lemma 11(b), we have an integer k and a sequence $\{v_l\}_l \subset E$ such that $\mathrm{Supp}(v_l) \subset \Omega_k$ for any $l \in \mathbb{N}$ and that $v_l \to v$ in $X_0(\Omega, K)$. So (60) leads to

$$\langle u + T_m(u), v\rangle_{0,\Omega,K} = \lim_{l\to\infty} \langle u + T_m(u), v_l\rangle_{0,\Omega,K} = 0 \quad \forall m \geq k; \tag{61}$$

that is, $\langle u, v\rangle_{0,\Omega,K} - \int_{\Omega_m} f(x, u(x))v(x)dx = 0$ $\forall m \geq k$. Letting $m \to \infty$, we get

$$\langle u, v\rangle_{0,\Omega,K} - \int_\Omega f(x, u(x))\, v(x)\, dx = 0, \tag{62}$$

which shows that u is a generalized solution. Note that u might be zero! But $u = 0$ is not a solution if $b = f(\cdot, 0)$ takes nonzero values on a nonzero measure set. The proof is completed. □

4. Proofs of Corollaries and Examples

Proof of Corollary 2. Let $\tilde{f}(x,t) = \lambda G(x)|t|^{\nu-2}t + f(x,t)$. It suffices to check that $\tilde{f}$ satisfies (15)–(17). Clearly, we can assume $\lambda \neq 0$. Let $\rho = l/r$. Since $1 < \nu < 2$, we have $(n-2s)\nu > n - 2s$ and thus

$$l > \frac{2n}{2n - (n-2s)\nu} > \frac{2n}{n+2s} = r. \tag{63}$$

Then, $\rho > 1$ and

$$\begin{aligned}1 < (\nu-1)\rho' &= (\nu - 1)\frac{\rho}{\rho - 1} = \frac{l}{l-r} < \frac{n+2s}{n-2s}\\ &= 2_s^* - 1.\end{aligned} \tag{64}$$

By Young's inequality, we obtain

$$\begin{aligned}|\lambda G(x)|\,|t|^{\nu-2}\,t &= |\lambda G(x)|\,|t|^{\nu-1}\\ &\leq \frac{|\lambda|}{\rho}|G(x)|^\rho + \frac{|\lambda|}{\rho'}|t|^{(\nu-1)\rho'}.\end{aligned} \tag{65}$$

Note that $G \in L^l_{\mathrm{loc}}(\Omega)$ and $r\rho = l$ imply $|G|^\rho \in L^r_{\mathrm{loc}}(\Omega)$. Let $\tilde{q} - 1 = \max\{q - 1, (\nu-1)\rho'\}$, which sits in $(1, 2_s^* - 1)$. From these and (15), it follows that

$$\begin{aligned}\left|\tilde{f}(x,t)\right| &\leq a(x) + \frac{|\lambda|}{\rho}|G(x)|^\rho + \frac{|\lambda|}{\rho'}|t|^{(\nu-1)\rho'}\\ &\quad + \alpha|t|^{q-1}\\ &\leq \left(a(x) + \frac{|\lambda|}{\rho}|G(x)|^\rho + C\right) + \tilde{\alpha}|t|^{\tilde{q}-1}\\ &\qquad \text{a.e. } x \in \Omega,\ \forall t \in \mathbb{R}\end{aligned} \tag{66}$$

for some constants $C > 0$ and $\tilde{\alpha} > 0$, where Young's inequality is used again. So $\tilde{f}$ satisfies (15). Moreover, $\tilde{b}(x) := \tilde{f}(x,0) = f(x,0) = b(x)$; that is, $\tilde{f}$ satisfies (16).

Finally, let us check that (17) holds for $\tilde{f}$. If $\lambda G \leq 0$, then

$$\begin{aligned}-\tilde{f}(x,t)\,t &= -\lambda G(x)|t|^\nu - f(x,t)\,t\\ &\geq -\beta|t|^p - c(x).\end{aligned} \tag{67}$$

For another case, observe that

$$\begin{aligned}-\tilde{f}(x,t)\,t &= -\lambda G(x)|t|^\nu - f(x,t)\,t\\ &\geq -\beta|t|^p - c(x) - (\lambda G)^+(x)|t|^\nu.\end{aligned} \tag{68}$$

Since $\kappa > 2/(2-\nu) > 1$, we may choose a real number σ in $(2/(2-\nu), \kappa)$. Let $\sigma' = \sigma/(\sigma - 1)$. By Young's inequality, we have

$$(\lambda G)^+(x)|t|^\nu \leq \frac{1}{\sigma}\left|(\lambda G)^+(x)\right|^\sigma + \frac{1}{\sigma'}|t|^{\nu\sigma'}. \tag{69}$$

Note that $|(\lambda G)^+|^\sigma \in L^1(\Omega)$ since $\sigma < \kappa$. Moreover, $1 < \nu\sigma'$ and

$$\frac{2}{2-\nu} < \sigma \Longleftrightarrow \frac{2-\nu}{2} > \frac{1}{\sigma} \Longleftrightarrow \frac{2}{\nu} > \frac{\sigma}{\sigma-1} = \sigma' \Longleftrightarrow \nu\sigma' < 2. \tag{70}$$

Let $\tilde{p} = \max\{p, \nu\sigma'\}$, which belongs to $[1,2)$. Using Young's inequality, we can derive

$$-\beta|t|^p - \frac{1}{\sigma'}|t|^{\nu\sigma'} \geq -\tilde{\beta}|t|^{\tilde{p}} - C \quad \forall t \in \mathbb{R} \tag{71}$$

for some constants $\tilde{\beta} > 0$ and $C > 0$. Hence, for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$, we have

$$\begin{aligned} -\tilde{f}(x,t)t &= -\lambda G(x)|t|^\nu - f(x,t)t \\ &\geq -\tilde{\beta}|t|^{\tilde{p}} - \tilde{c}(x), \end{aligned} \tag{72}$$

where $\tilde{c}(x) := c(x) + (1/\sigma)|(\lambda G)^+(x)|^\sigma + C$ belongs to $L^1(\Omega)$ as above. This shows that (17) is true for $\tilde{f}$. The desired conclusion follows from Theorem 1 immediately. □

Proof of Corollary 3. Let $\tilde{f}(x,t) = \lambda G(x)|t|^{\nu-2}t + f(x,t)$. Since $\lambda G \leq 0$, we see that $\tilde{f}$ satisfies (17) from the above proof. It remains to prove that $\tilde{f}$ satisfies (15)-(16). Let $\vartheta' = \vartheta/(\vartheta-1)$. Note that

$$\begin{aligned} &\vartheta > \frac{n+2s}{n+2s-(\nu-1)(n-2s)} \\ &\Longleftrightarrow \frac{n+2s-(\nu-1)(n-2s)}{n+2s} = 1 - \frac{(\nu-1)(n-2s)}{n+2s} \\ &> \frac{1}{\vartheta} \Longleftrightarrow 1 - \frac{1}{\vartheta} = \frac{\vartheta-1}{\vartheta} > \frac{(\nu-1)(n-2s)}{n+2s} \\ &\Longleftrightarrow \frac{\vartheta(\nu-1)}{\vartheta-1} = (\nu-1)\vartheta' < 2_s^* - 1 = \frac{n+2s}{n-2s}. \end{aligned} \tag{73}$$

By Young's inequality, we have

$$|G(x)||t|^{\nu-1} \leq \frac{|G(x)|^\vartheta}{\vartheta} + \frac{|t|^{(\nu-1)\vartheta'}}{\vartheta'}. \tag{74}$$

Now, $l \geq r\vartheta$ implies $l/\vartheta \geq r$ and $|G|^\vartheta \in L^{l/\vartheta}_{\text{loc}}(\Omega)$ because $G \in L^l_{\text{loc}}(\Omega)$. We obtain $|G|^\vartheta \in L^r_{\text{loc}}(\Omega)$. As in the proof of Corollary 2, using Young's inequality, we may derive from this and (73)-(74) that $\tilde{f}$ satisfies (15) and (16). □

Example 1. Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be as above. Consider

$$\begin{aligned} \mathfrak{L}_K u + G(x)u + h(u) &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega, \end{aligned} \tag{75}$$

where $G \leq 0$ belongs to $L^l_{\text{loc}}(\Omega)$ with $l \in ((n+2s)/4s, n/2s)$, $h \in C(\mathbb{R})$ is absolutely continuous, $h(0) \neq 0$, $\sup_{t\in\mathbb{R}} h(t)t < \infty$, and $|h'(t)| \leq e_1 + e_2|t|^\rho$, a.e., $t \in \mathbb{R}$, $0 \leq \rho < 4s/(n-2s)$. Then, (75) has a nontrivial generalized solution.

In fact, taking $\nu = 2$ in Corollary 3, we should require $\vartheta > (n+2s)/4s$. Since $l \in ((n+2s)/4s, n/2s)$, there is sufficiently small $\epsilon > 0$ such that

$$\begin{aligned} 1+\epsilon < \frac{n+2s}{4s} + \epsilon < l < \frac{n}{2s} < \frac{n}{2s} + r\epsilon \\ = r\frac{n+2s}{4s} + r\epsilon. \end{aligned} \tag{76}$$

This means that we can take $\vartheta = (n+2s)/4s + \epsilon$. Moreover, $1 \leq \rho + 1 < 1 + 4s/(n-2s) = 2_s^*$, and

$$\begin{aligned} |h(t) - h(0)| &= \left|\int_0^t h'(\tau)\,d\tau\right| \leq e_1|t| + \frac{e_2}{\rho+1}|t|^{\rho+1} \\ &\leq \frac{e_1\rho}{\rho+1} + \frac{e_1+e_2}{\rho+1}|t|^{\rho+1}, \\ -f(x,t)t &= -G(x)t^2 - h(t)t \geq -\sup_{t\in\mathbb{R}} h(t)t \\ &> -\infty. \end{aligned} \tag{77}$$

Hence, (15)–(17) are satisfied for $f(x,t) = G(x)t + h(t)$.

5. Proof of Theorem 4

Consider the product Hilbert space $\mathbf{H} = X_0(\Omega, K_1) \times X_0(\Omega, K_2)$ equipped with inner product

$$\begin{aligned} \langle u, v\rangle_{\mathbf{H}} &= \langle (u_1,u_2), (v_1,v_2)\rangle_{\mathbf{H}} \\ &= \langle u_1, v_1\rangle_{X_0,\Omega,K_1} + \langle u_2, v_2\rangle_{X_0,\Omega,K_2} \end{aligned} \tag{78}$$

for $u = (u_1,u_2), v = (v_1,v_2) \in \mathbf{H}$. The induced norm is

$$\|(u,v)\|_{\mathbf{H}} = \left(\|u\|^2_{0,\Omega,K_1} + \|v\|^2_{0,\Omega,K_2}\right)^{1/2}. \tag{79}$$

Let $\Omega = \cup_{k=1}^\infty \Omega_k$ and $E = \cup_m E_m$ be as in Section 3. For every integer $m \in \mathbb{N}$ let $\mathbf{E}_m = \cup_{k+l\leq m}(E_k \times E_l)$ and $\mathbf{E} = \cup_m \mathbf{E}_m$. Denote by $\mathbf{P}_m$ the orthogonal projection from $\mathbf{E}$ onto $\mathbf{E}_m$. Corresponding to Lemma 11, we have the following.

Lemma 13. *(a)* $\mathbf{E}$ *is dense in* $\mathbf{H}$.

(b) For each $u \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)$, *there are* $k \in \mathbb{N}$ *and a sequence* $\{u_m\}_m$ *in* $\mathbf{E}$ *such that the supports of all* u_m *are contained in* $\Omega_k \times \Omega_k$ *and that* $u_m \to u$ *in* $\mathbf{H}$ *as* $m \to \infty$.

(c) For every $m \in \mathbb{N}$, *and for every given* $u \in \mathbf{H}$, *there exists a unique* $\mathbf{T}_m(u)$ *in* $\mathbf{H}$ *such that*

$$\begin{aligned} \langle \mathbf{T}_m(u), v\rangle_{\mathbf{H}} = &-\int_{\Omega_m} f_1(x, u_2(x))\,v_1(x)\,dx \\ &-\int_{\Omega_m} f_2(x, u_1(x))\,v_2(x)\,dx \end{aligned} \tag{80}$$

for any $v = (v_1,v_2) \in \mathbf{H}$. *Moreover, if* $v = (v_1,v_2) \in C_0^\infty(\Omega_k) \times C_0^\infty(\Omega_k)$, *then* $\langle \mathbf{T}_m(u), v\rangle_{\mathbf{H}} = \langle \mathbf{T}_k(u), v\rangle_{\mathbf{H}}$ $\forall m \geq k$.

(d) Suppose that a sequence $\{u_k\}_k \subset \mathbf{H}$ weakly converges to u in $\mathbf{H}$. Then, $\{\mathbf{T}_m(u_k)\}_k$ weakly converges to $\mathbf{T}_m(u)$ in $\mathbf{H}$ for $m = 1, 2, \ldots$.

(e) For every given $u \in \mathbf{E}$ (the support of u must be contained in some $\Omega_{m_0} \times \Omega_{m_0}$ by the construction of $\mathbf{E}$), there exists a unique $\mathbf{T}(u) \in \mathbf{H}$ such that

$$\begin{aligned}\langle \mathbf{T}(u), v\rangle_{\mathbf{H}} &= -\int_{\Omega} f_1(x, u_2(x)) v_1(x)\, dx \\ &\quad - \int_{\Omega} f_2(x, u_1(x)) v_2(x)\, dx, \\ \langle \mathbf{T}(u), v\rangle_{\mathbf{H}} &= \langle \mathbf{T}_m(u), v\rangle \quad \forall v \in \mathbf{H}(\Omega),\ \forall m \ge m_0.\end{aligned} \tag{81}$$

(f) $\mathbf{P}_m \circ (\mathbf{T}|_{\mathbf{E}_m}) : \mathbf{E}_m \to \mathbf{E}_m$ is continuous for every $m \in \mathbb{N}$.

(g) There exists a constant $C > 0$ such that

$$\begin{aligned}\langle u + T(u), u\rangle_{\mathbf{H}} \ge \|u\|_{\mathbf{H}}^2 \Big[1 &- \left(\|c_1\|_{L^1(\Omega)} + \|c_2\|_{L^1(\Omega)}\right) \|u\|_{\mathbf{H}}^{-2} \\ &- C\left(\|u\|_{\mathbf{H}}^{p_1-2} + \|u\|_{\mathbf{H}}^{\hat{p}_1-2} + \|u\|_{\mathbf{H}}^{p_2-2} + \|u\|_{\mathbf{H}}^{\hat{p}_2-2}\right)\Big]\end{aligned} \tag{82}$$

for all $u \in E \setminus \{0\}$.

Proof. (a) and (b) follow from Lemma 11(a) and (b) immediately.

(c) By (24), we can write $f_j(x,t) = b_j(x) + f_{0,j}(x,t)\ \forall (x,t) \in \Omega \times \mathbb{R}$ as in the proof of Lemma 11(c); then, $f_{0,j}(x,0) = 0\ \forall x \in \Omega$, and

$$\begin{aligned}\left|f_{0,j}(x,t)\right| &\le a_j(x) + b_j(x) + \alpha_j |t|^{q_j-1} \\ &\qquad \forall (x,t) \in \Omega \times \mathbb{R},\ j = 1, 2.\end{aligned} \tag{83}$$

Moreover, for every given $m \in \mathbb{N}$ and $j = 1, 2$, we have continuous mappings

$$L^{r(q_j-1)}(\Omega) \ni u \longmapsto \mathfrak{F}_{m,j}(u) \in L^r(\Omega_m), \tag{84}$$

where $\mathfrak{F}_{m,j}(u)(x) = f_{0,j}(x, u(x))$ for $x \in \Omega_m$, $j = 1, 2$. Recall that $2_s^* = 2_s^*$. For $u_1, v_1 \in X_0(\Omega, K_1)$, $u_2, v_2 \in X_0(\Omega, K_2)$, by Lemma 9(b) and (c), we have

$$\begin{gathered} v_j \in L^{2_s^*}(\Omega_m), \\ u_j \in L^{r(q_i-1)}(\Omega), \\ i, j = 1, 2.\end{gathered} \tag{85}$$

Thus, $\mathfrak{F}_{m,1}(u_2) \in L^r(\Omega_m)$, $\mathfrak{F}_{m,2}(u_1) \in L^r(\Omega_m)$, and

$$\begin{aligned}\left|-\int_{\Omega_m} f_{0,1}(x, u_2(x)) v_1(x)\, dx\right| &= \left|\int_{\Omega_m} \mathfrak{F}_{m,1}(u_2) v_1(x)\, dx\right| \\ &\le \|\mathfrak{F}_{m,1}(u_2)\|_{L^r(\Omega_m)} \|v_1\|_{L^{2_s^*}}, \\ \left|-\int_{\Omega_m} f_{0,2}(x, u_1(x)) v_2(x)\, dx\right| &= \left|\int_{\Omega_m} \mathfrak{F}_{m,2}(u_1) v_2(x)\, dx\right| \\ &\le \|\mathfrak{F}_{m,2}(u_1)\|_{L^r(\Omega_m)} \|v_2\|_{L^{2_s^*}}.\end{aligned} \tag{86}$$

Using Lemma 9(c) again, there exist constants $C > 0$ such that $\|v_j\|_{L^{2_s^*}} \le C\|v_j\|_{0,\Omega,K_j}$ for all $v = (v_1, v_2) \in \mathbf{H}$, $j = 1, 2$. It follows that

$$\begin{aligned}&\left|-\int_{\Omega_m} f_{0,1}(x, u_2(x)) v_1(x)\, dx - \int_{\Omega} b_1(x) v_1(x)\, dx\right| \\ &\quad\le C\left(\|\mathfrak{F}_{m,1}(u_2)\|_{L^r(\Omega_m)} + \|b_1\|_{L^r(\Omega)}\right)\|v_1\|_{0,\Omega,K_1}, \\ &\left|-\int_{\Omega_m} f_{0,2}(x, u_1(x)) v_2(x)\, dx - \int_{\Omega} b_2(x) v_2(x)\, dx\right| \\ &\quad\le C\left(\|\mathfrak{F}_{m,2}(u_1)\|_{L^r(\Omega_m)} + \|b_2\|_{L^r(\Omega)}\right)\|v_2\|_{0,\Omega,K_2}.\end{aligned} \tag{87}$$

By the Riesz representative theorem, for each $j \in \{1, 2\}$, we have a unique $T_{mj}(u) \in X_{0,\Omega,K_j}$ such that

$$\langle T_{m1}(u), v_1\rangle_{0,\Omega,K_1} = -\int_{\Omega_m} f_1(x, u_2) v_1(x)\, dx, \tag{88}$$

$$\langle T_{m2}(u), v_2\rangle_{0,\Omega,K_2} = -\int_{\Omega_m} f_2(x, u_1) v_2(x)\, dx. \tag{89}$$

Setting $\mathbf{T}_m(u) := (T_{m1}(u), T_{m2}(u))$, we obtain

$$\begin{aligned}\langle \mathbf{T}_m(u), v\rangle_{\mathbf{H}} &= \langle T_{m1}(u), v_1\rangle_{0,\Omega,K_1} + \langle T_{m2}(u), v_2\rangle_{0,\Omega,K_2} \\ &= -\int_{\Omega_m} f_1(x, u_2) v_1(x)\, dx - \int_{\Omega_m} f_2(x, u_1) v_2(x)\, dx.\end{aligned} \tag{90}$$

Another claim can be proved as that of Lemma 11(c).

(d) Let $u_k = (u_k^1, u_k^2)$ for each $k \in \mathbb{N}$. Then, $\{u_k^1\}_k \subset X_{0,\Omega,K_1}$ weakly converges to u^1 in $X_0(\Omega, K_1)$ and $\{u_k^2\}_k \subset X_{0,\Omega,K_2}$ weakly converges to u^2 in $X_0(\Omega, K_2)$. For each $m \in \mathbb{N}$, by Lemma 11(d), $\{T_{m1}(u_k)\}_k$ weakly converges to $T_{m1}(u)$ in $X_0(\Omega, K_1)$ and $\{T_{m2}(u_k)\}_k$ weakly converges to $T_{m2}(u)$ in

$X_0(\Omega, K_2)$. Thus, $\{\mathbf{T}_m(u_k)\}_k$ weakly converges to $\mathbf{T}_m(u)$ in $\mathbf{H}$ for $m = 1, 2, \ldots$.

(e) Since $f_{0,j}(x, 0) = 0$ $\forall x \in \Omega$, $j = 1, 2$, and $\operatorname{Supp}(u) \subset \Omega_{m_0} \times \Omega_{m_0}$, for any integer $m \geq m_0$ and $v = (v_1, v_2) \in \mathbf{H}$, we derive from (89) that

$$\begin{aligned}
-\int_\Omega f_1(x, u_2(x)) v_1(x)\,dx &= -\int_\Omega f_{0,1}(x, u_2(x)) v_1(x)\,dx - \int_\Omega b_1(x) v_1(x)\,dx \\
&= -\int_{\Omega_m} f_{0,1}(x, u_2(x)) v_1(x)\,dx - \int_\Omega b_1(x) v_1(x)\,dx = \langle T_{m1}(u), v_1\rangle_{0,\Omega,K_1}, \\
-\int_\Omega f_2(x, u_1(x)) v_2(x)\,dx &= -\int_\Omega f_{0,2}(x, u_1(x)) v_2(x)\,dx - \int_\Omega b_2(x) v_2(x)\,dx \\
&= -\int_{\Omega_m} f_{0,2}(x, u_1(x)) v_2(x)\,dx - \int_\Omega b_2(x) v_2(x)\,dx = \langle T_{m2}(u), v_2\rangle_{0,\Omega,K_2}.
\end{aligned} \tag{91}$$

These show that

$$\begin{aligned}
X_0(\Omega, K_1) \ni v_1 &\longmapsto -\int_\Omega f_1(x, u_2(x)) v_1(x)\,dx, \\
X_0(\Omega, K_2) \ni v_2 &\longmapsto -\int_\Omega f_2(x, u_1(x)) v_2(x)\,dx
\end{aligned} \tag{92}$$

are two continuous linear functionals. Using the Riesz representative theorem again we obtain a unique $T_1(u) \in X_0(\Omega, K_1), T_2(u) \in X_0(\Omega, K_2)$ such that

$$\begin{aligned}
\langle T_1(u), v_1\rangle_{0,\Omega,K_1} &= -\int_\Omega f_1(x, u_2(x)) v_1(x)\,dx, \\
\langle T_2(u), v_2\rangle_{0,\Omega,K_2} &= -\int_\Omega f_2(x, u_1(x)) v_2(x)\,dx
\end{aligned} \tag{93}$$

for all $v = (v_1, v_2) \in \mathbf{H}$. Set $\mathbf{T}(u) := (T_1(u), T_2(u))$; then,

$$\begin{aligned}
\langle T(u), v\rangle_{\mathbf{H}} = -\int_\Omega f_1(x, u_2(x)) v_1(x)\,dx &- \int_\Omega f_2(x, u_1(x)) v_2(x)\,dx, \\
&\forall v \in \mathbf{H}.
\end{aligned} \tag{94}$$

Clearly, $\langle \mathbf{T}(u), v\rangle_{\mathbf{H}} = \langle \mathbf{T}_m(u), v\rangle_{\mathbf{H}}$ $\forall v \in \mathbf{H}$ for all $m \geq m_0$.

(f) follows the above (e) and Lemma 11(f) directly.

(g) Since $1 \leq p_j, \widehat{p}_j < 2 < 2_s^*$, by Lemma 9(b) and (c) there is a constant $C_0 > 0$ such that

$$\begin{aligned}
\|u_i\|_{L^{\widehat{p}_j}(\Omega)} + \|u_i\|_{L^{p_j}(\Omega)} \leq C_0 \|u_i\|_{0,\Omega,K_i} \\
\forall u = (u_1, u_2) \in \mathbf{H},\ i, j = 1, 2.
\end{aligned} \tag{95}$$

It follows from this and (25) that

$$\begin{aligned}
\langle u + \mathbf{T}(u), u\rangle_{\mathbf{H}} &= \|u\|_{\mathbf{H}}^2 - \int_\Omega f_1(x, u_2(x)) u_1(x)\,dx - \int_\Omega f_2(x, u_1(x)) u_2(x)\,dx \geq \|u\|_{\mathbf{H}}^2 \\
&- \int_\Omega \left(\beta_1 |u_2|^{p_1/2} |u_1|^{\widehat{p}_1/2} + |c_1|\right) dx - \int_\Omega \left(\beta_2 |u_1|^{p_2/2} |u_2|^{\widehat{p}_2/2} + |c_2|\right) dx \geq \|u\|_{\mathbf{H}}^2 \\
&- \|c_1\|_{L^1} - \|c_2\|_{L^1} - \frac{\beta_1}{2} \int_\Omega \left(|u_2|^{p_1} + |u_1|^{\widehat{p}_1}\right) dx - \frac{\beta_2}{2} \cdot \int_\Omega \left(|u_1|^{p_2} + |u_2|^{\widehat{p}_2}\right) dx \geq \|u\|_{\mathbf{H}}^2 - (\beta_1 + \beta_2) \\
&\cdot \left(\|u_1\|_{L^{\widehat{p}_1}}^{\widehat{p}_1} + \|u_1\|_{L^{p_2}}^{p_2} + \|u_2\|_{L^{p_1}}^{p_1} + \|u_2\|_{L^{\widehat{p}_2}}^{\widehat{p}_2}\right) - \|c_1\|_{L^1} - \|c_2\|_{L^1} \\
&\geq \|u\|_{\mathbf{H}}^2 - C\left(\|u_1\|_{0,\Omega,K_1}^{\widehat{p}_1} + \|u_1\|_{0,\Omega,K_1}^{p_2} + \|u_2\|_{0,\Omega,K_2}^{p_1} + \|u_2\|_{0,\Omega,K_2}^{\widehat{p}_2}\right) - \|c_1\|_{L^1} - \|c_2\|_{L^1} \geq \|u\|_{\mathbf{H}}^2 - C\left(\|u\|_{\mathbf{H}}^{p_1} + \|u\|_{\mathbf{H}}^{\widehat{p}_1} + \|u\|_{\mathbf{H}}^{p_2} + \|u\|_{\mathbf{H}}^{\widehat{p}_2}\right) - \|c_1\|_{L^1} - \|c_2\|_{L^1}.
\end{aligned} \tag{96}$$

Here, $C = (\beta_1 + \beta_2)\max\{C_0^{p_1}, C_0^{\widehat{p}_1}, C_0^{p_2}, C_0^{\widehat{p}_2}\}$. This leads to (g). □

Proof of Theorem 4. We replace the space $X_0(\Omega, K)$ in the proof of Theorem 1 by $\mathbf{H}$. Since $p_j < 2, \widehat{p}_j < 2$ for $j = 1, 2$, as in (54), we have $R > 0$ such that

$$\begin{aligned}
1 - C\left(R^{p_1 - 2} + R^{\widehat{p}_1 - 2} + R^{p_2 - 2} + R^{\widehat{p}_2 - 2}\right) \\
- \left(\|c_1\|_{L^1} + \|c_1\|_{L^1}\right) R^{-2} > \frac{1}{4}.
\end{aligned} \tag{97}$$

Then repeating the proof of Theorem 1, we get a $u = (u_1, u_2) \in \mathbf{H}$ such that $\langle u + \mathbf{T}(u), v\rangle_{\mathbf{H}} = 0$ for any $v = (\phi, \psi) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)$; namely, (22) holds. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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