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Research Article

On Tricomi Problem of Chaplygin's Hodograph Equation

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The existence and uniqueness results for the Tricomi problem of Chaplygin's hodograph equation are shown, in the case that the domain considered is close to the parabolic degenerate line, by adopting the energy integral methods and choosing judiciously suitable multipliers.

1. Introduction

In this paper we consider the Tricomi problem of the following second-order linear partial differential equation. Consider

$$L\phi \triangleq K(t)\,\phi_{\theta\theta} + \phi_{tt} - P(t)\,\phi_t = 0, \tag{1}$$

where

$$K(t) = \frac{t(2-t)}{(1-t)^2\left[1-\mu^2(1-t)^2\right]},$$
$$P(t) = \frac{t(2-t)}{(1-t)\left[1-\mu^2(1-t)^2\right]}, \tag{2}$$

and $\mu \in (0,1)$ is a constant. Equation (1) is of elliptic type for $0 < t < 1$ and $0 \le \theta < 2\pi$, hyperbolic type for $1 - 1/\mu < t < 0$ and $0 \le \theta < 2\pi$, and parabolic degenerate on the line $\{t = 0\}$. We are interested in this equation because it is actually an equivalent form of Chaplygin's hodograph equation (with $\Phi = \Phi(u,v)$ as the unknown). (In this paper we will use the subscripts like ϕ_t and ϕ_{tt} to denote the partial derivatives $\partial\phi/\partial t$ and $\partial^2\phi/\partial t^2$)

$$\left(c^2 - v^2\right)\Phi_{uu} + 2uv\Phi_{uv} + \left(c^2 - u^2\right)\Phi_{vv} = 0, \tag{3}$$

where the function $c = c(u,v)$ (called sonic speed in gas dynamics) is given by the Bernoulli law [1, page 23]

$$\mu^2\left(u^2 + v^2\right) + \left(1 - \mu^2\right)c^2 = c_*^2, \tag{4}$$

with c_* being a positive constant, $\mu = \sqrt{(\gamma-1)/(\gamma+1)}$, and $\gamma > 1$ the adiabatic exponent for polytropic gas. One can easily show that, by taking $t = 1 - \sqrt{u^2+v^2}/c_*$ and $\theta = \arctan(v/u)$, (3) is transformed to (1), with $\phi(t,\theta) = \Phi(c_*(1-t),\theta)$ (*cf.* [2, page 72]).

The significance of Chaplygin's hodograph equation (3) lies in the fact that it is the hodograph transform of the following compressible Euler equations of isentropic irrotational flows:

$$v_x - u_y = 0,$$
$$(\rho u)_x + (\rho v)_y = 0, \tag{5}$$

where ρ is the density of mass of the flow and (u,v) is the velocity of the flow along the (x,y) coordinates of the Euclidean plane. Since in this case the sonic speed $c = \rho^{(\gamma-1)/2}$, then ρ is a function of (u,v) given by the Bernoulli law. Some fundamental problems in gas dynamics, such as detached shocks in supersonic flow past blunt bodies and subsonic jets (*cf.* [1, 3]), could be considered more favorably by using hodograph equation (3) (or (1)) rather than Euler equations (5), because the latter are generally a quasi-linear mixed elliptic-hyperbolic system, which is still far beyond the ability of present-day analytical tools to study.

In a previous work [2], the authors have studied a mixed boundary value problem of (3) in the sonic circle $\{u^2 + v^2 < c_*^2\}$, with an artificial Dirichlet boundary condition on part of the sonic line $\{u^2 + v^2 = c_*^2\}$, to understand the regularity and behavior of solutions of (3) in the elliptic region and near the degenerate line. Now, we continue our project in this paper to investigate the *Tricomi problem* of (1), that is, to find a function $\phi = \phi(\theta, t)$ satisfying (in certain sense to be specified later) (1) in a planar domain D which is simply connected, containing a segment of the θ-axis, and bounded by the characteristic curves (by definition, a characteristic curve of (1) satisfies equation $-(\sqrt{-K(t)})^2(\mathrm{d}t)^2 + (\mathrm{d}\theta)^2 = 0$ for $t < 0$) Γ_2 and Γ_3 lying in the lower half plane $\{t < 0\}$ and a Jordan curve Γ_1 lying in the upper half plane $\{t > 0\}$, with Dirichlet conditions on Γ_1 and Γ_3 (see Figure 1). Here

$$\Gamma_3\colon \theta = -\int_0^t \sqrt{-K(r)}\mathrm{d}r, \quad t < 0 \tag{6}$$

emanates from the origin O and intersects the horizontal line $\{t = t_1\}$ at a point $P(\theta_1, t_1)$, where $t_1 < 0$ is sufficiently small. The characteristic curve

$$\Gamma_2\colon \theta = \int_0^t \sqrt{-K(r)}\mathrm{d}r + \theta_0, \quad t < 0 \tag{7}$$

emanates from the point P and intersects the θ-axis at a point $A(\theta_0, 0)$. The arc Γ_1 has two endpoints $O = (0,0)$ and $A = (\theta_0, 0)$. The Dirichlet conditions on Γ_1 and Γ_3 are, respectively,

$$\begin{aligned} \phi &= f(s) \quad \text{on } \Gamma_1, \\ \phi &= g(s) \quad \text{on } \Gamma_3, \end{aligned} \tag{8}$$

where s is the arc-length parameter of the boundary curve $\partial D = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ so that the point $(\theta(s), t(s))$ moves counterclockwise on ∂D as s increases. Then the outward unit normal along ∂D is given by $\mathbf{n} = (n_1, n_2) = (\mathrm{d}t/\mathrm{d}s, -\mathrm{d}\theta/\mathrm{d}s)$. Note that one can require $\Gamma_1 \cup \Gamma_3$ to be piecewise smooth except at the point O, where at best the curve is $C^{1,1/2}$. (We thank a referee for pointing out this fact. Here as usual we use $C^{k,\alpha}(\Omega)$ to denote the Hölder space of k-times continuously differential real-valued functions on Ω whose kth order derivatives are all Hölder continuous with the exponent $\alpha \in (0,1)$.) Let $\tilde{\phi}$ be a given function in the standard Sobolev space $H^2(D)$. The functions f and g are the traces of $\tilde{\phi}$ on Γ_1 and Γ_3, respectively. Then it is obvious that their union

$$(f \cup g)(s) = \begin{cases} f(s) & (\theta(s), t(s)) \in \Gamma_1, \\ g(s) & (\theta(s), t(s)) \in \Gamma_3 \end{cases} \tag{9}$$

belongs to $H^1(\Gamma_1 \cup \Gamma_3)$.

It is well known that the Tricomi problem was firstly proposed and studied by Tricomi in [4] for the now so-called Tricomi equation $t\phi_{\theta\theta} + \phi_{tt} = 0$, by using singular integral equations and the matching technique. Tricomi's study of

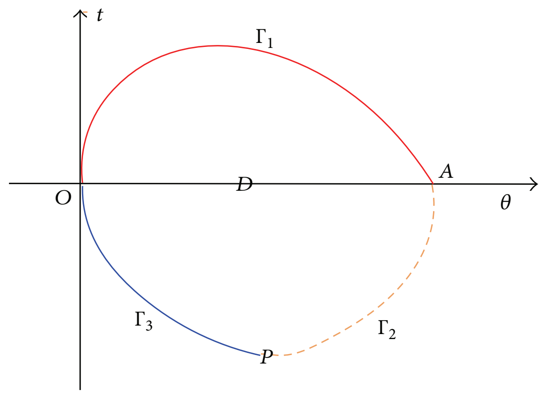

Figure 1: The domain D and its boundary in the formulations of the Tricomi problem.

this problem was mainly motivated to understand second-order mixed elliptic-hyperbolic type equations from a purely mathematical point of view. Later it was discovered that the Tricomi equation may be considered in certain sense as a simple approximation near the sonic line of Chaplygin's hodograph equation in transonic aerodynamics (*cf.* [5]) and the Tricomi problem is physically relevant to determining some flow field in transonic flows, such as the detached bow shock and the mixed subsonic-supersonic flow ahead of a blunt body [6]. More general linear mixed elliptic-hyperbolic equations and more general formulations of boundary conditions (such as generalized Tricomi problem, Frankl problem, and generalized Frankl problem) were also considered. For example, Morawetz [7] proved the uniqueness for smooth solutions using Noether's theorem on conservation laws $A = \iint (K(\sigma)\phi_\theta^2 + \phi_\sigma^2)\mathrm{d}\sigma\mathrm{d}\theta$ for the equation $K(\sigma)\phi_{\theta\theta} + \phi_{\sigma\sigma} = 0 (\sigma K(\sigma) \geq 0)$. Rassias [8] studied weak solutions for the equation

$$K(\sigma)\phi_{\theta\theta} + \phi_{\sigma\sigma} + r(\sigma,\theta)\phi = f(\sigma,\theta), \quad \sigma K(\sigma) \geq 0. \tag{10}$$

Osher [9] showed the existence for Lavrentiev-Bitsadze equation $\operatorname{sgn}(\sigma)\phi_{\theta\theta} + \phi_{\sigma\sigma} = 0$. Aziz and Schneider [10] investigated the existence of weak L^2 solutions of the Gellerstedt problem and the Gellerstedt-Neumann problems for the equation

$$K(\sigma)u_{\theta\theta} + u_{\sigma\sigma} + \lambda u = f(\sigma,\theta), \quad \lambda = \text{constant} < 0. \tag{11}$$

Lupo et al. [11] proved existence of weak solutions for Tricomi problem with closed Dirichlet boundary conditions. Lupo et al. [12] considered the existence, uniqueness, and qualitative properties of weak solutions to the degenerate hyperbolic Goursat problem on characteristic triangles for linear and semilinear equations of Tricomi type. See also, for example, [13–16] for works on the nonlinear Tricomi problems. We recommend the introduction in the monograph [17] for a review of the status of mixed-type equations around the 1970s. Morawetz [18] also reviewed the existence and uniqueness theorems for mixed-type equations and their applications to transonic flows, and Chen [19] introduced more recent progress.

However, to the best of our knowledge, there is not any result on the Tricomi problem of Chaplygin's hodograph equation (1), which is relevant to many physical problems in transonic aerodynamics. So we will devote this work to establishing some basic properties of such problems. The main result is the following theorem.

Theorem 1. *There is a positive constant ε_0 determined only by μ such that if the domain D is contained in the strip $\{|t| < \varepsilon_0\}$, then the Tricomi problem (1) and (8) has a quasi-regular distributional solution. Furthermore, the solution is unique in $C^2(D) \cap C^1(\overline{D})$.*

For the definition of quasi-regular distributional solution, see Definition 2. The constant ε_0 is given in (77).

Our proof depends on the classical energy methods, or the *a-b-c* method of multipliers (see [8, 20, 21]). Besides the method of singular integral equations, these seem to be the only general way to study well-posedness of mixed-type equations. (However, see also [22] for regularity of solutions of Tricomi equation by using the methods from harmonic analysis.) Although the idea of energy method is rather simple, it is usually very technical to choose appropriate multipliers to a physically interesting equation, like (1), as shown in this paper.

We remark that there is another type of mixed elliptic-hyperbolic equations, firstly studied by Maria Cinquini-Cibrario, now called Keldysh type, whose canonical form is $t^{2m+1}\phi_{tt} + \phi_{\theta\theta} = 0$ $(m = 0, 1, \ldots)$ (see [23, page 11]). An up-to-date review of studies of Keldysh-type mixed equations was presented in [24]. It is possible now to study directly many boundary value problems of quasi-linear Keldysh-type equations; for example, see [5, 25, 26] for studies of steady continuous subsonic-supersonic flows in (approximate) de Laval nozzles.

The rest of the paper is organized as follows. In Section 2 we will define a quasi-regular distributional solution to our Tricomi problem and show that it satisfies the equation and boundary conditions in the ordinary sense if it is a classical solution. We will establish the uniqueness of classical solutions in Section 3. Finally, in Section 4, the existence of a quasi-regular distributional solution is proved by the dual method in functional analysis.

2. Definition of Solutions

Denote the linear operator L^* by

$$L^*\psi \triangleq K(t)\psi_{\theta\theta} + \psi_{tt} + (P(t)\psi)_t, \quad \psi \in \operatorname{Dom}(L^*), \tag{12}$$

with

$$\operatorname{Dom}(L^*) \triangleq \left\{w \in C^2(\overline{D}) : w|_{\Gamma_1 \cup \Gamma_2} = 0\right\}. \tag{13}$$

It is a formal dual operator of L.

Definition 2. Let $F \in L^2(D)$ and $f = \widetilde{\phi}|_{\Gamma_1}$ and $g = \widetilde{\phi}|_{\Gamma_3}$ for some $\widetilde{\phi} \in H^2(D)$. A function $\phi \in L^2(D)$ is a *quasi-regular distributional solution* of the equation

$$L\phi = F \quad \text{in } D \tag{14}$$

subjected to boundary conditions (8), if

$$\begin{aligned}(F,\psi)_{L^2(D)} &= (\phi, L^*\psi) - \int_{\Gamma_1} (K\psi_\theta n_1 + \psi_t n_2) f \,\mathrm{d}s \\ &+ \int_{\Gamma_3} \left[(Kg_\theta n_1 + g_t n_2)\psi - (K\psi_\theta n_1 + \psi_t n_2) g \right. \\ &\left. - Pg\psi n_2\right] \mathrm{d}s\end{aligned} \tag{15}$$

for all $\psi \in \operatorname{Dom}(L^*) \subset L^2(D)$.

Now we show that a quasi-regular distributional solution ϕ satisfies (1) and boundary conditions (8) in the classical pointwise sense if it belongs to $C^2(D) \cap C^1(\overline{D})$. In fact, using integration by parts and (15), we get, for all $\psi \in \operatorname{Dom}(L^*)$, that

$$\begin{aligned}&(F,\psi)_{L^2(D)} + \int_{\Gamma_1} (K\psi_\theta n_1 + \psi_t n_2) f \,\mathrm{d}s \\ &- \int_{\Gamma_3} \left[(Kg_\theta n_1 + g_t n_2)\psi - (K\psi_\theta n_1 + \psi_t n_2) g \right. \\ &\left. - Pg\psi n_2\right] \mathrm{d}s = (\phi, L^*\psi) = (L\phi, \psi) \\ &- \oint_{\partial D} \left[(K\phi_\theta n_1 + \phi_t n_2)\psi - (K\psi_\theta n_1 + \psi_t n_2)\phi \right. \\ &\left. - P\phi\psi n_2\right] \mathrm{d}s.\end{aligned} \tag{16}$$

Choosing particularly that $\psi \in C_0^\infty(D)$, all the three boundary integrals vanish, and (16) is reduced to

$$(F,\psi)_{L^2(D)} = (L\phi, \psi) \quad \text{for any } \psi \in C_0^\infty(D). \tag{17}$$

Since $C_0^\infty(D)$ is dense in $L^2(D)$, we get

$$L\phi = F \quad \text{in the } L^2\text{-sense}, \tag{18}$$

and hence $L\phi = F$ almost everywhere in D.

Next, by employing (16) and (18), we have, for all $\psi \in \operatorname{Dom}(L^*)$,

$$\begin{aligned}&\oint_{\partial D} \left[(K\phi_\theta n_1 + \phi_t n_2)\psi - (K\psi_\theta n_1 + \psi_t n_2)\phi \right. \\ &\left. - P\phi\psi n_2\right] \mathrm{d}s = - \int_{\Gamma_1} (K\psi_\theta n_1 + \psi_t n_2) f \,\mathrm{d}s \\ &+ \int_{\Gamma_3} \left[(Kg_\theta n_1 + g_t n_2)\psi - (K\psi_\theta n_1 + \psi_t n_2) g \right. \\ &\left. - Pg\psi n_2\right] \mathrm{d}s.\end{aligned} \tag{19}$$

Since $\psi|_{\Gamma_1\cup\Gamma_2} = 0$, it follows that

$$\begin{aligned}
&\int_{\Gamma_3} [(K\phi_\theta n_1 + \phi_t n_2)\,\psi - (K\psi_\theta n_1 + \psi_t n_2)\,\phi \\
&\quad - P\phi\psi n_2]\,\mathrm{d}s - \int_{\Gamma_1\cup\Gamma_2} (K\psi_\theta n_1 + \psi_t n_2)\,\phi\,\mathrm{d}s \\
&\quad = -\int_{\Gamma_1} (K\psi_\theta n_1 + \psi_t n_2)\, f\,\mathrm{d}s \\
&\quad + \int_{\Gamma_3} [(Kg_\theta n_1 + g_t n_2)\,\psi - (K\psi_\theta n_1 + \psi_t n_2)\, g \\
&\quad - Pg\psi n_2]\,\mathrm{d}s.
\end{aligned} \tag{20}$$

Therefore,

$$\begin{aligned}
&\int_{\Gamma_3} [(K(\phi - g)_\theta\, n_1 + (\phi - g)_t\, n_2)\,\psi \\
&\quad - (K\psi_\theta n_1 + \psi_t n_2 + P\psi n_2)\,(\phi - g)]\,\mathrm{d}s \\
&\quad - \int_{\Gamma_1} (K\psi_\theta n_1 + \psi_t n_2)\,(\phi - f)\,\mathrm{d}s - \int_{\Gamma_2} (K\psi_\theta n_1 \\
&\quad + \psi_t n_2)\,\phi\,\mathrm{d}s = 0
\end{aligned} \tag{21}$$

for all $\psi \in \mathrm{Dom}(L^*)$. Taking that ψ vanishes in a neighborhood of Γ_1 and Γ_2, we infer that $\phi|_{\Gamma_3} = g$, and furthermore, for all $\psi \in \mathrm{Dom}(L^*)$, there holds

$$\begin{aligned}
&\int_{\Gamma_1} (K\psi_\theta n_1 + \psi_t n_2)\,(\phi - f)\,\mathrm{d}s \\
&\qquad + \int_{\Gamma_2} (K\psi_\theta n_1 + \psi_t n_2)\,\phi\,\mathrm{d}s = 0.
\end{aligned} \tag{22}$$

Recall that $\psi|_{\Gamma_1\cup\Gamma_2} = 0$, and we have

$$\begin{aligned}
0 = \mathrm{d}\psi = \psi_t \mathrm{d}t + \psi_\theta \mathrm{d}\theta = (\psi_t n_1 - \psi_\theta n_2)\,\mathrm{d}s& \\
\text{on } \Gamma_1 \cup \Gamma_2.&
\end{aligned} \tag{23}$$

Hence we get $\psi_t n_1 = \psi_\theta n_2$ on $\Gamma_1 \cup \Gamma_2$, or

$$\begin{aligned}
\psi_t &= N n_2, \\
\psi_\theta &= N n_1, \\
&\text{on } \Gamma_1 \cup \Gamma_2,
\end{aligned} \tag{24}$$

where N is a normalizing factor. Thus, we have

$$(K\psi_\theta n_1 + \psi_t n_2)|_{\Gamma_1\cup\Gamma_2} = [K n_1^2 + n_2^2]\, N \tag{25}$$

by using (24). Since $K(t) > 0$ on Γ_1 and Γ_2 is a characteristic curve, then $[Kn_1^2 + n_2^2]|_{\Gamma_1} > 0$ and $[Kn_1^2 + n_2^2]|_{\Gamma_2} = 0$. Thus, we have

$$\int_{\Gamma_1} (K n_1^2 + n_2^2)\, N\,(\phi - f)\,\mathrm{d}s = 0 \tag{26}$$

by using (22). Since N is an arbitrary function, we see $\phi|_{\Gamma_1} = f$.

3. Uniqueness of Classical Solutions

Assume that $\phi_1, \phi_2 \in C^2(D) \cap C^1(\overline{D})$ are two solutions of Tricomi problem (1) and (8), and take

$$\phi = \phi_1 - \phi_2. \tag{27}$$

Then ϕ solves

$$\begin{aligned}
L\phi = K(t)\,\phi_{\theta\theta} + \phi_{tt} - P(t)\,\phi_t = 0 \quad \text{in } D,& \\
\phi|_{\Gamma_1\cup\Gamma_3} = 0.&
\end{aligned} \tag{28}$$

We will show that $\phi \equiv 0$ in D.

Set

$$\begin{aligned}
I &\triangleq 2\iint_D L\phi \cdot [a(t,\theta)\,\phi + b(t,\theta)\,\phi_t \\
&\quad + c(t,\theta)\,\phi_\theta]\,\mathrm{d}t\,\mathrm{d}\theta = \iint_D \{2aK(t)\,\phi\phi_{\theta\theta} \\
&\quad + 2a\phi\phi_{tt} - 2aP(t)\,\phi\phi_t + 2bK(t)\,\phi_t\phi_{\theta\theta} + 2b\phi_t\phi_{tt} \\
&\quad - 2bP(t)\,\phi_t^2 + 2cK(t)\,\phi_\theta\phi_{\theta\theta} + 2c\phi_\theta\phi_{tt} \\
&\quad - 2cP(t)\,\phi_\theta\phi_t\}\,\mathrm{d}t\,\mathrm{d}\theta,
\end{aligned} \tag{29}$$

where $a(t,\theta)$, $b(t,\theta)$, and $c(t,\theta)$ are sufficiently smooth functions to be determined (*cf.* Remark 4). Since

$$\begin{aligned}
2aK(t)\,\phi\phi_{\theta\theta} &= (2aK\phi\phi_\theta)_\theta - 2aK\phi_\theta^2 - (a_\theta K\phi^2)_\theta \\
&\quad + a_{\theta\theta}K\phi^2, \\
2a\phi\phi_{tt} &= (2a\phi\phi_t)_t - 2a\phi_t^2 - (a_t\phi^2)_t + a_{tt}\phi^2, \\
2aP(t)\,\phi\phi_t &= (aP(t)\,\phi^2)_t - (aP(t))_t\,\phi^2, \\
2bK(t)\,\phi_t\phi_{\theta\theta} &= (2bK\phi_t\phi_\theta)_\theta - (bK\phi_\theta^2)_t + (bK)_t\,\phi_\theta^2 \\
&\quad - 2b_\theta K\phi_t\phi_\theta, \\
2b\phi_t\phi_{tt} &= (b\phi_t^2)_t - b_t\phi_t^2, \\
2cK(t)\,\phi_\theta\phi_{\theta\theta} &= (cK\phi_\theta^2)_\theta - c_\theta K\phi_\theta^2, \\
2c\phi_\theta\phi_{tt} &= (2c\phi_\theta\phi_t)_t - (c\phi_t^2)_\theta + c_\theta\phi_t^2 - 2c_t\phi_t\phi_\theta,
\end{aligned} \tag{30}$$

we obtain that

$$
\begin{aligned}
0 = I &= \iint_D \left[a_{\theta\theta}K + a_{tt} + (aP(t))_t\right]\phi^2 \,\mathrm{d}t\,\mathrm{d}\theta \\
&\quad - \iint_D \left\{\left[2aK - (bK)_t + c_\theta K\right]\phi_\theta^2\right. \\
&\quad + 2\left[b_\theta K + c_t + cP(t)\right]\phi_\theta\phi_t \\
&\quad \left.+ \left[2a + b_t + 2bP(t) - c_\theta\right]\phi_t^2\right\}\mathrm{d}t\,\mathrm{d}\theta \\
&\quad + \oint_{\partial D}\left\{2a\phi\left[K\phi_\theta n_1 + \phi_t n_2\right] - \left[Ka_\theta n_1 + a_t n_2\right]\phi^2\right. \\
&\quad \left.- aP(t)\phi^2 n_2\right\}\mathrm{d}s + \oint_{\partial D}\left\{\left[cn_1 - bn_2\right]K\phi_\theta^2\right. \\
&\quad \left.+ 2\left[bKn_1 + cn_2\right]\phi_\theta\phi_t + \left[bn_2 - cn_1\right]\phi_t^2\right\}\mathrm{d}s \\
&\quad \triangleq I_1 + I_2 + J_1 + J_2.
\end{aligned} \tag{31}
$$

The goal is to show that all integrals I_1, I_2, J_1, and J_2 are nonnegative by choosing suitable functions a, b, and c.

One observes that the integral $I_1 \geq 0$ if

$$a_{\theta\theta}K + a_{tt} + (aP(t))_t \geq 0 \quad \text{in } D, \tag{32}$$

and the integral $I_2 \geq 0$ if the following conditions hold in D:

$$
\begin{aligned}
&2aK - (bK)_t + c_\theta K \leq 0, \\
&\left[b_\theta K + c_t + cP(t)\right]^2 \\
&\quad - \left[2aK - (bK)_t + c_\theta K\right]\left[2a + b_t + 2bP(t) - c_\theta\right] \\
&\quad \leq 0.
\end{aligned} \tag{33}
$$

By using (28), we have

$$
\begin{aligned}
J_1 &= 2\int_{\Gamma_2} a\phi\left[K\phi_\theta n_1 + \phi_t n_2\right]\mathrm{d}s \\
&\quad - \int_{\Gamma_2}\left[Ka_\theta n_1 + a_t n_2\right]\phi^2\,\mathrm{d}s - \int_{\Gamma_2} aP(t)\phi^2 n_2\,\mathrm{d}s \\
&\triangleq J_{11} + J_{12} + J_{13}.
\end{aligned} \tag{34}
$$

Since $n_2 = -n_1\sqrt{-K}$ on Γ_2, it follows that

$$
\begin{aligned}
(\mathrm{d}\phi)\big|_{\Gamma_2} &= \phi_t\mathrm{d}t + \phi_\theta\mathrm{d}\theta = \left(\phi_t n_1 - \phi_\theta n_2\right)\mathrm{d}s \\
&= \left(\phi_t + \phi_\theta\sqrt{-K}\right)n_1\mathrm{d}s \\
&= \frac{\left(\sqrt{-K}\phi_t - K\phi_\theta\right)n_1\mathrm{d}s}{\sqrt{-K}} \\
&= -\frac{\left(K\phi_\theta n_1 + \phi_t n_2\right)\mathrm{d}s}{\sqrt{-K}}.
\end{aligned} \tag{35}
$$

Then

$$
\begin{aligned}
J_{11} &= 2\int_{\Gamma_2} a\phi\left[K\phi_\theta n_1 + \phi_t n_2\right]\mathrm{d}s = -2\int_{\Gamma_2} a\sqrt{-K}\phi\,\mathrm{d}\phi \\
&= -\int_{\Gamma_2} a\sqrt{-K}\mathrm{d}\left(\phi^2\right) \\
&= -\left[a\sqrt{-K}\phi^2\right]\Big|_A^P + \int_{\Gamma_2}\phi^2\,\mathrm{d}\left(a\sqrt{-K}\right).
\end{aligned} \tag{36}
$$

Recall that $\phi(A) = 0$ and $\phi(P) = 0$, we have

$$J_{11} = \int_{\Gamma_2}\phi^2\mathrm{d}\left(a\sqrt{-K}\right). \tag{37}$$

We also note that

$$
\begin{aligned}
(\mathrm{d}a)\big|_{\Gamma_2} &= a_t\mathrm{d}t + a_\theta\mathrm{d}\theta = \left(a_t n_1 - a_\theta n_2\right)\mathrm{d}s \\
&= \left(a_t + a_\theta\sqrt{-K}\right)n_1\mathrm{d}s \\
&= \frac{\left(\sqrt{-K}a_t - Ka_\theta\right)n_1\mathrm{d}s}{\sqrt{-K}} \\
&= -\frac{\left(Ka_\theta n_1 + a_t n_2\right)\mathrm{d}s}{\sqrt{-K}},
\end{aligned} \tag{38}
$$

and hence

$$J_{12} = -\int_{\Gamma_2}\left[Ka_\theta n_1 + a_t n_2\right]\phi^2\,\mathrm{d}s = \int_{\Gamma_2}\phi^2\sqrt{-K}\mathrm{d}a. \tag{39}$$

Therefore, we get

$$J_{11} + J_{12} = \int_{\Gamma_2}\phi^2\left[\mathrm{d}\left(a\sqrt{-K}\right) + \sqrt{-K}\mathrm{d}a\right] \tag{40}$$

by using (37) and (39). It follows that

$$J_1 = \int_{\Gamma_2}\phi^2\left[\mathrm{d}\left(a\sqrt{-K}\right) + \sqrt{-K}\mathrm{d}a - aP(t)n_2\mathrm{d}s\right] \tag{41}$$

by using (34). Since $n_2 < 0$ and $\mathrm{d}\theta = \sqrt{-K}\mathrm{d}t = -n_2\mathrm{d}s > 0$ on Γ_2, we have

$$
\begin{aligned}
\mathrm{d}\left(a\sqrt{-K}\right)\big|_{\Gamma_2} &= \left(a\sqrt{-K}\right)_t\mathrm{d}t + \left(a\sqrt{-K}\right)_\theta\mathrm{d}\theta \\
&= \left[a_t + \frac{aK'}{2K}\right]\sqrt{-K}\mathrm{d}t + \sqrt{-K}a_\theta\mathrm{d}\theta \\
&= \left\{a_t + \frac{aK'}{2K} + a_\theta\sqrt{-K}\right\}\mathrm{d}\theta, \\
\left(\sqrt{-K}\mathrm{d}a\right)\big|_{\Gamma_2} &= \sqrt{-K}\left(a_t\mathrm{d}t + a_\theta\mathrm{d}\theta\right) \\
&= \left(a_t + a_\theta\sqrt{-K}\right)\mathrm{d}\theta.
\end{aligned} \tag{42}
$$

Thus

$$
\begin{aligned}
&\mathrm{d}\left(a\sqrt{-K}\right) + \sqrt{-K}\mathrm{d}a - aP(t)n_2\mathrm{d}s \\
&\quad = \left\{2a_t + 2a_\theta\sqrt{-K} + \frac{aK'}{2K} + aP(t)\right\}\mathrm{d}\theta \quad \text{on } \Gamma_2.
\end{aligned} \tag{43}
$$

So $J_1 \geq 0$ by using (43) provided that

$$a_t + a_\theta \sqrt{-K} + a\frac{K' + 2KP(t)}{4K} \geq 0 \quad \text{on } \Gamma_2. \tag{44}$$

Next, observe that the integral $J_2 \geq 0$ if

$$J_2 = \oint_{\partial D} Q\,\mathrm{d}s = \int_{\Gamma_1 \cup \Gamma_3} Q\,\mathrm{d}s + \int_{\Gamma_2} Q\,\mathrm{d}s \triangleq J_{21} + J_{22} \geq 0, \tag{45}$$

where

$$Q = [cn_1 - bn_2] K\phi_\theta^2 + 2[bKn_1 + cn_2]\phi_\theta\phi_t + [bn_2 - cn_1]\phi_t^2 \tag{46}$$

is a quadratic form of ϕ_θ and ϕ_t. Since $\phi|_{\Gamma_1 \cup \Gamma_3} = 0$, we have

$$0 = \mathrm{d}\phi = \phi_t \mathrm{d}t + \phi_\theta \mathrm{d}\theta = (\phi_t n_1 - \phi_\theta n_2)\,\mathrm{d}s \quad \text{on } \Gamma_1 \cup \Gamma_3, \tag{47}$$

which implies that, by similar analysis as in Section 2, we can set

$$\phi_t = \widetilde{N} n_2, \quad \phi_\theta = \widetilde{N} n_1, \quad \text{on } \Gamma_1 \cup \Gamma_3, \tag{48}$$

with $\widetilde{N}$ being a normalizing factor. Thus, we obtain that

$$\begin{aligned} Q|_{\Gamma_1 \cup \Gamma_3} &= [cn_1 - bn_2] K\phi_\theta^2 + 2\left[bK\phi_\theta^2 n_2 + cn_1\phi_t^2\right] \\ &\quad + [bn_2 - cn_1]\phi_t^2 = [cn_1 + bn_2]\left[K\phi_\theta^2 + \phi_t^2\right] \\ &= [cn_1 + bn_2]\left[Kn_1^2 + n_2^2\right]\widetilde{N}^2. \end{aligned} \tag{49}$$

Since $K(t) > 0$ on Γ_1 and Γ_3 is characteristic, then $[Kn_1^2 + n_2^2]|_{\Gamma_1} > 0$ and $[Kn_1^2 + n_2^2]|_{\Gamma_3} = 0$. Thus, we have

$$Q|_{\Gamma_1 \cup \Gamma_3} = Q|_{\Gamma_1} = [cn_1 + bn_2]\left[Kn_1^2 + n_2^2\right]\widetilde{N}^2 \geq 0 \quad (\Longrightarrow J_{21} \geq 0) \tag{50}$$

provided that

$$cn_1 + bn_2 \geq 0 \quad \text{on } \Gamma_1. \tag{51}$$

Also, since Γ_2 is characteristic, we infer that $[Kn_1^2 + n_2^2]|_{\Gamma_2} = 0$. Moreover, we have

$$\begin{aligned} &[bKn_1 + cn_2]^2 - [cn_1 - bn_2]K \cdot [bn_2 - cn_1] \\ &\quad = \left(b^2K + c^2\right)\left(Kn_1^2 + n_2^2\right) = 0 \quad \text{on } \Gamma_2. \end{aligned} \tag{52}$$

Since $K|_{\Gamma_2} < 0$, we have $Q|_{\Gamma_2} \geq 0$ by using (46), and then $J_{22} \geq 0$, provided that

$$bn_2 - cn_1 \geq 0 \quad \text{on } \Gamma_2. \tag{53}$$

Recall that $n_2 = -n_1\sqrt{-K}$ on Γ_2 and $n_1 > 0$ on Γ_2; then (53) is equivalent to

$$c + b\sqrt{-K} \leq 0 \quad \text{on } \Gamma_2. \tag{54}$$

Therefore, by (32), (33), (44), (54), and (51), we summarize the requirements on the multipliers a, b, and c as follows:

$$a_{\theta\theta}K + a_{tt} + (aP(t))_t \geq 0 \quad \text{in } D, \tag{55a}$$

$$2aK - (bK)_t + c_\theta K \leq 0 \quad \text{in } D, \tag{55b}$$

$$\begin{aligned} &\left[b_\theta K + c_t + cP(t)\right]^2 \\ &\quad - \left[2aK - (bK)_t + c_\theta K\right]\left[2a + b_t + 2bP(t) - c_\theta\right] \\ &\quad \leq 0 \quad \text{in } D, \end{aligned} \tag{55c}$$

$$a_t + a_\theta\sqrt{-K} + a\frac{K' + 2KP}{4K} \geq 0 \quad \text{on } \Gamma_2, \tag{55d}$$

$$c + b\sqrt{-K} \leq 0 \quad \text{on } \Gamma_2, \tag{55e}$$

$$cn_1 + bn_2 \geq 0 \quad \text{on } \Gamma_1. \tag{55f}$$

Our task below is to find sufficient conditions such that inequalities (55a)–(55f) hold. Set $D^+ = D \cap \{t > 0\}$ to be the elliptic region and set $D^- = D \cap \{t < 0\}$ to be the hyperbolic region. We will actually choose

$$\begin{aligned} a &= e^{\lambda\theta}\left(t^2 - \sigma\right) \quad \text{in } D, \\ b &= c = 0 \quad \text{in } \overline{D^+}, \\ b &= \frac{4aK}{K' - 2KP(t)} \quad \text{in } D^-, \\ c &= -b\sqrt{-K} \quad \text{in } D^-, \end{aligned} \tag{56}$$

where λ can be taken as -1 and σ is to be $(1 - \mu^2)/4$. What is left is to choose ε_0 announced in Theorem 1 sufficiently small (depending only on μ that appeared in the coefficients of (1)), as computations shown below.

Elliptic Region D^+. First of all, we specify

$$c = b = 0 \quad \text{in } D^+ \tag{57}$$

to meet the requirement of (55f). Thus, remember that $K \geq 0$ in D^+, and inequalities (55a)–(55c) are reduced to

$$\begin{aligned} &a \leq 0, \\ &a_{\theta\theta}K + a_{tt} + (aP(t))_t \geq 0 \quad \text{in } D^+. \end{aligned} \tag{58}$$

Thus, if $a = e^{\lambda\theta}\varphi(t)$, (55a) is transformed to

$$\lambda^2\varphi(t)K + \varphi''(t) + \varphi(t)P' + \varphi'(t)P \geq 0. \tag{59}$$

Next, we choose $\varphi(t) = t^2 - \sigma < 0$. Then $\varphi'(t) = 2t$, $\varphi''(t) = 2$, and (59) is simplified as

$$2 + 2tP + \left(P' + \lambda^2 K\right)\left(t^2 - \sigma\right) \geq 0. \tag{60}$$

It is easy to see that this holds for sufficiently small $|t|$, provided that $\sigma P' < 3/2$, which is exactly

$$\sigma\left\{\left(1-\mu^2\right)\left[1+(1-t)^2\right]+\mu^2\left[1-(1-t)^2\right]^2\right\} < \frac{3}{2}(1-t)^2\left[1-\mu^2(1-t)^2\right]^2. \tag{61}$$

By fixing $\sigma = (1-\mu^2)/4$, a sufficient condition for this inequality is to take a small ε_1 (depending only on μ) and then require that

$$|t| < \min\left\{\varepsilon_1, \sqrt{\sigma}\right\}. \tag{62}$$

Hyperbolic Region D^-. Now we choose

$$c = -b\sqrt{-K} \quad \text{in } D^- \tag{63}$$

to satisfy (55e). Then

$$b_\theta K + c_t + cP(t) = \frac{1}{2\sqrt{-K}}\left[\left(b_t K + bK' - c_\theta K\right) + \left(b_t K + 2bKP(t) - c_\theta K\right)\right], \tag{64}$$

and (55c) becomes

$$\begin{aligned} 0 &\geq \left[b_\theta K + c_t + cP(t)\right]^2 - \left[2aK - (bK)_t + c_\theta K\right]\left[2a + b_t + 2bP(t) - c_\theta\right] = \frac{1}{-4K}\left\{\left[\left(b_t K + bK' - c_\theta K\right) - \left(b_t K + 2bKP(t) - c_\theta K\right)\right]^2 \right.\\ &\quad - 8aK\left[\left(b_t K + bK' - c_\theta K\right) \left. - \left(b_t K + 2bKP(t) - c_\theta K\right)\right] + 16a^2K^2\right\} \\ &= \frac{1}{-4K}\left\{\left[bK' - 2bKP(t)\right]^2 - 8aK\left(bK' - 2bKP(t)\right) + 16a^2K^2\right\} = \frac{1}{-4K}\left[bK' - 2bKP(t) - 4aK\right]^2. \end{aligned} \tag{65}$$

Therefore, we must choose

$$b = \frac{4aK}{K' - 2KP(t)} \quad \text{in } D^-. \tag{66}$$

Note that, by using (2), direct computation yields

$$K'(t) = 2\frac{1-\mu^2+\mu^2t^2(2-t)^2}{(1-t)^3\left[1-\mu^2(1-t)^2\right]^2} > 0, \tag{67}$$

$$K'(t) - 2K(t)P(t) = \frac{2\left(1-\mu^2\right)\left[1-t^2(2-t)^2\right]}{(1-t)^3\left[1-\mu^2(1-t)^2\right]^2} > 0 \tag{68}$$

for $|t| < \varepsilon_2$, where ε_2 is a small positive constant determined by μ. Hence (66) is well defined.

Next, we still choose $a = e^{\lambda\theta}(t^2-\sigma)$ in D^-. Since condition (62) is valid as required, then, as shown above, (55a) holds automatically. Hence, (55d) becomes

$$e^{\lambda\theta}2t + \lambda e^{\lambda\theta}\left(t^2-\sigma\right)\sqrt{-K} + e^{\lambda\theta}\left(t^2-\sigma\right)\frac{K'+2KP}{4K} \geq 0 \quad \text{on } \Gamma_2. \tag{69}$$

That is,

$$8tK + 4\lambda\left(t^2-\sigma\right)\sqrt{-K}K + \left(t^2-\sigma\right)\left(K'+2KP\right) \leq 0 \quad \text{on } \Gamma_2. \tag{70}$$

By fixing $\lambda = -1$, this is valid for $|t| < \varepsilon_3$ with ε_3 being a small positive constant. Here we still used continuity and the facts that $K' > 0$ and $KP \geq 0$ on Γ_2.

In order to get (55b), we need to have

$$2aK - b_tK - bK' + b_\theta(-K)^{3/2} \leq 0. \tag{71}$$

Since (66) implies that

$$\begin{aligned} 2aK - bK' &= -2aK - 2bKP(t) = -2K(a + bP(t)) \\ &= -2aK\frac{K'+2KP}{K'-2KP} \leq 0, \end{aligned} \tag{72}$$

we only need to guarantee that

$$\begin{aligned} b_\theta &\leq 0 \\ b_t &\leq 0, \\ &\text{in } D^-. \end{aligned} \tag{73}$$

In fact, by using (68), we have

$$b = \frac{2e^{\lambda\theta}}{1-\mu^2}\cdot\frac{\left(t^2-\sigma\right)t(2-t)\left[1-\mu^2(1-t)^2\right]}{(1-t)\left(1+2t-t^2\right)}. \tag{74}$$

It is obvious that

$$b_\theta = \frac{2\lambda e^{\lambda\theta}}{1-\mu^2}\cdot\frac{\left(t^2-\sigma\right)t(2-t)\left[1-\mu^2(1-t)^2\right]}{(1-t)\left(1+2t-t^2\right)} \leq 0 \quad \text{for } \lambda = -1. \tag{75}$$

Direct computation yields that

$$b_t = \frac{2e^{\lambda\theta}\left(t^2-\sigma\right)}{1-\mu^2} \cdot \left\{ \frac{\left[2t^2(2-t)/\left(t^2-\sigma\right)+(2-2t)\right]\left[1-\mu^2(1-t)^2\right]+2\mu^2 t(2-t)(1-t)}{(1-t)\left(1+2t-t^2\right)} - \frac{t(2-t)\left[1-\mu^2(1-t)^2\right]\left[1-6t+3t^2\right]}{(1-t)^2\left(1+2t-t^2\right)^2} \right\} < 0 \tag{76}$$

for $-\varepsilon_4 < t < 0$ as desired. Here ε_4 is a small positive constant determined by μ.

Finally we see that the positive constant ε_0 should be chosen so that

$$\varepsilon_0 \le \min\left\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \sqrt{\sigma}\right\}. \tag{77}$$

This finishes the choice of multipliers and we proved that $I_1 = 0$, $I_2 = 0$, $J_1 = 0$, and $J_2 = 0$. (To guarantee existence claimed in Section 4, ε_0 might need to be chosen further smaller, according to the construction of multipliers in Section 4.1, but anyway it is in essence determined by the parameter μ that appeared in (1)).

Finally, observe that (62) actually guarantees the stronger property that

$$a_{\theta\theta}K + a_{tt} + (aP(t))_t > 0 \quad \text{in } D. \tag{78}$$

Hence, $I_1 = 0$ implies that $\phi \equiv 0$ as desired.

Remark 3. Note that in the above we have chosen b and c to be only continuous across the degenerate line $\{t = 0\}$. This is harmless for our earlier computations since, by applying integration by parts separately in D^+ and D^- and then summing up, the resultant line integrals on $D \cap \{t = 0\}$ are cancelled.

Remark 4. There are some other ways to choose the multipliers. For example, we may set

$$\begin{aligned} &a = b = c = 0 \quad \text{in } \overline{D^+}, \\ &a = e^{\lambda\theta} t^{\sigma} \quad \text{in } D^-, \\ &b = \frac{4aK}{K' - 2KP(t)} \quad \text{in } D^-, \\ &c = -b\sqrt{-K} \quad \text{in } D^-, \end{aligned} \tag{79}$$

where $\lambda \le 0$ and $\sigma = 1/(2k+1)$ $(k \in \mathbb{N})$.

The other way is to choose

$$\begin{aligned} &a = b = c = 0 \quad \text{in } \overline{D^+}, \\ &a = e^{\lambda\theta}\arctan(\sigma t) \quad \text{in } D^-, \\ &b = \frac{4aK}{K' - 2KP(t)} \quad \text{in } D^-, \\ &c = -b\sqrt{-K} \quad \text{in } D^-, \end{aligned} \tag{80}$$

where $\lambda \le 0$ and $\sigma > 2/\sqrt{1-\mu^2}$ is sufficiently large.

However, in both cases, as before, we need D to be quite close to the line $\{(t,\theta) : t = 0\}$. So the restriction on smallness of ε_0 required in Theorem 1 is still not removed.

4. Existence of Quasi-Regular Distributional Solutions

In this section, we firstly indicate how to obtain a priori estimate for our Tricomi problem and then use this estimate to show the existence of a quasi-regular distributional solution by a dual method in functional analysis.

4.1. A Priori Estimate. We now prove that

$$\|\psi\|_{W^{1,2}(D)} \le C\left\|L^*\psi\right\|_{L^2(D)}, \quad \forall \psi \in \operatorname{Dom}\left(L^*\right). \tag{81}$$

Similar to the analysis in the previous section, we have

$$\begin{aligned} I^* &\triangleq 2\iint_D L^*\psi \cdot \left[a^*(t,\theta)\psi + b^*(t,\theta)\psi_t \right. \\ &\quad \left. + c^*(t,\theta)\psi_\theta\right] \mathrm{d}t\,\mathrm{d}\theta = \iint_D \left\{2a^*K\psi\psi_{\theta\theta} \right. \\ &\quad + 2a^*\psi\psi_{tt} + 2a^*P\psi\psi_t + 2a^*P'\psi^2 + 2b^*K\psi_t\psi_{\theta\theta} \\ &\quad + 2b^*\psi_t\psi_{tt} + 2b^*P\psi_t^2 + 2b^*P'\psi\psi_t + 2c^*K\psi_\theta\psi_{\theta\theta} \\ &\quad \left. + 2c^*\psi_\theta\psi_{tt} + 2c^*P\psi_\theta\psi_t + 2c^*P'\psi\psi_\theta\right\} \mathrm{d}t\,\mathrm{d}\theta \\ &= \iint_D \left[a^*P' + a^*_{\theta\theta}K + a^*_{tt} - a^*_t P - \left(b^*P'\right)_t \right. \\ &\quad \left. - c^*_\theta P'\right]\psi^2 \mathrm{d}t\,\mathrm{d}\theta \\ &\quad + \iint_D \left\{\left[-2a^*K + \left(b^*K\right)_t - c^*_\theta K\right]\psi_\theta^2 \right. \\ &\quad + 2\left[c^*P - b^*_\theta K - c^*_t\right]\psi_\theta\psi_t \\ &\quad \left. + \left[2b^*P - 2a^* - b^*_t + c^*_\theta\right]\psi_t^2\right\} \mathrm{d}t\,\mathrm{d}\theta \\ &\quad + \oint_{\partial D} \left\{2a^*\psi\left[K\psi_\theta n_1 + \psi_t n_2\right] \right. \\ &\quad - \left[Ka^*_\theta n_1 + a^*_t n_2\right]\psi^2 \\ &\quad \left. + \left[a^*Pn_2 + b^*P'n_2 + c^*P'n_1\right]\psi^2\right\} \mathrm{d}s \end{aligned}$$

$$+\oint_{\partial D}\{[c^*n_1-b^*n_2]K\psi_\theta^2+2[b^*Kn_1+c^*n_2]\psi_\theta\psi_t+[b^*n_2-c^*n_1]\psi_t^2\}\,ds \triangleq I_1^*+I_2^*+J_1^*+J_2^*. \quad (82)$$

Since $\psi|_{\Gamma_1\cup\Gamma_2}=0$, it follows that

$$J_1^*=\int_{\Gamma_3}\{2a^*\psi[K\psi_\theta n_1+\psi_t n_2]-[Ka_\theta^* n_1+a_t^* n_2]\psi^2+[a^*Pn_2+b^*P'n_2+c^*P'n_1]\psi^2\}\,ds \triangleq J_{11}^*+J_{12}^*+J_{13}^*. \quad (83)$$

Observing that $d\theta=-\sqrt{-K}dt$, or $-n_2ds=-\sqrt{-K}n_1ds$, and hence $n_2=\sqrt{-K}n_1$ on Γ_3, we have

$$(d\psi)|_{\Gamma_3}=(\psi_t n_1-\psi_\theta n_2)\,ds=\frac{(K\psi_\theta n_1+\psi_t n_2)\,ds}{\sqrt{-K}}. \quad (84)$$

Then

$$J_{11}^*=2\int_{\Gamma_3}a^*\sqrt{-K}\psi\,d\psi = \left[a^*\sqrt{-K}\psi^2\right]\Big|_O^P-\int_{\Gamma_3}\psi^2\,d\left(a^*\sqrt{-K}\right). \quad (85)$$

Remember $\psi|_O=0$ and $\psi|_P=0$, and we get

$$J_{11}^*=-\int_{\Gamma_3}\psi^2\,d\left(a^*\sqrt{-K}\right). \quad (86)$$

Next, using

$$(da^*)|_{\Gamma_3}=(a_t^*n_1-a_\theta^*n_2)\,ds=\frac{(Ka_\theta^*n_1+a_t^*n_2)\,ds}{\sqrt{-K}}, \quad (87)$$

we may write

$$J_{12}^*=-\int_{\Gamma_3}[Ka_\theta^*n_1+a_t^*n_2]\psi^2ds=-\int_{\Gamma_3}\psi^2\sqrt{-K}da^*. \quad (88)$$

Henceforth, using (83)–(88),

$$J_1^*=\int_{\Gamma_3}\left[-d\left(a^*\sqrt{-K}\right)-\sqrt{-K}da^*+\left(a^*Pn_2+b^*P'n_2+c^*P'n_1\right)ds\right]\psi^2\ge 0, \quad (89)$$

provided that

$$-d\left(a^*\sqrt{-K}\right)-\sqrt{-K}da^*+\left(a^*Pn_2+b^*P'n_2+c^*P'n_1\right)ds\ge 0 \quad \text{on } \Gamma_3. \quad (90)$$

Since $n_2<0$ and $d\theta=-\sqrt{-K}dt=-n_2ds>0$ on Γ_3, we have

$$d\left(a^*\sqrt{-K}\right)\Big|_{\Gamma_3}=-\left\{a_t^*+\frac{a^*K'}{2K}-a_\theta^*\sqrt{-K}\right\}d\theta, \qquad \left(\sqrt{-K}da^*\right)\Big|_{\Gamma_3}=\left(-a_t^*+a_\theta^*\sqrt{-K}\right)d\theta. \quad (91)$$

Thus, using (90) and $d\theta|_{\Gamma_3}\ge 0$, $J_1^*\ge 0$ provided that

$$2a_t^*+\frac{a^*K'}{2K}-2a_\theta^*\sqrt{-K}-a^*P-b^*P'-\frac{c^*P'}{\sqrt{-K}}\ge 0 \quad \text{on } \Gamma_3. \quad (92)$$

The integral J_2 is nonnegative if

$$J_2^*=\oint_{\partial D}Q^*\,ds\equiv\int_{\Gamma_1\cup\Gamma_2}Q^*\,ds+\int_{\Gamma_3}Q^*\,ds\triangleq J_{21}^*+J_{22}^*\ge 0, \quad (93)$$

where

$$Q^*=[c^*n_1-b^*n_2]K\psi_\theta^2+2[b^*Kn_1+c^*n_2]\psi_\theta\psi_t+[b^*n_2-c^*n_1]\psi_t^2 \quad (94)$$

is a quadratic form with respect to ψ_θ and ψ_t. Similarly to get (24), we have

$$\begin{aligned}Q^*|_{\Gamma_1\cup\Gamma_2}&=[c^*n_1-b^*n_2]K\psi_\theta^2+2\left[b^*K\psi_\theta^2n_2+c^*n_1\psi_t^2\right]+[b^*n_2-c^*n_1]\psi_t^2\\&=[c^*n_1+b^*n_2]\left[K\psi_\theta^2+\psi_t^2\right]\\&=[c^*n_1+b^*n_2]\left[Kn_1^2+n_2^2\right]N^2.\end{aligned} \quad (95)$$

Since $K(t)>0$ on Γ_1 and Γ_2 is a characteristic curve, then $[Kn_1^2+n_2^2]|_{\Gamma_1}>0$ and $[Kn_1^2+n_2^2]|_{\Gamma_2}=0$. Thus, we have

$$Q^*|_{\Gamma_1\cup\Gamma_2}=Q^*|_{\Gamma_1}=[c^*n_1+b^*n_2]\left[Kn_1^2+n_2^2\right]N^2\ge 0 \quad (\Longrightarrow J_{21}^*\ge 0) \quad (96)$$

provided that

$$c^*n_1+b^*n_2\ge 0 \quad \text{on } \Gamma_1. \quad (97)$$

Since Γ_3 is characteristic, then $[Kn_1^2+n_2^2]|_{\Gamma_3}=0$. Moreover, we have

$$[b^*Kn_1+c^*n_2]^2-[c^*n_1-b^*n_2]K\cdot[b^*n_2-c^*n_1]=\left[(b^*)^2K+(c^*)^2\right]\left(Kn_1^2+n_2^2\right)=0 \quad (98)$$

on Γ_3. Because $K|_{\Gamma_3}<0$, we see $Q^*|_{\Gamma_3}\ge 0$ (then $J_{22}^*\ge 0$) provided that

$$b^*n_2-c^*n_1\ge 0 \quad \text{on } \Gamma_3. \quad (99)$$

Since $n_2=n_1\sqrt{-K}$ on Γ_3, then (99) is equivalent to

$$\left(b^*\sqrt{-K}-c^*\right)n_1\ge 0 \quad \text{on } \Gamma_3. \quad (100)$$

Note that $n_1<0$ on Γ_3, so (99) or (100) is equivalent to

$$c^*-b^*\sqrt{-K}\ge 0 \quad \text{on } \Gamma_3. \quad (101)$$

Therefore we conclude, with the further help of Young's inequality, that

$$\begin{aligned} I_1^* + I_2^* \le I^* &\le \iint_D 2\left|L^*\psi\right| \cdot \left(\left|a^*\psi\right| + \left|b^*\psi_t + c^*\psi_\theta\right|\right) dt\, d\theta \\ &\le \iint_D \left\{\lambda_1 \left|a^*\psi\right|^2 + \lambda_2 \left|b^*\psi_t + c^*\psi_\theta\right|^2 + \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)\left|L^*\psi\right|^2\right\} dt\, d\theta \end{aligned} \tag{102}$$

provided that (92), (97), and (101) hold. (Here λ_1 and λ_2 are two positive constants that can be chosen arbitrarily small.) Therefore, we have the a priori estimate (81), if (92), (97), (101), and

$$a^*P' + a^*_{\theta\theta}K + a^*_{tt} - a^*_t P - \left(b^*P'\right)_t - c^*_\theta P' - \lambda_1 \left(a^*\right)^2 \ge \epsilon_1 > 0 \quad \text{in } D, \tag{103a}$$

$$N_1 \triangleq -2a^*K + \left(b^*K\right)_t - c^*_\theta K - \lambda_2 \left(c^*\right)^2 \ge 0 \quad \text{in } D, \tag{103b}$$

$$N_2 \triangleq 2b^*P - 2a^* - b^*_t + c^*_\theta - \lambda_2 \left(b^*\right)^2 \ge \epsilon_2 > 0 \quad \text{in } D, \tag{103c}$$

$$N_1 N_2 - \left[c^*P - b^*_\theta K - c^*_t - \lambda_2 b^* c^*\right]^2 \ge 0 \quad \text{in } D \tag{103d}$$

are valid for some multipliers a^*, b^*, and c^*. Here ϵ_1 and ϵ_2 are two positive constants.

As a matter of fact, we can choose the functions a^*, b^*, and c^* such that

$$\begin{aligned} a^* &= e^{\lambda\theta}\left(t^2 - \sigma\right) \quad \text{in } D, \\ b^* &= c^* = 0 \quad \text{in } \overline{D^+}, \\ b^* &= \frac{4a^*K}{K' + 2KP(t)} \quad \text{in } D^-, \\ c^* &= b^*\sqrt{-K} \quad \text{in } D^-, \end{aligned} \tag{104}$$

to guarantee that conditions (92), (97), (101), and (103a)–(103d) hold, if $\lambda < 0$ and $0 < \sigma < (1-\mu^2)/2$ were chosen similarly as in Section 3, and ε_0 are taken appropriately small. The verification is very similar to that in the previous section and therefore we omit the details.

4.2. The Proof of the Existence. By our assumption, there exists a function $\widetilde{\phi} \in H^2(D)$ such that $\widetilde{\phi}|_{\Gamma_1} = f$ and $\widetilde{\phi}|_{\Gamma_3} = g$. Next, take

$$\overline{\phi} = \phi - \widetilde{\phi}, \tag{105}$$

and then $\overline{\phi}$ satisfies

$$\begin{aligned} L\overline{\phi} &= L\left(\phi - \widetilde{\phi}\right) = -L\widetilde{\phi} \in L^2(D) \quad \text{in } D, \\ \overline{\phi}\Big|_{\Gamma_1 \cup \Gamma_3} &= 0. \end{aligned} \tag{106}$$

Next, we show that there is a quasi-regular distributional solution $\overline{\phi} \in L^2(D)$ of Tricomi problem (106).

In fact, let $\operatorname{ran}(L^*)$ be the range (image) of the operator L^* defined on $\operatorname{Dom}(L^*)$. For $F \triangleq -L\widetilde{\phi} \in L^2(D)$, we define a linear functional on $\operatorname{ran}(L^*)$ by

$$\begin{aligned} \mathcal{T}:\ \operatorname{ran}\left(L^*\right) &\longrightarrow \mathbb{R}, \\ L^*\psi &\longmapsto (F, \psi) \\ &\quad \text{for any } \psi \in \operatorname{Dom}\left(L^*\right). \end{aligned} \tag{107}$$

Here we consider $\operatorname{ran}(L^*)$ as a linear subspace of $L^2(D)$. Using estimate (81), we have

$$\begin{aligned} \left|\mathcal{T}\left(L^*\psi\right)\right| = \left|F, \psi\right| &\le \|F\|_{L^2(D)} \|\psi\|_{L^2(D)} \\ &\le \|F\|_{L^2(D)} \|\psi\|_{W^{1,2}(D)} \\ &\le C \|F\|_{L^2(D)} \left\|L^*\psi\right\|_{L^2(D)}. \end{aligned} \tag{108}$$

Thus, the functional $\mathcal{T}$ is bounded.

Since $\operatorname{ran}(L^*)$ is a linear subspace of $L^2(D)$, by Hahn-Banach theorem, there exists a linear functional $\overline{\mathcal{T}}$: $L^2(D) \to \mathbb{R}$ as an extension of $\mathcal{T}$ that preserves the operator norm.

Thus, by Riesz representation theorem, there is $\overline{F} \in L^2(D)$ such that

$$\begin{aligned} \overline{\mathcal{T}}(w) &= \iint_D w\overline{F}\, dt\, d\theta = \left(\overline{F}, w\right), \\ \left\|\overline{\mathcal{T}}\right\| &= \left\|\overline{F}\right\|_{L^2(D)}. \end{aligned} \tag{109}$$

Therefore,

$$\left(\overline{F}, w\right) = \overline{\mathcal{T}}(w) = \mathcal{T}(w) \quad \forall w \in \operatorname{ran}\left(L^*\right). \tag{110}$$

Take $w = L^*\psi$. Then, for all $\psi \in \operatorname{Dom}(L^*)$, we have

$$\left(\overline{F}, L^*\psi\right) = \overline{\mathcal{T}}\left(L^*\psi\right) = \mathcal{T}\left(L^*\psi\right) = (F, \psi). \tag{111}$$

Therefore, $\overline{F}$ is a quasi-regular distributional solution of Tricomi problem (106) by Definition 2.

Finally, by using (105), it is obvious that $\phi = \overline{\phi} + \widetilde{\phi} \in L^2(D)$ is a quasi-regular distributional solution of (1) with boundary conditions (8). This finishes the proof of Theorem 1.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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