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Research Article

On the Relation between Phase-Type Distributions and Positive Systems

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The relation between phase-type representation and positive system realization in both the discrete and continuous time is discussed. Using the Perron-Frobenius theorem of nonnegative matrix theory, a transformation from positive realization to phase-type realization is derived under the excitability condition. In order to explain the connection, some useful properties and characteristics such as irreducibility, excitability, transparency, and order reduction for positive realization and phase-type representation are discussed. In addition, the connection between the phase-type renewal process and the feedback positive system is discussed in the stabilization concept.

1. Introduction

Positive system problems have been developed in applications areas such as biological models, production systems, and economic applications. The realization problem for positive system has extensively been considered in many research papers as [1–5]. Many research activities and applications have been devoted to the field of phase-type distributions [6–8]. The positive representation problem finding a Markov chain associated with phase-type distribution has considerable connections with the positive realization problem in the control theory [9].

We will discuss the relationship between phase-type representation and positive realization by using the Perron-Frobenius theorem introduced in [10–13]. The Perron-Frobenius theorem is an important concept for the study of positive systems. For example, the Perron-Frobenius theorem can be used to derive a transformation from positive realization to phase-type representation. We use tools and results within the broad research area of nonnegative matrix theory, which enable us to explore the characteristics and properties of a Metzler matrix and nonnegative matrix. Metzler matrices are replaced by Z-matrices, in particular, by the M-matrices introduced in [10–13].

The connection between phase-type and positive realization has restrictively been proved in irreducible representation cases by remarking that it can be easily simplified to an irreducible case by discarding some states [9]. Under the irreducible assumption, it is proven that the positive realization can be transformed into a phase-type representation [9]. We will show that a positive realization normalized by a positive number can be transformed into a phase-type representation. We modify the correspondence between positive realizations and phase-type representations under more general assumptions. We use excitable systems as a subclass of the positive systems introduced in [2, 14]. However, the phase-type representation has a benefit that the number of free parameters in the representation can be reduced, compared with the general positive realization.

We will discuss the properties and characteristics, such as irreducibility, excitability, transparency, and stabilization introduced in [3, 8, 15–18]. Excitability and transparency are similar to the reachability and observability of positive linear systems [17, 18]. There exist unreachable and unobservable positive states that are excitable and transparent [2, 14]. The properties of excitability and transparency are discussed furthermore. We will demonstrate how to discard some unnecessary states when a representation is not irreducible.

The common connection between the phase-type renewal process introduced in [7, 8] and the state feedback control introduced in [15, 16] will be handled. In addition, the relation and the common characteristics between discrete phase-type distributions and discrete-time positive systems are discussed in a similar manner to that applied to the continuous case.

An outline of the paper is as follows. In Section 2, some relevant background materials, including definitions and preliminary results are provided. Section 3 addresses the relationship between phase-type distributions and positive realizations in the continuous-time domain and discusses the common properties and characteristics, such as irreducibility, excitability, transparency, and stabilization. Section 4 discusses the relationship and the common characteristics between discrete phase-type and discrete-time positive systems in a similar manner to the continuous case. Finally, the conclusions follow.

2. Preliminaries

Before proceeding, we introduce some basic notations. An $n\times n$ nonnegative matrix A is denoted by $A \geq 0$ if its entries are nonnegative and at least one entry is positive. $A = [a_{ij}]$ is defined by a strict positive matrix (i.e., $A > 0$) if all entries $a_{ij} > 0$. The associated directed graph, $G(A)$, consists of n vertices $\{v_1, \ldots, v_n\}$, where $a_{ij} \neq 0$ denotes an edge from v_i to v_j. A nonnegative matrix A is said to be reducible if there exists a permutation matrix P such that

$$PAP^T = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad (1)$$

where A_{11} and A_{22} are square matrices and P^T is defined by the transpose of P. Otherwise, A is called irreducible. It is called a Frobenius normal form of A if there exists a suitable permutation P such that PAP^T is in block triangular of the form

$$PAP^T = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1p} \\ 0 & A_{22} & \cdots & A_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{pp} \end{bmatrix}, \quad (2)$$

where A_{ii} is square and either irreducible or a 1×1 null matrix. The spectral radius of A, denoted by $\rho(A)$, is defined by the largest absolute eigenvalue of A. The spectrum of A, denoted by $\sigma(A)$, is defined by the set of eigenvalues of A. The dominant eigenvalue of A is called by the maximal real among the real eigenvalues of A, denoted by $\lambda_{\max}(A)$.

A matrix $A \in \mathbb{R}^{n\times n}$ is said to be a Metzler matrix if all of its off-diagonal elements are nonnegative. If A is a Metzler matrix, then there exist a nonnegative matrix $B \geq 0$ and some $\alpha > 0$ such that $A = B - \alpha I$. The real dominant eigenvalue of A is defined by $\lambda_{\max}(A) = \rho(B) - \alpha$ if $\lambda_{\max}(A) \geq \mathrm{Re}(\lambda)$ for all $\lambda \in \sigma(A)$. There is a long stream of research dealing with M-matrices and Z-matrices instead of Metzler matrices [10]. Metzler matrices are replaced by Z-matrices; that is, $-A$ is a Z-matrix if A is a Metzler matrix. In particular, a matrix A is called an M-matrix if any matrix A is expressed in the form $A = \eta I - B$ where $B \geq 0$ and $\eta \geq \rho(B)$.

Basic definitions and results of cone theory may be needed within this paper. A set $\mathfrak{K}$ is said to be a cone if $\alpha\mathfrak{K} \subset \mathfrak{K}$ for all $\alpha \geq 0$. A cone is convex if for any two points in $\mathfrak{K}$ it contains the line segment between them. A convex cone $\mathfrak{K}$ is solid if the interior of $\mathfrak{K}$ is nonempty. It is pointed if $\mathfrak{K} \cap (-\mathfrak{K}) = \{0\}$. A closed pointed solid convex cone is called a proper cone. A cone $\mathfrak{K}$ is said to be polyhedral if it can be expressed as the set of nonnegative combinations of a finite set of generating vectors. We adopt the notation $\mathfrak{K} = \mathrm{Cone}(C)$ if $\mathfrak{K}$ coincides with the set of nonnegative combinations of the $n \times k$ matrix C.

We discuss the phase-type distribution for a random variable $X \geq 0$ in the terms of a continuous-time Markov process. A continuous-time Markov process is defined on $n+1$ finite state space. The row vector β gives the initial probabilities with i state probability β_i. A phase-type distribution is defined as the distribution of the time to absorption in a continuous-time Markov chain (CTMC) with one absorbing state [8]. If the $n+1$ state is an absorbing state and all other states are transient, we define a phase-type infinitesimal generator matrix of the Markov chain, denoted by $(\overline{A}, \overline{b}, \overline{\beta})$, such that

$$\overline{A} = \begin{bmatrix} A & -A\mathbb{1} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \qquad \overline{b} = \mathbf{e}_{n+1}, \qquad \overline{\beta} = [\beta \;\; \beta_{n+1}], \quad (3)$$

where $\mathbf{0}$ refers to the column vector, row vector, or matrix with all entries equal to zero in the case without ambiguity, $\mathbf{e}_i$ is the corresponding row vector whose ith entry is one and the others are zero, and $\mathbb{1}$ is the column vector with all entries being one. We can see that $\beta_{n+1} = 0$ if $\beta\mathbb{1} = 1$, and $\beta\mathbb{1} + \beta_{n+1} = 1$ otherwise. Phase-type distributions are commonly represented by a vector-matrix tuple (β, A) that describes the transient part of the CTMC. The vector-matrix tuple (β, A) is a phase-type (Markovian) representation of a phase-type distribution if and only if $\beta\mathbb{1} \leq 1$ and $\beta \geq 0$, A is a Metzler matrix with $A\mathbb{1} \leq 0$ and $A\mathbb{1} \neq \mathbf{0}$, and A is nonsingular.

The probability density function (PDF), cumulative distribution function (CDF), and Laplace-Stieltjes transform (LST) of the PDF, respectively, are defined by

$$f(x) = \beta \exp(Ax)(-A\mathbb{1}), \qquad F(x) = 1 - \beta \exp(Ax)\mathbb{1}, \qquad f^*(s) = E(\exp(sX)) = \beta(sI - A)^{-1}(-A\mathbb{1}), \quad (4)$$

where $E(\cdot)$ is an expectation. The ME (matrix exponential) distribution is a generalization of the phase-type distribution. A distribution function is called an ME distribution if there exists the triple (A, u, β) such that $F(x) = 1 - \beta\exp(Ax)u$, and there is no restriction on the elements of (A, u, β).

We note that these representations of phase-type distributions are equivalent to the state space realizations of linear systems in control. Let us consider single-input, single output linear time-invariant systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t). \tag{5}$$

The linear system in (5) is said to be a positive linear system provided that, for any nonnegative input and nonnegative initial state, the state trajectory and the output are always nonnegative. Let a transfer function $H(s) = C(sI - A)^{-1}B$ be defined by the Laplace transform of the impulse response function $h(t) = C\exp(At)B$ for $t \geq 0$ and, otherwise, $h(t) = 0$. A triple (A, B, C) is said to be a positive realization of $H(s)$ in a continuous-time linear positive system if and only if A is a Metzler matrix, $B \geq 0$, and $C \geq 0$. A triple (A, B, C) is denoted by a minimal realization if (A, B, C) is jointly completely controllable and completely observable.

An integral function of $h(t)$ is defined by $h_I(t) = \int_0^t h(t)dt$ and $h_I(0) = 0$. We can see that an augmented realization $(\overline{A}, \overline{B}, \overline{C})$ of $h_I(t)$ is defined by

$$\overline{A} = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}, \qquad \overline{B} = \mathbf{e}_{n+1}, \qquad \overline{C} = \begin{pmatrix} C & 0 \end{pmatrix}. \tag{6}$$

The augmented realization $(\overline{A}, \overline{B}, \overline{C})$ presents a state space realization of the integral function, such as $h_I(t) = \overline{C}\exp(\overline{A}t)\overline{B}$. The augmented realization $(\overline{A}, \overline{B}, \overline{C})$ is closely related to representation (3). We note that the positive realization of the integration of a positive system is closely related to the representation of the phase-type distribution.

3. Phase-Type Representation and the Positive Realization

3.1. Some Connections. Using a generalized version of the Perron-Frobenius theorem of nonnegative matrix theory, we derived a transformation from positive realization into phase-type realization under a constraint. The Perron-Frobenius theorem asserts that a real square matrix with positive entries has a unique largest real eigenvalue and that the corresponding eigenvector has strictly positive components [10, 19]. The Perron-Frobenius results of reducible matrices are characterized by being weaker than those of irreducible matrices. The general Perron-Frobenius theorem for reducible matrices is introduced from the result in Chapter 8 in [19] as follows.

Theorem 1 (Perron-Frobenius [19]). *Let $A \geq 0$ be an $n \times n$ matrix. Then, A has a nonnegative real eigenvalue equal to its spectral radius $\rho(A)$ and there is an eigenvector $x \geq 0$ corresponding to $\rho(A)$.*

The solvability problem of the matrix equation $(I - A)y = b$ with constraint $y \geq 0$ was originally solved by Carlson [11]. Several generalized versions have been discussed by researchers [12, 13, 20]. We note that the excitability is closely related to the existence of the strict positive solution of $(I - A)y = b$ (i.e., $y > 0$) as a more general case. For a discrete-time positive system with realization (A, b, c), (A, b) is defined to be excitable if there is an n such that $\sum_{m=0}^{n} A^m b > 0$, and (c, A) is defined to be transparent if there is an n such that $\sum_{m=0}^{n} cA^m > 0$ [2, 14]. We consider some properties of excitability.

Lemma 2. *Assume that $A \geq 0$ with $\rho(A) < 1$. Let $b \geq 0$. Then the following statements are equivalent:*

(1) *There is a $y > 0$ such that $(I - A)y = b$ and $y = \lim_{n\to\infty} \sum_{m=0}^{n} A^m b$.*

(2) *(A, b) is excitable.*

Proof. (1) ⇒ (2): we define $y_n = \sum_{m=0}^{n} A^m b$. Assume that y is strictly positive and $\lim_{n\to\infty} y_n = y$. Therefore, there is an n such that $\|y_k - y\|_\infty < \epsilon$ for all $k \geq n$ and $\epsilon = \|y\|_\infty/2$ where $\|y\|_\infty$ is defined by ∞-norm of y. Because $y_n > 0$ for sufficiently large n, it follows that (A, b) is excitable.

(2) ⇒ (1): because A is stable (i.e., $\rho(A) < 1$), the sum $\sum_{k=1}^{n} A^k$ converges uniformly to $(I - A)^{-1}$ as $n \to \infty$. We have $y = \lim_{n\to\infty} y_n$. Because each entry of y_n is monotonically increasing with respect to n and excitable, we can see that y is strictly positive. □

Consider a continuous-time system with a positive realization (A, B, C). Since A is a Metzler matrix, we can choose an $\alpha > 0$ such that $(I + \alpha A) \geq 0$ and $\rho(I + \alpha A) < 1$. We can define the excitability and the transparency of continuous-time positive linear systems in a similar form to discrete-time ones. The pair (A, B) is excitable if there is an integer $n > 0$ such that

$$z_n = \sum_{k=0}^{n} (I + \alpha A)^k B > 0 \tag{7}$$

for some $\alpha > 0$. The pair (C, A) is transparent if there exists an n such that

$$w_n = \sum_{k=0}^{n} C(I + \alpha A)^k > 0. \tag{8}$$

for some $\alpha > 0$.

Lemma 3. *For the continuous time with a positive realization (A, B, C), assume that (A, B) is excitable, A is an asymptotically stable Metzler matrix (i.e., $\lambda_{\max}(A) < 0$), and an augmented realization $(\overline{A}, \overline{B}, \overline{C})$ is given as (6). Then there is a positive eigenvector v of the spectral radius $\sigma(\overline{A}) = 0$ (i.e., $\overline{A}v = 0$) such that v is strictly positive (i.e., $v > 0$).*

Proof. We can choose a sufficiently large $\eta > 0$ such that a positive matrix $\widetilde{A} = \overline{A} + \eta I$ satisfies $\rho(\widetilde{A}) = \eta$, and $\eta > |x|$ for all $x \in \sigma(A)$. By using the Perron-Frobenius Theorem 1 for the augmented nonnegative matrix $\widetilde{A}$ with the order $n+1$, there exists an eigenvector $v \geq 0$ corresponding to η such that $\widetilde{A}v = \eta v$. If the last entry of v is zero, $v_{n+1} = 0$, then it induces that a vector $v^* \triangleq \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}^T$ is an eigenvector of $A + \eta I$ corresponding to an eigenvalue η. It contradicts the fact that all the eigenvalues of $A+\eta I$ are less than η. Therefore, we have $v_{n+1} > 0$.

Set $A_+ = I + A/\eta$. We have $A_+ v^* + v_{n+1}B/\eta = v^*$ since $\widetilde{A}v = \eta v$. Thus, A_+ is stable and positive. By using Lemma 2, the excitable condition (7) implies a strict positive solution $v^* = v_{n+1}(I - A_+)^{-1}B/\eta > 0$. Therefore, it is shown that v is strictly positive; that is, $v > 0$. □

We will show that a positive realization of continuous-time positive system can be transformed into a phase-type representation normalized by a positive number. Under the irreducible assumption, it was proven that the positive realization can be transformed into phase-type representation [9]. We modify the proof as a generalized version of the correspondence between positive realizations and phase-type representation. We can see that a phase-type representation is a special positive realization with excitable constraint.

Theorem 4. *Consider the continuous-time positive system with the positive realization (A, B, C) such that (A, B) is excitable, and A is an asymptotically stable and Metzler matrix. Then it is transformed into a phase-type infinitesimal generator matrix $(\overline{A}_+, \overline{B}_+, \overline{C}_+)$ such that*

$$\overline{A}_+ = \begin{bmatrix} A_+ & -A_+\mathbb{1} \\ 0 & 0 \end{bmatrix}, \qquad \overline{B}_+ = \mathbf{e}_{n+1}, \qquad \overline{C}_+ = \begin{bmatrix} C_+ & 0 \end{bmatrix}, \tag{9}$$

where $A_+\mathbb{1} = -B_+$ and $C_+\mathbb{1} > 0$.

Proof. First, let us define an augmented realization $(\overline{A}, \overline{B}, \overline{C})$ in the form of (6). Using Lemma 3, the Perron eigenvector v of $\overline{A}$ corresponding to the Perron root $\rho(\overline{A}) = 0$ is strictly positive. Set $\overline{U} = \operatorname{diag}(v_1, \ldots, v_n, v_{n+1})/v_{n+1}$. Thus, we can derive that $\overline{A}_+ = \overline{U}^{-1}\overline{A}\,\overline{U}$ with $\overline{A}_+\mathbb{1} = 0$, $\overline{B}_+ = \overline{U}^{-1}\overline{B} = \mathbf{e}_{n+1}$, and $\overline{C}_+ = \overline{C}\,\overline{U}$. Therefore, we can verify that $A_+\mathbb{1} + B_1 = 0$ where $A_+ = U^{-1}AU$, $B_+ = U^{-1}B$, $C_+ = CU$, and $U = \operatorname{diag}(v_1, \ldots, v_n)/v_{n+1}$. □

An important consequence of the above theorem is that an excitable positive realization can be transformed into the form of phase-type representation. Therefore, it is remarked that he concept of positive realizations is a superset of phase-type representations.

3.2. Common Properties and Characteristics. We discuss the properties and characteristics, such as stability, irreducibility, excitability, and transparency, in positive systems and phase-type distributions. A positive system with a positive realization (A, B, C) is said to be irreducible if (A, B) is excitable and (C, A) is transparent in the terminology introduced in [2, 14]. We note that the properties and characteristics of excitability and transparency are closely related to those of the reachability and observability of positive linear systems [17, 18]. A phase-type representation (α, A) in whose graph all the state vertices are connected to the initial vertex and to the absorbing vertex is called irreducible [8]. We note that the irreducible representation is closely related to the irreducibility of the phase-type renewal process in the Markovian point process introduced in [7, 8]. Renewal processes provide simple models of point processes, which may describe an ordered set of points. We consider a renewal process with a phase-type distribution for the interrenewal intervals. In [8], the phase-type representation (α, A) for the distribution function $F(x)$ is called irreducible if Q^* is irreducible where an infinitesimal generator Q^* is defined by

$$Q^* = A - A\mathbb{1}\alpha. \tag{10}$$

We may associate a Markov process with a phase-type renewal process. A renewal function denoted by $R(x)$ for a phase-type distribution $F(x)$ is defined by the expected number of renewals in the interval $[0, x]$; that is, $R(x) = E[N(x)]$, where $N(x)$ denotes the number of renewals. A renewal density is defined by $r(x) = dR(x)/dx$. The Laplace transform of the renewal density $r(x)$ is denoted by $r^*(s)$, which is rewritten by

$$r^*(s) = \frac{f^*(s)}{(1 - f^*(s))} = \left(-1 + \left(1 - f^*(s)\right)^{-1}\right). \tag{11}$$

In view of control theory, the equation in (11) is equivalent to the positive feedback control. Because the state space realization of the inverse system $(1 - f^*(s))^{-1}$ is given by $(1 - f^*(s))^{-1} = 1 + \alpha(sI - Q^*)^{-1}(-A\mathbb{1})$, we can see that its renewal density is given by $r(x) = \alpha\exp(Q^*x)(-A\mathbb{1})$, which is equal to the results in [7]. For an irreducible representation (α, A), the vectors $\alpha\exp(Tx)$ and $\exp(Ax)(-A\mathbb{1})$ are strictly positive [8]. We note that these results are related to the excitability and transparency of the positive linear system.

The irreducibility of a positive system can be defined in a similar manner. A positive system is irreducible if each state variable influences and is influenced by another variable [2]. It is defined by an irreducible realization for the positive system with a positive realization (A, B, C) if Q is irreducible, where Q is defined by

$$Q = A + BC. \tag{12}$$

For the open-loop system (5), the associated closed loop system (positive feedback system) is given by

$$\dot{x}(t) = Ax(t) + By(t) = Ax(t) + BCx(t) = Qx(t), \tag{13}$$

where the linear state-feedback law is $u(t) = Cx(t)$ and C is the constant feedback gain row vector. The closed loop system (13) is positive if and only if Q is a Metzler matrix. The stabilization problem of positive systems has recently been discussed in [3, 15, 16]. It is known an unstable open positive system (5) cannot be stabilized by linear state-feedback if the restriction on nonnegative control in the closed loop is imposed [15, 16].

The properties and characteristics of excitability and transparency are closely related to the reachability and observability of positive linear systems [5, 17, 18]. A reachable set $\mathscr{R}$ is the set of all points which the states approach from the origin by nonnegative inputs within finite time. It was shown that $\mathscr{R} = \operatorname{cl}[\operatorname{cone}\{x \mid x = \exp(At)B, t \geq 0\}]$, where

$\mathrm{cl}(S)$ is a closure set of S [17]. An observable set $\mathcal{S}$ is the set of initial states in which the output is nonnegative for all $t \geq 0$. The observable set $\mathcal{S}$ can be defined by $\mathcal{S} = \{x \mid C\exp(At)x \geq 0, \forall t \geq 0\}$.

Theorem 5 (see [17]). *Let a transfer function $H(s)$ be a strictly proper rational function with degree n, whose realization is given by (A, B, C). Then, $H(s)$ has a positive realization (A_*, B_+, C_+) with a Metzler matrix $A_* = A_+ - \eta I$ if and only if there exists a generator matrix P and $\lambda \geq 0$ such that a polyhedral cone $\mathcal{P} = \mathrm{cone}(P)$ satisfies*

(1) $(A + \lambda I)\mathcal{P} \subset \mathcal{P}$;

(2) $\mathcal{R} \subset \mathcal{P} \subset \mathcal{S}$,

where $P \in \mathbb{R}^{n\times m}$, $n \leq m$, $\mathcal{R}$ is a reachable set, and $\mathcal{S}$ is an observable set.

A positive system (A, B, C) is said to be reducible otherwise. When a representation is not irreducible, it can be simplified by discarding some states. Our next question is how to discard some unnecessary states when a representation is not irreducible. We discuss an order reduction algorithm of the asymptotical stable and unexcitable positive realization. Let a set $\langle n\rangle = \{1, \ldots, n\}$. For a column vector b with n entries, we define a support set

$$\mathrm{supp}(b) = \{i \in \langle n\rangle \mid b_i \neq 0\}. \tag{14}$$

Theorem 6. *Assume that a positive realization (A_*, b_+, c_+) is unexcitable and A_* is asymptotically stable. Then there is a permutation matrix $M = [M_1\ \ M_2]$ such that*

$$\begin{gathered} A_* M = [M_1\ \ M_2]\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22}\end{bmatrix}, \\ b_+ = [M_1\ \ M_2]\begin{bmatrix} e_1 \\ 0\end{bmatrix}, \\ c_+[M_1\ \ M_2] = [f_1\ \ f_2], \end{gathered} \tag{15}$$

where A_{11} is an $n_1 \times n_1$ matrix. Furthermore, we have a reduced positive realization (A_{11}, e_1, f_1) such that

$$A_* M_1 = M_1 A_{11}, \qquad b_+ = M_1 e_1, \qquad c_+ M_1 = f_1. \tag{16}$$

Proof. Because A_* is a Metzler matrix and asymptotical stable, there is an η such that $A_+ = A_* + \eta I$ is a positive matrix. Define $z_m = \sum_{k=0}^{m}(I + \alpha A)^k B$ with $\alpha = 1/\eta$. There is an integer $N > 0$ such that $\mathrm{supp}(z_m) = \mathrm{supp}(z_{m+1})$ for all $m \geq N$. A support set for z_N is defined by $\overline{\mathcal{Z}} = \mathrm{supp}(z_N)$. Let n_1 be the element number of $\overline{\mathcal{Z}}$. We can find a permutation matrix M such that $\mathrm{supp}(M^T z_N) = \langle n_1\rangle$. Set $A_1 = M^T A M = \left[\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{smallmatrix}\right]$ where the size of A_{11} is n_1. If A_{21} is a nonzero matrix, then we have $\mathrm{supp}(A_1 M^T z_N) \not\subset \langle n_1\rangle$, but this contradicts the definition of z_m. We can see that there is a permutation matrix $M = [M_1\ \ M_2]$ such that $A_* M = [M_1\ \ M_2]\left[\begin{smallmatrix} A_{11} & A_{12} \\ 0 & A_{22}\end{smallmatrix}\right]$ and $\mathrm{supp}(e_1) \subset \langle n_1\rangle$. We have $\mathcal{R} \subset \mathrm{cone}(M_1)$. Because A_* is a Metzler matrix and $c_+ > 0$, we can see $\mathrm{cone}(M_1) \subset \mathcal{S}$ by the definition of $\mathcal{S}$. By using Theorem 5, we can derive (16). □

We discussed the method to remove unnecessary states in the unexcitable case. When the transposed realization is given by (A^T, C^T, B^T), the concept of the transparency can be interpreted by that of the excitability. A removing method of the unnecessary state in the nontransparent case is similar to that in the unexcitable case. We illustrate the previous theorems by means of an example.

Example 7. A positive state-space realization (A, B, C) is given by

$$\begin{gathered} A = \begin{bmatrix} -2 & 1 & 1 & 1 \\ 0 & -3 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & -4 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \\ C = \begin{bmatrix} 2 & 4 & 1 & 3 \end{bmatrix}. \end{gathered} \tag{17}$$

We can see that $Q = A + BC$ is reducible. Set $\alpha = 1/5$. We have $z_5 = [4.11392\ \ 0\ \ 5.48992\ \ 0]^T$. The support set is $\overline{\mathcal{Z}} = \{1, 3\}$. We obtain

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{18}$$

By using Theorem 6 and removing the unexcitable part, we obtain an excitable positive realization (A_{11}, e_1, f_1) such that $A_{11} = \left[\begin{smallmatrix} -2 & 1 \\ 1 & -1\end{smallmatrix}\right]$, $e_1 = [1\ \ 1]$, and $f_1 = [2\ \ 1]^T$. By using Theorem 4, the phase-type representation (A, β) can be obtained as follows:

$$A = \begin{bmatrix} -2.000 & 1.500 \\ 0.666 & -1.000 \end{bmatrix}, \qquad \beta = [4\ \ 3]. \tag{19}$$

4. Discrete Phase-Type Distributions and Discrete-Time Positive Systems

A discrete phase-type (DPT) distribution is the distribution of the time until one absorbing state in a discrete-state discrete-time Markov chain (DTMC) with n transients states and one absorbing state [8, 21]. DPT distributions have received little attention in applied stochastic modeling. The main research activity has addressed continuous phase-type distributions. Let τ be the time till absorption into state $n+1$ in the DTMC. We say that τ is a random variable of order n and representation (A, b, β) [8]. The DPT representation has the following properties. A positive matrix $A = [a_{ij}]$ is an $(n \times n)$ matrix grouping the transition probabilities among the transient states. A column vector $b = [b_i]$ is a positive n-dimensional column vector grouping the probabilities from any state to the absorbing state. Thus, we have $\sum_{j=1}^{n} a_{ij} = 1 - b_i$. It means that $(I - A)\mathbb{1} = b$. An n vector β is defined by the initial probability vector. The augmented matrix tuple $(\widehat{A}, \widehat{b}, \widehat{\beta})$ is denoted by the augmented phase-type representation where the one-step transition probability matrix $\widehat{A}$ of the corresponding DTMC can be partitioned by

$$\widehat{A} = \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix}, \quad (20)$$

the augmented initial vector is defined by $\widehat{\beta} = [\beta \ \ 0]$, and $\widehat{b} = \mathbf{e}_{n+1}$. Its probability generating function is defined by $P(z) = \beta(z^{-1}I - A)^{-1}b$ [8].

We can also discuss the realization between the DPH distributions and discrete-time positive systems in a similar manner. The discrete-time linear system is represented by

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \quad (21)$$

where $A \in R_{+}^{n\times n}$, $B \in R_{+}^{n}$, and $C^{T} \in R_{+}^{n}$. The matrix tuple (A, B, C) denotes the positive realization (21). In the next theorem, we show that the positive realization can be transformed into a DPH representation multiplied by a positive scalar (i.e., β is not necessarily a probability vector).

Theorem 8. *Assume that a realization (A, B, C) is denoted by a positive realization satisfying (21) and (A, B) is excitable (essential reachable) and stable. Then there is a nonsingular matrix M such that the realization $(\widetilde{A}, \widetilde{b}, \widetilde{\beta})$, which is defined by $\widetilde{\beta} = CM$, $\widetilde{A} = M^{-1}AM$, and $\widetilde{b} = M^{-1}B$, has the properties of the DPH representation such as $\widetilde{b} = (I - \widetilde{A})\mathbb{1}$ and $\beta \geq 0$.*

Proof. Because A is positive, stable, and excitable, the absolute values of its eigenvalues are less than 1 and we can use Lemma 2. Thus, we obtain the fact that the entries of $x = (I - A)^{-1}B$ are positive. A similarity transform matrix M is defined by a diagonal matrix, $M = \mathrm{diag}(x)$. Compute a new realization $(\widetilde{A}, \widetilde{b}, \widetilde{\beta})$. We obtain the facts that $(I - \widetilde{A})\mathbb{1} = \widetilde{b} \geq 0$, $\widetilde{\beta} \geq 0$, and $A \geq 0$. Therefore, we can verify that the new realization satisfies the discrete phase-type distribution properties. □

We can easily deploy the properties and characteristics, such as irreducibility, excitability, transparency, and order reduction, in the discrete domain in a similar manner as in the continuous case. We omit the detailed exploration for the discrete case in this paper.

5. Conclusions

We considered the relation between the positive realization and the phase-type representation in continuous time and discrete time, respectively. Using the Perron-Frobenius theorem, it was shown that a phase-type representation is a special case with excitable constraint of the positive realization. We discussed their common properties and characteristics, such as irreducibility, excitability, transparency, stabilization, and order reduction. The connection between the phase-type renewal process and the feedback control of positive system was discussed. A lot of open problems related to positive system still remain and should be addressed in future research. The communities of control and probability theory can work together on solving the remaining same open problems.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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