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*Research Article*

# Normal Forms of Hopf Bifurcation for a Reaction-Diffusion System Subject to Neumann Boundary Condition

**Cun-Hua Zhang and Xiang-Ping Yan**

*Department of Mathematics, Lanzhou Jiaotong University, Lanzhou 730070, China*

Correspondence should be addressed to Xiang-Ping Yan; xpyan72@163.com

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A reaction-diffusion system coupled by two equations subject to homogeneous Neumann boundary condition on one-dimensional spatial domain $(0, \ell\pi)$ with $\ell > 0$ is considered. According to the normal form method and the center manifold theorem for reaction-diffusion equations, the explicit formulas determining the properties of Hopf bifurcation of spatially homogeneous and nonhomogeneous periodic solutions of system near the constant steady state $(0, 0)$ are obtained.

## 1. Introduction

As an important dynamic bifurcation phenomenon in dynamical systems, Hopf bifurcation of periodic solutions has attracted great interest of many authors in the last several decades [1–8]. In general, the study of Hopf bifurcation includes the existence and the properties such as the direction of bifurcation and the stability of bifurcating periodic solutions. In application, however, it is more difficult to determine the properties of Hopf bifurcation than to find the existence of a Hopf bifurcation. An approach applied to determine the properties of Hopf bifurcation is to derive the projected equation of original equations on the associated center manifold, that is, the so-called normal form. Then one may explore the local dynamical behaviors of a higher dimensional or even infinitely dimensional dynamical system near a certain nonhyperbolic steady state according to the normal form obtained. The normal form of Hopf bifurcation in ordinary differential equations (ODEs) with or without delays has been established well [1, 3, 5] since in this case the equilibrium is always constant and there are also no effects of spatial diffusion.

Under some certain conditions, the reaction-diffusion equations under the homogeneous Neumann boundary condition may have the constant steady state and thus one can study the Hopf bifurcation of system at this constant steady state. Compared with the ODEs, it is more difficult to derive the normal form of Hopf bifurcation for reaction-diffusion equations at the constant steady state. Although Hassard et al. [3] established the method computing the normal form of Hopf bifurcation in reaction-diffusion equations with the homogeneous Neumann boundary condition and also considered the Hopf bifurcation of spatially homogeneous periodic solutions in Brusselator system, using the same method, Jin et al. [9] and Ruan [10] as well as Yi et al. [11, 12] considered the Hopf bifurcation of spatially homogeneous periodic solutions for Gierer-Meinhardt system and CIMA reaction, respectively. There are few results regarding Hopf bifurcation of spatially nonhomogeneous periodic solutions for spatially homogeneous reaction-diffusion equations [7].

Based on the reason mentioned above, in this paper we consider the normal form of Hopf bifurcation of reaction-diffusion equations at the constant steady state following the idea in [3]. In order to have a clearer structure, we are concerned with the following general reaction-diffusion system coupled by two equations defined on one-dimensional spatial domain $(0, \ell\pi)$ with $\ell > 0$ and subject to Neumann boundary conditions; that is,

$$\begin{aligned} u_t &= d_1 u_{xx} + f_1(\lambda, u, v), \quad x \in (0, \ell\pi), \ t > 0, \\ v_t &= d_2 v_{xx} + f_2(\lambda, u, v), \quad x \in (0, \ell\pi), \ t > 0, \\ u_x(0,t) &= v_x(0,t) = u_x(\ell\pi, t) = v_x(\ell\pi, t) = 0, \\ & \qquad t > 0, \end{aligned}$$

$$u(x,0) = u_0(x),$$
$$v(x,0) = v_0(x),$$
$$x \in (0, \ell\pi), \tag{1}$$

in which $d_1, d_2 > 0$ are the diffusion coefficients, $\lambda \in \mathbb{R}$ is the parameter, and $f_1, f_2 : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ are $C^r$ ($r \geq 5$) functions with $f_k(\lambda, 0, 0) = 0$ ($k = 1, 2$) for any $\lambda \in \mathbb{R}$. Although Yi et al. [7] described the algorithm determining the properties of Hopf bifurcation of spatially homogeneous and nonhomogeneous periodic solutions for (1) at $(0,0)$ and also considered the Hopf bifurcation of a diffusive predator-prey system with Holling type-II functional response and subject to the homogeneous Neumann boundary condition, they did not give the normal form of Hopf bifurcation of spatially homogeneous and nonhomogeneous periodic solutions of the general reaction-diffusion system (1) at $(0,0)$.

This paper is organized as follows. In the next section, following the abstract method according to [3], we describe the algorithm determining the properties of Hopf bifurcation of spatially homogeneous and nonhomogeneous periodic solutions for system (1) at the constant steady state $(0,0)$. In Section 3, the explicit formulas determining the properties of Hopf bifurcation of spatially homogeneous periodic solutions for system (1) at $(0,0)$ are obtained. The explicit formulas determining the properties of Hopf bifurcation of spatially nonhomogeneous periodic solutions for (1) at $(0,0)$ are also derived in Section 4.

## 2. Algorithm Determining the Properties of Hopf Bifurcation

In this section, we will describe the explicit algorithm determining the direction of Hopf bifurcation and the stability of the bifurcating periodic solutions of system (1) at $(0,0)$.

Define the real-valued Sobolev space $X$ by

$$X = \left\{(u,v) \in H^2(0,\ell\pi) \times H^2(0,\ell\pi) \mid u_x = v_x = 0,\ x = 0, \ell\pi\right\}. \tag{2}$$

In terms of $X$, the complex-valued Sobolev space $X_{\mathbb{C}}$ is given by

$$X_{\mathbb{C}} = X \oplus iX = \{x_1 + ix_2,\ x_1, x_2 \in X\}, \tag{3}$$

and the inner product $\langle\cdot,\cdot\rangle$ on $X_{\mathbb{C}}$ is defined by

$$\langle U_1, U_2\rangle = \int_0^{\ell\pi} (\overline{u}_1 u_2 + \overline{v}_1 v_2)\,dx, \quad \text{for } U_1 = (u_1, v_1) \in X_{\mathbb{C}},\ U_2 = (u_2, v_2) \in X_{\mathbb{C}}. \tag{4}$$

Let $A(\lambda) = f_{1u}(\lambda,0,0)$, $B(\lambda) = f_{1v}(\lambda,0,0)$, $C(\lambda) = f_{2u}(\lambda,0,0)$, and $D(\lambda) = f_{2v}(\lambda,0,0)$ and define the linear operator $L(\lambda)$ with the domain $D_{L(\lambda)} = X_{\mathbb{C}}$ by

$$L(\lambda) = \begin{pmatrix} d_1 \dfrac{\partial^2}{\partial x^2} + A(\lambda) & B(\lambda) \\ C(\lambda) & d_2 \dfrac{\partial^2}{\partial x^2} + D(\lambda) \end{pmatrix}. \tag{5}$$

Assume that, for some $\lambda_0 \in \mathbb{R}$, the following condition holds:

(H) There exists a neighborhood $O$ of $\lambda_0$ such that, for $\lambda \in O$, $L(\lambda)$ has a pair of simple and continuously differentiable eigenvalues $\alpha(\lambda) \pm i\omega(\lambda)$ with $\alpha(\lambda_0) = 0$, $\omega(\lambda_0) = \omega_0 > 0$, and $\alpha'(\lambda_0) \neq 0$. In addition, all other eigenvalues of $L(\lambda)$ have nonzero real parts for $\lambda \in O$.

Then from [3, 7] we know that system (1) undergoes a Hopf bifurcation at $(0,0)$ when $\lambda$ crosses through $\lambda_0$.

Define the second-order matrix sequence $L_j(\lambda)$ by

$$L_j(\lambda) = \begin{pmatrix} A(\lambda) - \dfrac{d_1 j^2}{\ell^2} & B(\lambda) \\ C(\lambda) & D(\lambda) - \dfrac{d_1 j^2}{\ell^2} \end{pmatrix}, \quad j \in \mathbb{N}_0. \tag{6}$$

Then the characteristic equation of $L_j(\lambda)$ is

$$\beta^2 - \beta T_j(\lambda) + D_j(\lambda) = 0, \quad j \in \mathbb{N}_0, \tag{7}$$

where

$$\begin{aligned} T_j(\lambda) &= A(\lambda) + D(\lambda) - \frac{(d_1 + d_2) j^2}{\ell^2}, \\ D_j(\lambda) &= \frac{d_1 d_2 j^4}{\ell^4} - (d_1 D(\lambda) + d_2 A(\lambda)) \frac{j^2}{\ell^2} \\ &\quad + A(\lambda) D(\lambda) - B(\lambda) C(\lambda). \end{aligned} \tag{8}$$

The eigenvalues of $L(\lambda)$ can be determined by the eigenvalues of $L_j(\lambda)$ ($j \in \mathbb{N}_0$) and we have the following conclusion.

**Lemma 1.** *If $\beta(\lambda) \in \mathbb{C}$ is an eigenvalue of the operator $L(\lambda)$, then there exists some $n \in \mathbb{N}_0$ such that $\beta(\lambda)$ is the eigenvalue of $L_n(\lambda)$ and vice versa.*

*Proof.* It is well known that the eigenvalue problem

$$\begin{gathered} -\varphi'' = \mu\varphi, \\ x \in (0, \ell\pi), \\ \varphi'(0) = \varphi'(\ell\pi) = 0 \end{gathered} \tag{9}$$

has eigenvalues $j^2/\ell^2$ ($j = 0, 1, 2, \ldots$) with eigenfunctions $\cos(j/\ell)x$. Assume that $\beta(\lambda) \in \mathbb{C}$ is an eigenvalue of the operator $L(\lambda)$ and the corresponding eigenfunction is $(\phi, \psi) \in X_{\mathbb{C}}$; that is,

$$L(\lambda) \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \lambda \begin{pmatrix} \phi \\ \psi \end{pmatrix}. \tag{10}$$

Notice that $(\phi, \psi) \in X_{\mathbb{C}}$ can be represented as

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \sum_{j=0}^{\infty} \begin{pmatrix} a_j \\ b_j \end{pmatrix} \cos \frac{j}{\ell} x, \tag{11}$$

where $a_j, b_j \in \mathbb{C}$ $(j \in \mathbb{N}_0)$. Then (10) can be written into

$$\sum_{j=0}^{\infty} L_j(\lambda) \begin{pmatrix} a_j \\ b_j \end{pmatrix} \cos \frac{j}{\ell} x = \beta(\lambda) \sum_{j=0}^{\infty} \begin{pmatrix} a_j \\ b_j \end{pmatrix} \cos \frac{j}{\ell} x. \tag{12}$$

From the orthogonality of the function sequence $\{\cos(j/\ell)x\}_{j=0}^{\infty}$, one can get from (12) that, for each $j \in \mathbb{N}_0$,

$$L_j(\lambda) \begin{pmatrix} a_j \\ b_j \end{pmatrix} = \beta(\lambda) \begin{pmatrix} a_j \\ b_j \end{pmatrix}. \tag{13}$$

Since $(\phi, \psi) \in X_{\mathbb{C}}$ is the eigenfunction of $L(\lambda)$ corresponding to the eigenvalue $\beta(\lambda)$, $(\phi, \psi) \neq 0$ and so there must be some $n \in \mathbb{N}_0$ such that $0 \neq (a_n, b_n) \in \mathbb{C} \times \mathbb{C}$. Therefore, $\beta(\lambda)$ is the eigenvalue of the matrix $L_n$.

If $\beta(\lambda)$ is the eigenvalue of some matrix $L_n$, then there exists a nonzero vector $(a_n, b_n) \in \mathbb{C} \times \mathbb{C}$ such that (13) holds. Let

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} a_n \\ b_n \end{pmatrix} \cos \frac{n}{\ell} x. \tag{14}$$

Then $(\phi, \psi) \neq 0$ and

$$\begin{aligned} L(\lambda) \begin{pmatrix} \phi \\ \psi \end{pmatrix} &= L(\lambda) \begin{pmatrix} a_n \\ b_n \end{pmatrix} \cos \frac{n}{\ell} x = L_n \begin{pmatrix} a_n \\ b_n \end{pmatrix} \cos \frac{n}{\ell} x \\ &= \lambda \begin{pmatrix} a_n \\ b_n \end{pmatrix} \cos \frac{n}{\ell} x = \beta(\lambda) \begin{pmatrix} \phi \\ \psi \end{pmatrix}. \end{aligned} \tag{15}$$

This demonstrates that $\beta(\lambda)$ is an eigenvalue of $L(\lambda)$ and thus the proof is complete. □

Lemma 1 shows that, under assumption (H), there is a unique $n \in \mathbb{N}_0$ such that $\pm i\omega_0$ are purely imaginary eigenvalues of $L_n(\lambda_0)$; that is, $T_n(\lambda_0) = 0$ and $D_n(\lambda_0) > 0$. Furthermore, it is easy to see that $T_j(\lambda_0) \neq 0$ for any $j \neq n$. Therefore, $L_j(\lambda_0)$ $(j \neq n)$ has eigenvalues with zero real parts if and only if $D_j(\lambda_0) = 0$. Assume that $\beta(\lambda) = \alpha(\lambda) + i\omega(\lambda)$ is the eigenvalue of $L(\lambda)$ for $\lambda$ sufficiently approaching $\lambda_0$. Then by the smoothness of $f_k$ $(k = 1, 2)$ we know that $\beta(\lambda)$ is also the eigenvalue of $L_n(\lambda)$; namely, $\beta(\lambda)$ satisfies the following equation:

$$\beta^2 - \beta T_n(\lambda) + D_n(\lambda) = 0. \tag{16}$$

Under the assumption (H), differentiating the above equation with respect to $\lambda$ at $\lambda_0$ yields

$$\frac{d\alpha(\lambda_0)}{d\lambda} = \frac{1}{2} \left[ A'(\lambda_0) + D'(\lambda_0) \right]. \tag{17}$$

Based on the above discussion, condition (H) has the following equivalent form:

$$\begin{gathered} T_n(\lambda_0) = 0, \\ D_n(\lambda_0) > 0, \\ A'(\lambda_0) + D'(\lambda_0) \neq 0 \\ \text{for some } n \in \mathbb{N}_0, \ D_j(\lambda_0) \neq 0 \text{ for any } j \in \mathbb{N}_0. \end{gathered} \tag{18}$$

Then we know that $\omega_0 = \sqrt{D_n(\lambda_0)}$ and $B(\lambda_0), C(\lambda_0)$ cannot be equal to zero simultaneously when the hypothesis (H) is satisfied. Therefore, the eigenvector of $L_n(\lambda_0)$ corresponding to the eigenvalue $i\omega_0$ can be chosen as

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} 1 \\ \dfrac{i\omega_0 - A(\lambda_0) + d_1 n^2/\ell^2}{B(\lambda_0)} \end{pmatrix} \tag{19}$$

and thus the eigenfunction of $L(\lambda_0)$ corresponding to the eigenvalue $i\omega_0$ has the form

$$\begin{aligned} q &= \begin{pmatrix} a_n \\ b_n \end{pmatrix} \cos \frac{n}{\ell} x \\ &= \begin{pmatrix} 1 \\ \dfrac{i\omega_0 - A(\lambda_0) + d_1 n^2/\ell^2}{B(\lambda_0)} \end{pmatrix} \cos \frac{n}{\ell} x. \end{aligned} \tag{20}$$

Let the linear operator $L^*(\lambda_0)$ with the domain $D_{L^*(\lambda_0)} = X_{\mathbb{C}}$ be defined by

$$L^*(\lambda_0) = \begin{pmatrix} d_1 \dfrac{\partial^2}{\partial x^2} + A(\lambda_0) & C(\lambda_0) \\ B(\lambda_0) & d_2 \dfrac{\partial^2}{\partial x^2} + D(\lambda_0) \end{pmatrix}. \tag{21}$$

Then $L^*(\lambda_0)$ is the adjoint operator of the operator $L(\lambda_0)$ such that $\langle U, L(\lambda_0)V \rangle = \langle L^*(\lambda_0)U, V \rangle$ with $U, V \in X_{\mathbb{C}}$. Similar to the choice of the eigenfunction $q$ of the operator $L(\lambda_0)$ corresponding to the eigenvalue $i\omega_0$, we can choose

$$\begin{aligned} q^* &= \begin{pmatrix} a_n^* \\ b_n^* \end{pmatrix} \cos \frac{n}{\ell} x \\ &= \begin{pmatrix} \dfrac{\omega_0 + i\left(A(\lambda_0) - d_1 n^2/\ell^2\right)}{2\omega_0 \int_0^{\ell\pi} \cos^2(n/\ell)x\,dx} \\ -i\dfrac{B(\lambda_0)}{2\omega_0 \int_0^{\ell\pi} \cos^2(n/\ell)x\,dx} \end{pmatrix} \cos \frac{n}{\ell} x \end{aligned} \tag{22}$$

such that

$$\begin{gathered} L^*(\lambda_0) q^* = -i\omega_0 q, \\ \langle q^*, q \rangle = 1, \\ \langle q^*, \overline{q} \rangle = 0. \end{gathered} \tag{23}$$

Define $X^C$ and $X^S$ by $X^C = \{zq + \overline{zq} \mid z \in \mathbb{C}\}$ and $X^S = \{U \in X \mid \langle q^*, U \rangle = 0\}$, respectively. Then $X$ can be decomposed as the direct sum of $X^C$ and $X^S$; that is, $X = X^C \oplus X^S$. Thus, for any $U = (u, v) \in X$, there exists $z \in \mathbb{C}$ and $w = (w_1, w_2) \in X_s$ such that

$$\begin{gathered} U = zq + \overline{z}\,\overline{q} + w, \\ \text{or } \begin{cases} u = z a_n \cos \dfrac{n}{\ell} x + \overline{z}\,\overline{a}_n \cos \dfrac{n}{\ell} x + w_1, \\ v = z b_n \cos \dfrac{n}{\ell} x + \overline{z}\overline{b}_n \cos \dfrac{n}{\ell} x + w_2. \end{cases} \end{gathered} \tag{24}$$

Define $F(\lambda, U)$ by

$$F(\lambda, U) = \begin{pmatrix} f_1(\lambda, u, v) - A(\lambda) u - B(\lambda) v \\ f_2(\lambda, u, v) - C(\lambda) u - D(\lambda) v \end{pmatrix}. \tag{25}$$

Then system (1) can be rewritten into the following abstract form:

$$\frac{dU}{dt} = L(\lambda) U + F(\lambda, U). \tag{26}$$

When $\lambda = \lambda_0$, system (26) is reduced to

$$\frac{dU}{dt} = L(\lambda_0) U + F_0(U), \tag{27}$$

where $F_0(U) = F(\lambda, U)|_{\lambda=\lambda_0}$. In terms of (23) and decomposition (24), system (27) can be transformed into the following system in $(z, w)$ coordinates:

$$\begin{aligned} \frac{dz}{dt} &= i\omega_0 z + \langle q^*, F_0 \rangle, \\ \frac{dw}{dt} &= L(\lambda_0) w + H(z, \bar{z}, w), \end{aligned} \tag{28}$$

where

$$\begin{aligned} H(z, \bar{z}, w) &= F_0 - \langle q^*, F_0 \rangle q - \langle \bar{q}^*, F_0 \rangle \bar{q}, \\ F_0 &= F_0(zq + \bar{z}\bar{q} + w). \end{aligned} \tag{29}$$

For $X = (x_1, x_2)$, $Y = (y_1, y_2)$, and $Z = (z_1, z_2) \in X_{\mathbb{C}}$, define the symmetric multilinear forms $Q(X, Y)$ and $C(X, Y, Z)$, respectively, by

$$Q(X, Y) = \begin{pmatrix} \displaystyle\sum_{k,j=1}^{2} \left.\frac{\partial^2 f_1(\lambda_0, \xi_1, \xi_2)}{\partial \xi_k \partial \xi_j}\right|_{\xi_1=\xi_2=0} x_k y_j \\ \displaystyle\sum_{k,j=1}^{2} \left.\frac{\partial^2 f_2(\lambda_0, \xi_1, \xi_2)}{\partial \xi_k \partial \xi_j}\right|_{\xi_1=\xi_2=0} x_k y_j \end{pmatrix}, \tag{30}$$

$$\begin{aligned} &C(X, Y, Z) \\ &= \begin{pmatrix} \displaystyle\sum_{k,j,l=1}^{2} \left.\frac{\partial^3 f_1(\lambda_0, \xi_1, \xi_2)}{\partial \xi_k \partial \xi_j \partial \xi_l}\right|_{\xi_1=\xi_2=0} x_k y_j z_l \\ \displaystyle\sum_{k,j,l=1}^{2} \left.\frac{\partial^3 f_2(\lambda_0, \xi_1, \xi_2)}{\partial \xi_k \partial \xi_j \partial \xi_l}\right|_{\xi_1=\xi_2=0} x_k y_j z_l \end{pmatrix}. \end{aligned} \tag{31}$$

Then, for $U = (u, v) \in X$, we have

$$F_0(U) = \frac{1}{2} Q(U, U) + \frac{1}{6} C(U, U, U) + O(|U|^4). \tag{32}$$

For the simplicity of notations, we will use $Q_{XY}$ and $C_{XYZ}$ to denote $Q(X, Y)$ and $C(X, Y, Z)$, respectively.

Let

$$\begin{aligned} H(z, \bar{z}, w) = &\frac{H_{20}}{2} z^2 + H_{11} z\bar{z} + \frac{H_{02}}{2} \bar{z}^2 + O(|z|^3) \\ &+ O(|z| |w|). \end{aligned} \tag{33}$$

Then from (29) and (32), one can get

$$\begin{aligned} H_{20} &= Q_{qq} - \langle q^*, Q_{qq} \rangle q - \langle \bar{q}^*, Q_{qq} \rangle \bar{q}, \\ H_{11} &= Q_{q\bar{q}} - \langle q^*, Q_{q\bar{q}} \rangle q - \langle \bar{q}^*, Q_{q\bar{q}} \rangle \bar{q}. \end{aligned} \tag{34}$$

From the center manifold theorem in [3], we can rewrite $w$ in the form

$$w = \frac{w_{20}}{2} z^2 + w_{11} z\bar{z} + \frac{w_{02}}{2} \bar{z}^2 + O(|z|^3). \tag{35}$$

The second equation of (28), (33), and (35) yields

$$\begin{aligned} w_{20} &= [2i\omega_0 I - L(\lambda_0)]^{-1} H_{20}, \\ w_{11} &= -[L(\lambda_0)]^{-1} H_{11}. \end{aligned} \tag{36}$$

Substituting (35) into the first equation of (28) gives the equation of reaction-diffusion system (1) restricted on the center manifold at $(\lambda_0, 0, 0)$ as

$$\frac{dz}{dt} = i\omega_0 z + \sum_{2 \le k+j \le 3} \frac{g_{kj}}{k! j!} z^k \bar{z}^j + O(|z|^4), \tag{37}$$

where $g_{20} = \langle q^*, Q_{qq} \rangle$, $g_{11} = \langle q^*, Q_{q\bar{q}} \rangle$, $g_{02} = \langle q^*, Q_{\bar{q}\bar{q}} \rangle$, and

$$g_{21} = 2 \langle q^*, Q_{w_{11} q} \rangle + \langle q^*, Q_{w_{20} \bar{q}} \rangle + \langle q^*, C_{qq\bar{q}} \rangle. \tag{38}$$

The dynamics of (28) can be determined by the dynamics of (37).

In addition, it can be observed from [3] that when $\lambda$ approaches sufficiently $\lambda_0$, the Poincaré normal form of (26) has the form

$$\dot{z} = (\alpha(\lambda) + i\omega(\lambda)) z + z \sum_{j=1}^{M} c_j(\lambda) (z\bar{z})^j, \tag{39}$$

where $z$ is a complex variable, $M \ge 1$, and $c_j(\lambda)$ are complex-valued coefficients with

$$\begin{aligned} c_1(\lambda_0) &= \frac{i}{2\omega_0} \left( g_{20} g_{11} - 2 |g_{11}|^2 - \frac{1}{3} |g_{02}|^2 \right) + \frac{g_{21}}{2} \\ &= \frac{i}{2\omega_0} \left( \langle q^*, Q_{qq} \rangle \langle q^*, Q_{q\bar{q}} \rangle - 2 \left| \langle q^*, Q_{q\bar{q}} \rangle \right|^2 \right. \\ &\quad \left. - \frac{1}{3} \left| \langle q^*, Q_{\bar{q}\bar{q}} \rangle \right|^2 \right) + \langle q^*, Q_{w_{11} q} \rangle + \frac{1}{2} \langle q^*, Q_{w_{20} \bar{q}} \rangle \\ &\quad + \frac{1}{2} \langle q^*, C_{qq\bar{q}} \rangle. \end{aligned} \tag{40}$$

The direction of Hopf bifurcation and the stability of the bifurcating periodic solutions of (1) at $(\lambda_0, 0, 0)$ can be determined by the sign of $\operatorname{Re} c_1(\lambda_0)$ and we have the following conclusion.

**Theorem 2.** *Assume that condition (H) (or equivalently (18)) holds. Then system (1) undergoes a supercritical (or subcritical) Hopf bifurcation at* $(0, 0)$ *when* $\lambda = \lambda_0$ *if*

$$\frac{1}{\alpha'(\lambda_0)} \operatorname{Re} c_1(\lambda_0) < 0 \quad (\textit{resp.} > 0). \tag{41}$$

*In addition, if all other eigenvalues of* $L(\lambda_0)$ *have negative real parts, then the bifurcating periodic solutions are stable (resp., unstable) when* $\operatorname{Re} c_1(\lambda_0) < 0$ *(resp.,* $> 0$*).*

## 3. Spatially Homogeneous Hopf Bifurcation

From the description in the previous section we know that Hopf bifurcation of (1) at $(\lambda_0, 0, 0)$ is spatially homogeneous if condition (18) holds when $n = 0$. In the present section, we compute $\operatorname{Re} c_1(\lambda_0)$ in (40) in order to determine the direction of spatially homogeneous Hopf bifurcation and the stability of bifurcating periodic solutions of (1) at $(\lambda_0, 0, 0)$ following the algorithm described in this pervious section.

**Lemma 3.** *If condition (18) is satisfied when $n = 0$, then $H_{20} = H_{11} = 0$.*

*Proof.* From (20) and (22) one can see

$$\begin{aligned} a_0 &= 1, \\ b_0 &= \frac{i\omega_0 - A(\lambda_0)}{B(\lambda_0)}, \\ a_0^* &= \frac{\omega_0 + iA(\lambda_0)}{2\ell\pi\omega_0}, \\ b_0^* &= -\frac{iB(\lambda_0)}{2\ell\pi\omega_0}, \end{aligned} \tag{42}$$

where

$$\begin{aligned} \omega_0 &= \sqrt{A(\lambda_0) D(\lambda_0) - B(\lambda_0) C(\lambda_0)} \\ &= \sqrt{-A^2(\lambda_0) - B(\lambda_0) C(\lambda_0)}. \end{aligned} \tag{43}$$

Let all the partial derivatives of $f_k(\lambda, u, v)$ $(k = 1, 2)$ be evaluated at $(\lambda_0, 0, 0)$, and let $c_{k0}$, $d_{k0}$, and $e_{k0}$ $(k = 1, 2)$ be defined, respectively, by

$$\begin{aligned} c_{k0} &= f_{kuu} + 2f_{kuv}b_0 + f_{kvv}b_0^2, \\ d_{k0} &= f_{kuu} + f_{kuv}\left(\overline{b}_0 + b_0\right) + f_{kvv}\left|b_0\right|^2, \\ e_{k0} &= f_{kuuu} + f_{kuuv}\left(2b_0 + \overline{b}_0\right) + f_{kuvv}\left(2\left|b_0\right|^2 + b_0^2\right) \\ &\quad + f_{kvvv}\left|b_0\right|^2 b_0. \end{aligned} \tag{44}$$

Then from (30) and (31) we can get

$$\begin{aligned} Q_{qq} &= \begin{pmatrix} c_{10} \\ c_{20} \end{pmatrix}, \\ Q_{q\overline{q}} &= \begin{pmatrix} d_{10} \\ d_{20} \end{pmatrix}, \\ C_{qq\overline{q}} &= \begin{pmatrix} e_{10} \\ e_{20} \end{pmatrix}. \end{aligned} \tag{45}$$

Therefore,

$$\begin{aligned} \left\langle q^*, Q_{qq} \right\rangle &= \int_0^{\ell\pi} \left(\overline{a}_0^* c_{10} + \overline{b}_0^* c_{20}\right) \\ &= \ell\pi \left(\overline{a}_0^* c_{10} + \overline{b}_0^* c_{20}\right), \\ \left\langle \overline{q}^*, Q_{qq} \right\rangle &= \int_0^{\ell\pi} \left(a_0^* c_{10} + b_0^* c_{20}\right) \\ &= \ell\pi \left(a_0^* c_{10} + b_0^* c_{20}\right), \\ \left\langle q^*, Q_{q\overline{q}} \right\rangle &= \int_0^{\ell\pi} \left(\overline{a}_0^* d_{10} + \overline{b}_0^* d_{20}\right) \\ &= \ell\pi \left(\overline{a}_0^* d_{10} + \overline{b}_0^* d_{20}\right), \\ \left\langle \overline{q}^*, Q_{q\overline{q}} \right\rangle &= \int_0^{\ell\pi} \left(a_0^* d_{10} + b_0^* d_{20}\right) \\ &= \ell\pi \left(a_0^* d_{10} + b_0^* d_{20}\right). \end{aligned} \tag{46}$$

From (34) and (46), one can obtain

$$\begin{aligned} &H_{20} \\ &= \begin{pmatrix} c_{10} - \ell\pi\left[\left(a_0^* + \overline{a}_0^*\right) c_{10} + \left(b_0^* + \overline{b}_0^*\right) c_{20}\right] \\ c_{20} - \ell\pi\left[\left(\overline{a}_0^* b_0 + a_0^* \overline{b}_0\right) c_{10} + \left(\overline{b}_0^* b_0 + b_0^* \overline{b}_0\right) c_{20}\right] \end{pmatrix}, \\ &H_{11} \\ &= \begin{pmatrix} d_{10} - \ell\pi\left[\left(a_0^* + \overline{a}_0^*\right) d_{10} + \left(b_0^* + \overline{b}_0^*\right) d_{20}\right] \\ d_{20} - \ell\pi\left[\left(\overline{a}_0^* b_0 + a_0^* \overline{b}_0\right) d_{10} + \left(\overline{b}_0^* b_0 + b_0^* \overline{b}_0\right) d_{20}\right] \end{pmatrix}. \end{aligned} \tag{47}$$

Notice from (42) that

$$\begin{aligned} a_0^* + \overline{a}_0^* &= \overline{b}_0^* b_0 + b_0^* \overline{b}_0 = \frac{1}{\ell\pi}, \\ b_0^* + \overline{b}_0^* &= \overline{a}_0^* b_0 + a_0^* \overline{b}_0 = 0. \end{aligned} \tag{48}$$

The conclusion follows by substituting (48) into (47). □

Lemma 3 and (36) imply $w_{20} = w_{11} = 0$ when $n = 0$ in (18) and thus we have

$$\begin{aligned} g_{21} = \left\langle q^*, C_{qq\overline{q}} \right\rangle &= \int_0^{\ell\pi} \left(a_0^* e_{10} + b_0^* e_{20}\right) \\ &= \ell\pi \left(\overline{a}_0^* e_{10} + \overline{b}_0^* e_{20}\right). \end{aligned} \tag{49}$$

It follows from (40) that

$$\begin{aligned} 2\operatorname{Re} c_1(\lambda_0) = \operatorname{Re}\bigg(&\frac{i}{\omega_0} \left\langle q^*, Q_{qq} \right\rangle \left\langle q^*, Q_{q\overline{q}} \right\rangle \\ &+ \left\langle q^*, C_{qq\overline{q}} \right\rangle\bigg) \end{aligned}$$

$$= \operatorname{Re}\left[ i\frac{\ell\pi\left(\overline{a}_0^* c_{10} + \overline{b}_0^* c_{20}\right)\ell\pi\left(\overline{a}_0^* d_{10} + \overline{b}_0^* d_{20}\right)}{\omega_0} + \ell\pi\left(\overline{a}_0^* e_{10} + \overline{b}_0^* e_{20}\right)\right]. \quad (50)$$

We represent $A(\lambda_0)$, $B(\lambda_0)$, and $C(\lambda_0)$ by $A, B$, and $C$, respectively, for the simplicity of notations and under assumption (18) with $n = 0$, substituting $b_0$ in (42) into (44) yields that, for $k = 1, 2$,

$$\begin{aligned}
\operatorname{Re} c_{k0} &= \frac{f_{kuu}B^2 - 2f_{kuv}AB + f_{kvv}\left(2A^2 + BC\right)}{B^2},\\
\operatorname{Im} c_{k0} &= 2\omega_0 \frac{f_{kuv}B - f_{kvv}A}{B^2},\\
d_{k0} &= \frac{f_{kuu}A - 2f_{kuv}B - f_{kvv}C}{B},\\
\operatorname{Re} e_{k0} &= \frac{f_{kuuu}B^2 - 3f_{kuuv}AB + 2f_{kuvv}A^2 + f_{kvvv}AC}{B^2},\\
\operatorname{Im} e_{k0} &= \omega_0 \frac{f_{kuuv}B - 2f_{kuvv}A - f_{kvvv}C}{B^2}.
\end{aligned} \quad (51)$$

From (42), (50), and (51) one can derive

$$2\operatorname{Re} c_1(\lambda_0) = \frac{\mathscr{C}_0(\lambda_0)}{4\omega_0^2 B^2}, \quad (52)$$

where

$$\begin{aligned}
\mathscr{C}_0(\lambda_0) = {} & \{[f_{1uu}A^2 + (f_{2uu} - 2f_{1uv})AB - f_{1vv}AC\\
& - 2f_{2uv}B^2 - f_{2vv}BC](f_{1uu}B + f_{1vv}C + 2f_{2uv}B\\
& - 2f_{2vv}A) + (f_{1uu}A - 2f_{1uv}B - f_{1vv}C)[f_{1uu}AB\\
& + 2f_{1uv}BC - f_{1vv}AC + f_{2uu}B^2 - 2f_{2uv}AB\\
& + f_{2vv}\left(2A^2 + BC\right)] - 2\left(A^2 + BC\right)\\
& \cdot [(f_{1uuu} + f_{2uuv})B^2 - 2(f_{1uuv} + f_{2uvv})AB\\
& - f_{2vvv}BC]\}.
\end{aligned} \quad (53)$$

Thus we have the following result.

**Theorem 4.** *Assume that condition (18) is satisfied when $n = 0$ and $\mathscr{C}_0(\lambda_0)$ is defined by (53). Then the spatially homogeneous Hopf bifurcation of system (1) at $(\lambda_0, 0, 0)$ is supercritical (resp., subcritical) if*

$$\frac{\mathscr{C}_0(\lambda_0)}{A'(\lambda_0) + D'(\lambda_0)} < 0 \quad (\textit{resp.} > 0). \quad (54)$$

*Moreover, if each eigenvalue of $L_j(\lambda_0)$ has negative real parts for all $j \in \mathbb{N}$, then the above spatially homogeneous bifurcating periodic solutions are stable (resp., unstable) when*

$$\mathscr{C}_0(\lambda_0) < 0 \quad (\textit{resp.} > 0). \quad (55)$$

## 4. Spatially Nonhomogeneous Hopf Bifurcation

Notice that the spatially nonhomogeneous periodic solutions of (1) at $(\lambda_0, 0, 0)$ from Hopf bifurcation are unstable. Accordingly, in this section we will calculate $\operatorname{Re} c_1(\lambda_0)$ in (40) in order to determine the direction of Hopf bifurcation of spatially nonhomogeneous periodic solutions of system (1) at $(\lambda_0, 0, 0)$. To this end, we always assume that $n \in \mathbb{N}$ in (18) throughout this section and still represent $A(\lambda_0)$, $B(\lambda_0)$, and $C(\lambda_0)$ by $A, B$, and $C$, respectively. Thus $q^*$ defined in (22) has the form

$$q^* = \begin{pmatrix} a_n^* \\ b_n^* \end{pmatrix} \cos\frac{n}{\ell}x = \begin{pmatrix} \dfrac{\omega_0 + i\left(A - d_1 n^2/\ell^2\right)}{\ell\pi\omega_0} \\ -i\dfrac{B}{\ell\pi\omega_0} \end{pmatrix} \cos\frac{n}{\ell}x, \quad (56)$$

where

$$\begin{aligned}
\omega_0 &= \sqrt{\left(A - \frac{d_1 n^2}{\ell^2}\right)\left(D - \frac{d_2 n^2}{\ell^2}\right) - BC}\\
&= \sqrt{-\left(A - \frac{d_1 n^2}{\ell^2}\right)^2 - BC}.
\end{aligned} \quad (57)$$

Since when $n \in \mathbb{N}$,

$$\int_0^{\ell\pi} \cos^3\frac{n}{\ell}x\,dx = 0, \quad (58)$$

one can obtain

$$\begin{aligned}
\left\langle q^*, Q_{qq}\right\rangle &= \left\langle q^*, Q_{q\overline{q}}\right\rangle = \left\langle \overline{q}^*, Q_{qq}\right\rangle = \left\langle \overline{q}^*, Q_{q\overline{q}}\right\rangle\\
&= 0.
\end{aligned} \quad (59)$$

Thus, in order to calculate $\operatorname{Re} c_1(\lambda_0)$, it remains to compute

$$\begin{aligned}
&\left\langle q^*, Q_{w_{11}q}\right\rangle,\\
&\left\langle q^*, Q_{w_{20}\overline{q}}\right\rangle,\\
&\left\langle q^*, C_{qq\overline{q}}\right\rangle.
\end{aligned} \quad (60)$$

Let all the second- and third-order partial derivatives of $f_k(\lambda, u, v)$ $(k = 1, 2)$ with respect to $u$ and $v$ be evaluated at $(\lambda_0, 0, 0)$ and let

$$\begin{aligned} c_{kn} &= f_{kuu} + 2 f_{kuv} b_n + f_{kvv} b_n^2, \\ d_{kn} &= f_{kuu} + f_{kuv} \left(\overline{b}_n + b_n\right) + f_{kvv} \left|b_n\right|^2, \\ e_{kn} &= f_{kuuu} + f_{kuuv} \left(2b_n + \overline{b}_n\right) + f_{kuvv} \left(2\left|b_n\right|^2 + b_n^2\right) \\ &\quad + f_{kvvv} \left|b_n\right|^2 b_n, \\ & \qquad k = 1, 2. \end{aligned} \tag{61}$$

Then from (30) and (31), one can observe

$$\begin{aligned} Q_{qq} &= \begin{pmatrix} c_{1n} \\ c_{2n} \end{pmatrix} \cos^2 \frac{n}{\ell} x = \frac{1}{2} \begin{pmatrix} c_{1n} \\ c_{2n} \end{pmatrix} \left(1 + \cos \frac{2n}{\ell} x\right), \\ Q_{q\overline{q}} &= \begin{pmatrix} d_{1n} \\ d_{2n} \end{pmatrix} \cos^2 \frac{n}{\ell} x = \frac{1}{2} \begin{pmatrix} d_{1n} \\ d_{2n} \end{pmatrix} \left(1 + \cos \frac{2n}{\ell} x\right), \\ C_{qq\overline{q}} &= \begin{pmatrix} e_{1n} \\ e_{2n} \end{pmatrix} \cos^3 \frac{n}{\ell} x. \end{aligned} \tag{62}$$

In view of (34), (59), and (62), we have

$$\begin{aligned} H_{20} &= Q_{qq} = \frac{1}{2} \begin{pmatrix} c_{1n} \\ c_{2n} \end{pmatrix} \left(1 + \cos \frac{2n}{\ell} x\right), \\ H_{11} &= Q_{q\overline{q}} = \frac{1}{2} \begin{pmatrix} d_{1n} \\ d_{2n} \end{pmatrix} \left(1 + \cos \frac{2n}{\ell} x\right). \end{aligned} \tag{63}$$

Equalities (63) show that the calculation of $[2i\omega_0 - L(\lambda_0)]^{-1}$ and $[L(\lambda_0)]^{-1}$ will be restricted on the subspaces spanned by eigenmodes 1 and $\cos(2n/\ell)x$.

Let

$$\begin{aligned} \alpha_1 &= \frac{\left(12 d_1 d_2 - 3 d_1^2\right) n^4}{\ell^4} - \frac{3\left(d_2 - d_1\right) A n^2}{\ell^2} - 3\omega_0^2, \\ \alpha_2 &= \frac{6\left(d_1 + d_2\right) n^2 \omega_0}{\ell^2}, \\ \alpha_3 &= \frac{d_1^2 n^4}{\ell^4} + \frac{\left(d_2 - d_1\right) A n^2}{\ell^2} - 3\omega_0^2, \\ \alpha_4 &= -\frac{2\left(d_1 + d_2\right) n^2 \omega_0}{\ell^2}. \end{aligned} \tag{64}$$

Then, under condition (18), one can derive

$$\begin{aligned} \left[2i\omega_0 I - L_{2n}\left(\lambda_0\right)\right]^{-1} &= \frac{1}{\alpha_1 + i\alpha_2} \\ &\cdot \begin{pmatrix} 2i\omega_0 + A + \dfrac{\left(3d_2 - d_1\right) n^2}{\ell^2} & B \\ C & 2i\omega_0 - A + \dfrac{4 d_1 n^2}{\ell^2} \end{pmatrix}, \\ \left[2i\omega_0 I - L_0\left(\lambda_0\right)\right]^{-1} &= \frac{1}{\alpha_3 + i\alpha_3} \\ &\cdot \begin{pmatrix} 2i\omega_0 + A - \dfrac{\left(d_1 + d_2\right) n^2}{\ell^2} & B \\ C & 2i\omega_0 - A \end{pmatrix}. \end{aligned} \tag{65}$$

From (36) and (61), we have

$$\begin{aligned} w_{20} &= \left\{ \frac{\left[2i\omega_0 I - L_{2n}\left(\lambda_0\right)\right]^{-1}}{2} \cos \frac{2n}{\ell} x \right. \\ &\quad \left. + \frac{\left[2i\omega_0 I - L_0\left(\lambda_0\right)\right]^{-1}}{2} \right\} \begin{pmatrix} c_{1n} \\ c_{2n} \end{pmatrix} = \frac{1}{2\left(\alpha_1 + i\alpha_2\right)} \\ &\quad \cdot \begin{pmatrix} \left[2i\omega_0 + A + \dfrac{\left(3d_2 - d_1\right) n^2}{\ell^2}\right] c_{1n} + B c_{2n} \\ C c_{1n} + \left(2i\omega_0 - A + \dfrac{4 d_1 n^2}{\ell^2}\right) c_{2n} \end{pmatrix} \\ &\quad \cdot \cos \frac{2n}{\ell} x + \frac{1}{2\left(\alpha_3 + i\alpha_4\right)} \\ &\quad \cdot \begin{pmatrix} \left[2i\omega_0 + A - \dfrac{\left(d_1 + d_2\right) n^2}{\ell^2}\right] c_{1n} + B c_{2n} \\ C c_{1n} + \left(2i\omega_0 - A\right) c_{2n} \end{pmatrix}. \end{aligned} \tag{66}$$

Similarly, we can get

$$\begin{aligned} w_{11} &= \frac{1}{2\alpha_5} \begin{pmatrix} \left[A + \dfrac{\left(3d_2 - d_1\right) n^2}{\ell^2}\right] d_{1n} + B d_{2n} \\ C d_{1n} + \left(\dfrac{4 d_1 n^2}{\ell^2} - A\right) d_{2n} \end{pmatrix} \\ &\quad \cdot \cos \frac{2n}{\ell} x + \frac{1}{2\alpha_6} \\ &\quad \cdot \begin{pmatrix} \left[A - \dfrac{\left(d_1 + d_2\right) n^2}{\ell^2}\right] d_{1n} + B d_{2n} \\ C d_{1n} - A d_{2n} \end{pmatrix}, \end{aligned} \tag{67}$$

where

$$\begin{aligned} \alpha_5 &= \frac{\left(12 d_1 d_2 - 3 d_1^2\right) n^4}{\ell^4} - \frac{3\left(d_2 - d_1\right) A n^2}{\ell^2} + \omega_0^2, \\ \alpha_6 &= \frac{d_1^2 n^4}{\ell^4} + \frac{\left(d_2 - d_1\right) A n^2}{\ell^2} + \omega_0^2. \end{aligned} \tag{68}$$

From (30) we have

$$Q_{w_{20}\bar{q}} = \begin{pmatrix} f_{1uu}\xi + f_{1uv}\eta + f_{1vv}\gamma \\ f_{2uu}\xi + f_{2uv}\eta + f_{2vv}\gamma \end{pmatrix} \cos\frac{n}{\ell}x\cos\frac{2n}{\ell}x + \begin{pmatrix} f_{1uu}\tau + f_{1uv}\chi + f_{1vv}\zeta \\ f_{2uu}\tau + f_{2uv}\chi + f_{2vv}\zeta \end{pmatrix} \cos\frac{n}{\ell}x,$$
$$Q_{w_{11}q} = \begin{pmatrix} f_{1uu}\tilde{\xi} + f_{1uv}\tilde{\eta} + f_{1vv}\tilde{\gamma} \\ f_{2uu}\tilde{\xi} + f_{2uv}\tilde{\eta} + f_{2vv}\tilde{\gamma} \end{pmatrix} \cos\frac{n}{\ell}x\cos\frac{2n}{\ell}x + \begin{pmatrix} f_{1uu}\tilde{\tau} + f_{1uv}\tilde{\chi} + f_{1vv}\tilde{\zeta} \\ f_{2uu}\tilde{\tau} + f_{2uv}\tilde{\chi} + f_{2vv}\tilde{\zeta} \end{pmatrix} \cos\frac{n}{\ell}x, \tag{69}$$

with

$$\begin{aligned}
\xi &= \frac{\left(2i\omega_0 + A + (3d_2 - d_1)n^2/\ell^2\right)c_{1n} + Bc_{2n}}{2(\alpha_1 + i\alpha_2)},\\
\eta &= \frac{\left(2i\omega_0 + A + (3d_2 - d_1)n^2/\ell^2\right)\left(d_1n^2/\ell^2 - A - i\omega_0\right) + BC}{2B(\alpha_1 + i\alpha_2)} \cdot c_{1n} + \frac{\left(5d_1n^2/\ell^2 - 2A + i\omega_0\right)}{2(\alpha_1 + i\alpha_2)}c_{2n},\\
\gamma &= \frac{\left[Cc_{1n} + \left(2i\omega_0 - A + 4d_1n^2/\ell^2\right)c_{2n}\right]\left(d_1n^2/\ell^2 - A - i\omega_0\right)}{2B(\alpha_1 + i\alpha_2)},\\
\tau &= \frac{\left(2i\omega_0 + A - (d_1 + d_2)n^2/\ell^2\right)c_{1n} + Bc_{2n}}{2(\alpha_3 + i\alpha_4)},\\
\chi &= \frac{\left(2i\omega_0 + A - (d_1 + d_2)n^2/\ell^2\right)\left(d_1n^2/\ell^2 - A - i\omega_0\right) + BC}{2B(\alpha_3 + i\alpha_4)} \cdot c_{1n} + \frac{\left(d_1n^2/\ell^2 - 2A + i\omega_0\right)}{2(\alpha_3 + i\alpha_4)}c_{2n},\\
\zeta &= \frac{\left[Cc_{1n} + (2i\omega_0 - A)c_{2n}\right]\left(d_1n^2/\ell^2 - A - i\omega_0\right)}{2B(\alpha_3 + i\alpha_4)},\\
\tilde{\xi} &= \frac{\left(A + (3d_2 - d_1)n^2/\ell^2\right)d_{1n} + Bd_{2n}}{2\alpha_5},\\
\tilde{\eta} &= \frac{\left(A + (3d_2 - d_1)n^2/\ell^2\right)\left(i\omega_0 - A + d_1n^2/\ell^2\right) + BC}{2B\alpha_5}d_{1n} + \frac{\left(i\omega_0 - 2A + 5d_1n^2/\ell^2\right)}{2\alpha_5}d_{2n},\\
\tilde{\gamma} &= \frac{\left(i\omega_0 - A + d_1n^2/\ell^2\right)\left[Cd_{1n} + \left(4d_1n^2/\ell^2 - A\right)d_{2n}\right]}{2B\alpha_5},\\
\tilde{\tau} &= \frac{\left(A - (d_1 + d_2)n^2/\ell^2\right)d_{1n} + Bd_{2n}}{2\alpha_6},\\
\tilde{\chi} &= \frac{\left(A - (d_1 + d_2)n^2/\ell^2\right)\left(i\omega_0 - A + d_1n^2/\ell^2\right) + BC}{2B\alpha_6}d_{1n} + \frac{\left(i\omega_0 - 2A + d_1n^2/\ell^2\right)}{2\alpha_6}d_{2n},\\
\tilde{\zeta} &= \frac{\left(i\omega_0 - A + d_1n^2/\ell^2\right)(Cd_{1n} - Ad_{2n})}{2B\alpha_6}.
\end{aligned} \tag{70}$$

Notice that, for $n \in \mathbb{N}$,

$$\int_0^{\ell\pi} \cos^2\frac{n}{\ell}x\,dx = \frac{\ell\pi}{2}, \qquad \int_0^{\ell\pi} \cos\frac{2n}{\ell}x\cos^2\frac{n}{\ell}x\,dx = \frac{\ell\pi}{4}. \tag{71}$$

It follows from (69) that

$$\begin{aligned}
\left\langle q^*, Q_{w_{20}\bar{q}}\right\rangle &= \frac{\ell\pi}{4}\left[\bar{a}_n^*\left(f_{1uu}\xi + f_{1uv}\eta + f_{1vv}\gamma\right) + \bar{b}_n^*\left(f_{2uu}\xi + f_{2uv}\eta + f_{2vv}\gamma\right)\right]\\
&\quad + \frac{\ell\pi}{2}\left[\bar{a}_n^*\left(f_{1uu}\tau + f_{1uv}\chi + f_{1vv}\zeta\right) + \bar{b}_n^*\left(f_{2uu}\tau + f_{2uv}\chi + f_{2vv}\zeta\right)\right],\\
\left\langle q^*, Q_{w_{11}q}\right\rangle &= \frac{\ell\pi}{4}\left[\bar{a}_n^*\left(f_{1uu}\tilde{\xi} + f_{1uv}\tilde{\eta} + f_{1vv}\tilde{\gamma}\right) + \bar{b}_n^*\left(f_{2uu}\tilde{\xi} + f_{2uv}\tilde{\eta} + f_{2vv}\tilde{\gamma}\right)\right]\\
&\quad + \frac{\ell\pi}{2}\left[\bar{a}_n^*\left(f_{1uu}\tilde{\tau} + f_{1uv}\tilde{\chi} + f_{1vv}\tilde{\zeta}\right) + \bar{b}_n^*\left(f_{2uu}\tilde{\tau} + f_{2uv}\tilde{\chi} + f_{2vv}\tilde{\zeta}\right)\right].
\end{aligned} \tag{72}$$

Substituting $b_n = (i\omega_0 - A(\lambda_0) + d_1n^2/\ell^2)/B(\lambda_0)$ into (61) gives

$$\operatorname{Re} c_{kn} = \frac{f_{kuu}B^2 - 2f_{kuv}\left(A - d_1n^2/\ell^2\right)B + f_{kvv}\left[2\left(A - d_1n^2/\ell^2\right)^2 + BC\right]}{B^2},$$
$$\operatorname{Im} c_{kn} = \frac{2\omega_0\left[f_{kuv}B - f_{kvv}\left(A - d_1n^2/\ell^2\right)\right]}{B^2},$$

$$d_{kn} = \frac{f_{kuu}\left(A - d_1 n^2/\ell^2\right) - 2 f_{kuv} B - f_{kvv} C}{B},$$

$$\operatorname{Re} e_{kn} = \frac{f_{kuuu} B^2 - 3 f_{kuuv}\left(A - d_1 n^2/\ell^2\right) B + 2 f_{kuvv}\left(A - d_1 n^2/\ell^2\right)^2 + f_{kvvv}\left(A - d_1 n^2/\ell^2\right) C}{B^2},$$

$$\operatorname{Im} e_{kn} = \frac{\omega_0\left[f_{kuuv} B - 2 f_{kuvv}\left(A - d_1 n^2/\ell^2\right) - f_{kvvv} C\right]}{B^2},$$

$$k = 1, 2. \tag{73}$$

Then from (70) and (73), we have

$$\operatorname{Re}\xi = \frac{\left[\left(A + (3d_2 - d_1) n^2/\ell^2\right)\operatorname{Re} c_{1n} + B \operatorname{Re} c_{2n} - 2\omega_0 \operatorname{Im} c_{1n}\right]\alpha_1 + \left[\left(A + (3d_2 - d_1) n^2/\ell^2\right)\operatorname{Im} c_{1n} + B \operatorname{Im} c_{2n} + 2\omega_0 \operatorname{Re} c_{1n}\right]\alpha_2}{2\left(\alpha_1^2 + \alpha_2^2\right)},$$

$$\operatorname{Im}\xi = \frac{\left[\left(A + (3d_2 - d_1) n^2/\ell^2\right)\operatorname{Im} c_{1n} + B \operatorname{Im} c_{2n} + 2\omega_0 \operatorname{Re} c_{1n}\right]\alpha_1 - \left[\left(A + (3d_2 - d_1) n^2/\ell^2\right)\operatorname{Re} c_{1n} + B \operatorname{Re} c_{2n} - 2\omega_0 \operatorname{Im} c_{1n}\right]\alpha_2}{2\left(\alpha_1^2 + \alpha_2^2\right)},$$

$$\operatorname{Re}\eta = \frac{\left[\left(A + (3d_2 - d_1) n^2/\ell^2\right)\left(d_1 n^2/\ell^2 - A\right) + BC + 2\omega_0^2\right]\operatorname{Re} c_{1n} - 3\omega_0\left(\left(d_1 - d_2\right) n^2/\ell^2 - A\right)\operatorname{Im} c_{1n} + B\left(5 d_1 n^2/\ell^2 - 2A\right)\operatorname{Re} c_{2n} - B\omega_0 \operatorname{Im} c_{2n}}{2B\left(\alpha_1^2 + \alpha_2^2\right)} \cdot \alpha_1$$
$$+ \frac{\left[\left(A + (3d_2 - d_1) n^2/\ell^2\right)\left(d_1 n^2/\ell^2 - A\right) + BC + 2\omega_0^2\right]\operatorname{Im} c_{1n} + 3\omega_0\left(\left(d_1 - d_2\right) n^2/\ell^2 - A\right)\operatorname{Re} c_{1n} + B\left(5 d_1 n^2/\ell^2 - 2A\right)\operatorname{Im} c_{2n} + B\omega_0 \operatorname{Re} c_{2n}}{2B\left(\alpha_1^2 + \alpha_2^2\right)} \cdot \alpha_2,$$

$$\operatorname{Im}\eta = \frac{\left[\left(A + (3d_2 - d_1) n^2/\ell^2\right)\left(d_1 n^2/\ell^2 - A\right) + BC + 2\omega_0^2\right]\operatorname{Im} c_{1n} + 3\omega_0\left(\left(d_1 - d_2\right) n^2/\ell^2 - A\right)\operatorname{Re} c_{1n} + B\left(5 d_1 n^2/\ell^2 - 2A\right)\operatorname{Im} c_{2n} + B\omega_0 \operatorname{Re} c_{2n}}{2B\left(\alpha_1^2 + \alpha_2^2\right)} \cdot \alpha_1$$
$$- \frac{\left[\left(A + (3d_2 - d_1) n^2/\ell^2\right)\left(d_1 n^2/\ell^2 - A\right) + BC + 2\omega_0^2\right]\operatorname{Re} c_{1n} - 3\omega_0\left(\left(d_1 - d_2\right) n^2/\ell^2 - A\right)\operatorname{Im} c_{1n} + B\left(5 d_1 n^2/\ell^2 - 2A\right)\operatorname{Re} c_{2n} - B\omega_0 \operatorname{Im} c_{2n}}{2B\left(\alpha_1^2 + \alpha_2^2\right)} \cdot \alpha_2,$$

$$\operatorname{Re}\gamma = \frac{\left(d_1 n^2/\ell^2 - A\right)\left[C \operatorname{Re} c_{1n} - \left(A - 4 d_1 n^2/\ell^2\right)\operatorname{Re} c_{2n} - 2\omega_0 \operatorname{Im} c_{2n}\right] + \omega_0\left[C \operatorname{Im} c_{1n} - \left(A - 4 d_1 n^2/\ell^2\right)\operatorname{Im} c_{2n} + 2\omega_0 \operatorname{Re} c_{2n}\right]}{2B\left(\alpha_1^2 + \alpha_2^2\right)}\alpha_1$$
$$+ \frac{\left(d_1 n^2/\ell^2 - A\right)\left[C \operatorname{Im} c_{1n} - \left(A - 4 d_1 n^2/\ell^2\right)\operatorname{Im} c_{2n} + 2\omega_0 \operatorname{Re} c_{2n}\right] - \omega_0\left[C \operatorname{Re} c_{1n} - \left(A - 4 d_1 n^2/\ell^2\right)\operatorname{Re} c_{2n} - 2\omega_0 \operatorname{Im} c_{2n}\right]}{2B\left(\alpha_1^2 + \alpha_2^2\right)}\alpha_2,$$

$$\operatorname{Im}\gamma = \frac{\left(d_1 n^2/\ell^2 - A\right)\left[C \operatorname{Im} c_{1n} - \left(A - 4 d_1 n^2/\ell^2\right)\operatorname{Im} c_{2n} + 2\omega_0 \operatorname{Re} c_{2n}\right] - \omega_0\left[C \operatorname{Re} c_{1n} - \left(A - 4 d_1 n^2/\ell^2\right)\operatorname{Re} c_{2n} - 2\omega_0 \operatorname{Im} c_{2n}\right]}{2B\left(\alpha_1^2 + \alpha_2^2\right)}\alpha_1$$

$$
\begin{aligned}
& - \frac{\left(d_1 n^2/\ell^2 - A\right)\left[C \operatorname{Re} c_{1n} - \left(A - 4d_1 n^2/\ell^2\right)\operatorname{Re} c_{2n} - 2\omega_0 \operatorname{Im} c_{2n}\right] + \omega_0 \left[C \operatorname{Im} c_{1n} - \left(A - 4d_1 n^2/\ell^2\right)\operatorname{Im} c_{2n} + 2\omega_0 \operatorname{Re} c_{2n}\right]}{2B\left(\alpha_1^2 + \alpha_2^2\right)} \alpha_2, \\
\operatorname{Re}\tau = {} & \frac{\left[\left(A + \left(d_1 + d_2\right) n^2/\ell^2\right)\operatorname{Re} c_{1n} + B \operatorname{Re} c_{2n} - 2\omega_0 \operatorname{Im} c_{1n}\right]\alpha_3 + \left[\left(A + \left(d_1 + d_2\right) n^2/\ell^2\right)\operatorname{Im} c_{1n} + B \operatorname{Im} c_{2n} + 2\omega_0 \operatorname{Re} c_{1n}\right]\alpha_4}{2\left(\alpha_3^2 + \alpha_4^2\right)}, \\
\operatorname{Im}\tau = {} & \frac{\left[\left(A + \left(d_1 + d_2\right) n^2/\ell^2\right)\operatorname{Im} c_{1n} + B \operatorname{Im} c_{2n} + 2\omega_0 \operatorname{Re} c_{1n}\right]\alpha_3 - \left[\left(A + \left(d_1 + d_2\right) n^2/\ell^2\right)\operatorname{Re} c_{1n} + B \operatorname{Re} c_{2n} - 2\omega_0 \operatorname{Im} c_{1n}\right]\alpha_4}{2\left(\alpha_3^2 + \alpha_4^2\right)}, \\
\operatorname{Re}\chi & \\
= {} & \frac{\left[\left(A - \left(d_1 + d_2\right) n^2/\ell^2\right)\left(d_1 n^2/\ell^2 - A\right) + BC + 2\omega_0^2\right]\operatorname{Re} c_{1n} - \omega_0 \left(\left(3d_1 + d_2\right) n^2/\ell^2 - 3A\right)\operatorname{Im} c_{1n} + B\left(d_1 n^2/\ell^2 - 2A\right)\operatorname{Re} c_{2n} - B\omega_0 \operatorname{Im} c_{2n}}{2B\left(\alpha_3^2 + \alpha_4^2\right)} \\
& \cdot \alpha_3 \\
& + \frac{\left[\left(A - \left(d_1 + d_2\right) n^2/\ell^2\right)\left(d_1 n^2/\ell^2 - A\right) + BC + 2\omega_0^2\right]\operatorname{Im} c_{1n} + \omega_0 \left(\left(3d_1 + d_2\right) n^2/\ell^2 - 3A\right)\operatorname{Re} c_{1n} + B\left(d_1 n^2/\ell^2 - 2A\right)\operatorname{Im} c_{2n} + B\omega_0 \operatorname{Re} c_{2n}}{2B\left(\alpha_3^2 + \alpha_4^2\right)} \\
& \cdot \alpha_4, \\
\operatorname{Im}\chi & \\
= {} & \frac{\left[\left(A - \left(d_1 + d_2\right) n^2/\ell^2\right)\left(d_1 n^2/\ell^2 - A\right) + BC + 2\omega_0^2\right]\operatorname{Im} c_{1n} + \omega_0 \left(\left(3d_1 + d_2\right) n^2/\ell^2 - 3A\right)\operatorname{Re} c_{1n} + B\left(d_1 n^2/\ell^2 - 2A\right)\operatorname{Im} c_{2n} + B\omega_0 \operatorname{Re} c_{2n}}{2B\left(\alpha_3^2 + \alpha_4^2\right)} \\
& \cdot \alpha_3 \\
& - \frac{\left[\left(A - \left(d_1 + d_2\right) n^2/\ell^2\right)\left(d_1 n^2/\ell^2 - A\right) + BC + 2\omega_0^2\right]\operatorname{Re} c_{1n} - \omega_0 \left(\left(3d_1 + d_2\right) n^2/\ell^2 - 3A\right)\operatorname{Im} c_{1n} + B\left(d_1 n^2/\ell^2 - 2A\right)\operatorname{Re} c_{2n} - B\omega_0 \operatorname{Im} c_{2n}}{2B\left(\alpha_3^2 + \alpha_4^2\right)} \\
& \cdot \alpha_4, \\
\operatorname{Re}\zeta = {} & \frac{\left(d_1 n^2/\ell^2 - A\right)\left(C \operatorname{Re} c_{1n} - A \operatorname{Re} c_{2n} - 2\omega_0 \operatorname{Im} c_{2n}\right) + \omega_0 \left(C \operatorname{Im} c_{1n} - A \operatorname{Im} c_{2n} + 2\omega_0 \operatorname{Re} c_{2n}\right)}{2B\left(\alpha_3^2 + \alpha_4^2\right)} \alpha_3 \\
& + \frac{\left(d_1 n^2/\ell^2 - A\right)\left(C \operatorname{Im} c_{1n} - A \operatorname{Im} c_{2n} + 2\omega_0 \operatorname{Re} c_{2n}\right) - \omega_0 \left(C \operatorname{Re} c_{1n} - A \operatorname{Re} c_{2n} - 2\omega_0 \operatorname{Im} c_{2n}\right)}{2B\left(\alpha_3^2 + \alpha_4^2\right)} \alpha_4, \\
\operatorname{Im}\zeta = {} & \frac{\left(d_1 n^2/\ell^2 - A\right)\left(C \operatorname{Im} c_{1n} - A \operatorname{Im} c_{2n} + 2\omega_0 \operatorname{Re} c_{2n}\right) - \omega_0 \left(C \operatorname{Re} c_{1n} - A \operatorname{Re} c_{2n} - 2\omega_0 \operatorname{Im} c_{2n}\right)}{2B\left(\alpha_3^2 + \alpha_4^2\right)} \alpha_3 \\
& - \frac{\left(d_1 n^2/\ell^2 - A\right)\left(C \operatorname{Re} c_{1n} - A \operatorname{Re} c_{2n} - 2\omega_0 \operatorname{Im} c_{2n}\right) + \omega_0 \left(C \operatorname{Im} c_{1n} - A \operatorname{Im} c_{2n} + 2\omega_0 \operatorname{Re} c_{2n}\right)}{2B\left(\alpha_3^2 + \alpha_4^2\right)} \alpha_4, \\
\operatorname{Re}\widetilde{\eta} = {} & \frac{\left[\left(A + \left(3d_2 - d_1\right) n^2/\ell^2\right)\left(d_1 n^2/\ell^2 - A\right) + BC\right] d_{1n} + B\left(5d_1 n^2/\ell^2 - 2A\right) d_{2n}}{2B\alpha_5}, \\
\operatorname{Re}\widetilde{\eta} = {} & \frac{\omega_0 \left[\left(A + \left(3d_2 - d_1\right) n^2/\ell^2\right) d_{1n} + B d_{2n}\right]}{2\alpha_5}, \\
\operatorname{Re}\widetilde{\gamma} = {} & \frac{\left(d_1 n^2/\ell^2 - A\right)\left[C d_{1n} + \left(4d_1 n^2/\ell^2 - A\right) d_{2n}\right]}{2B\alpha_5}, \\
\operatorname{Im}\widetilde{\gamma} = {} & \frac{\omega_0 \left[C d_{1n} + \left(4d_1 n^2/\ell^2 - A\right) d_{2n}\right]}{2B\alpha_5}, \\
\operatorname{Re}\widetilde{\chi} = {} & \frac{\left[\left(A - \left(d_1 + d_2\right) n^2/\ell^2\right)\left(d_1 n^2/\ell^2 - A\right) + BC\right] d_{1n} + B\left(d_1 n^2/\ell^2 - 2A\right) d_{2n}}{2B\alpha_6}, \\
\operatorname{Im}\widetilde{\chi} = {} & \frac{\omega_0 \left[\left(A - \left(d_1 + d_2\right) n^2/\ell^2\right) d_{1n} + B d_{2n}\right]}{2\alpha_6 B\left(\lambda_0\right)}, \\
\operatorname{Re}\widetilde{\zeta} = {} & \frac{\left(C d_{1n} - A d_{2n}\right)\left(d_1 n^2/\ell^2 - A\right)}{2B\alpha_6}, \\
\operatorname{Im}\widetilde{\zeta} = {} & \frac{\omega_0 \left(C d_{1n} - A d_{2n}\right)}{2B\alpha_6}.
\end{aligned}
\tag{74}
$$

Since $\ell\pi\bar{a}_n^* = 1 - i(A - d_1 n^2/\ell^2)/\omega_0$ and $\ell\pi\bar{b}_n^* = -iB/\omega_0$, one can get

$$\begin{aligned}
\operatorname{Re}\langle q^*, Q_{w_{20}\bar{q}}\rangle &= \frac{1}{4}\left[f_{1uu}(\operatorname{Re}\xi + 2\operatorname{Re}\tau)\right.\\
&\quad + f_{1uv}(\operatorname{Re}\eta + 2\operatorname{Re}\chi) + f_{1vv}(\operatorname{Re}\gamma + 2\operatorname{Re}\zeta)]\\
&\quad + \frac{(A - d_1 n^2/\ell^2)}{4\omega_0}\left[f_{1uu}(\operatorname{Im}\xi + 2\operatorname{Im}\tau)\right.\\
&\quad + f_{1uv}(\operatorname{Im}\eta + 2\operatorname{Im}\chi) + f_{1vv}(\operatorname{Im}\gamma + 2\operatorname{Im}\zeta)]\\
&\quad + \frac{B}{4\omega_0}\left[f_{2uu}(\operatorname{Im}\xi + 2\operatorname{Im}\tau)\right.\\
&\quad + f_{2uv}(\operatorname{Im}\eta + 2\operatorname{Im}\chi) + f_{2vv}(\operatorname{Im}\gamma + 2\operatorname{Im}\zeta)],\\
\operatorname{Re}\langle q^*, Q_{w_{11}q}\rangle &= \frac{1}{4}\left[f_{1uu}(\tilde{\xi} + 2\tilde{\tau})\right.\\
&\quad + f_{1uv}(\operatorname{Re}\tilde{\eta} + 2\operatorname{Re}\tilde{\chi}) + f_{1vv}(\operatorname{Re}\tilde{\gamma} + 2\operatorname{Re}\tilde{\zeta})]\\
&\quad + \frac{(A - d_1 n^2/\ell^2)}{4\omega_0}\left[f_{1uu}(\tilde{\xi} + 2\tilde{\tau})\right.\\
&\quad + f_{1uv}(\operatorname{Re}\tilde{\eta} + 2\operatorname{Re}\tilde{\chi}) + f_{1vv}(\operatorname{Re}\tilde{\gamma} + 2\operatorname{Re}\tilde{\zeta})]\\
&\quad + \frac{B}{4\omega_0}\left[f_{2uv}(\operatorname{Im}\tilde{\eta} + 2\operatorname{Im}\tilde{\chi})\right.\\
&\quad + f_{2vv}(\operatorname{Im}\tilde{\gamma} + 2\operatorname{Im}\tilde{\zeta})].
\end{aligned} \tag{75}$$

In addition, it follows from $\int_0^{\ell\pi}\cos^4(n/\ell)x\,dx = 3\ell\pi/8$ and (62) that

$$\langle q^*, C_{qq\bar{q}}\rangle = \frac{3\ell\pi}{8}\left(\bar{a}_n^* e_{1n} + \bar{b}_n^* e_{2n}\right). \tag{76}$$

Therefore,

$$\begin{aligned}
\operatorname{Re}\langle q^*, C_{qq\bar{q}}\rangle &= \frac{3}{8B^2}\left\{f_{1uuu}B^2\right.\\
&\quad - 3f_{1uuv}\left(A - \frac{d_1 n^2}{\ell^2}\right)B + 2f_{1uvv}\left(A - \frac{d_1 n^2}{\ell^2}\right)^2\\
&\quad + f_{1vvv}\left(A - \frac{d_1 n^2}{\ell^2}\right)C + \left(A - \frac{d_1 n^2}{\ell^2}\right)\\
&\quad \cdot\left[f_{1uuv}B - 2f_{1uvv}\left(A - \frac{d_1 n^2}{\ell^2}\right) - f_{1vvv}C\right]\\
&\quad \left. + B\left[f_{2uuv}B - 2f_{2uvv}\left(A - \frac{d_1 n^2}{\ell^2}\right) - f_{2vvv}C\right]\right\}.
\end{aligned} \tag{77}$$

Now, by (40), we have

$$\begin{aligned}
\operatorname{Re}c_1(\lambda_0) &= \operatorname{Re}\langle q^*, Q_{w_{11}q}\rangle + \frac{1}{2}\operatorname{Re}\langle q^*, Q_{w_{20}\bar{q}}\rangle + \frac{1}{2}\\
&\quad \cdot \operatorname{Re}\langle q^*, C_{qq\bar{q}}\rangle\\
&= \frac{1}{8}\left[f_{1uu}\left(\operatorname{Re}\xi + 2\operatorname{Re}\tau + 2\tilde{\xi} + 4\tilde{\tau}\right)\right.\\
&\quad + f_{1uv}(\operatorname{Re}\eta + 2\operatorname{Re}\chi + 2\operatorname{Re}\tilde{\eta} + 4\operatorname{Re}\tilde{\chi})\\
&\quad + f_{1vv}\left(\operatorname{Re}\gamma + 2\operatorname{Re}\zeta + 2\operatorname{Re}\tilde{\gamma} + 4\operatorname{Re}\tilde{\zeta}\right)]\\
&\quad + \frac{(A - d_1 n^2/\ell^2)}{8\omega_0}\left[f_{1uu}\left(\operatorname{Im}\xi + 2\operatorname{Im}\tau + 2\tilde{\xi} + 4\tilde{\tau}\right)\right.\\
&\quad + f_{1uv}(\operatorname{Im}\eta + 2\operatorname{Im}\chi + 2\operatorname{Re}\tilde{\eta} + 4\operatorname{Re}\tilde{\chi})\\
&\quad + f_{1vv}\left(\operatorname{Im}\gamma + 2\operatorname{Im}\zeta + 2\operatorname{Re}\tilde{\gamma} + 4\operatorname{Re}\tilde{\zeta}\right)]\\
&\quad + \frac{B}{8\omega_0}\left[f_{2uu}(\operatorname{Im}\xi + 2\operatorname{Im}\tau)\right.\\
&\quad + f_{2uv}(\operatorname{Im}\eta_I + 2\operatorname{Im}\chi + 2\operatorname{Im}\tilde{\eta} + 4\operatorname{Im}\tilde{\chi})\\
&\quad + f_{2vv}\left(\operatorname{Im}\gamma + \operatorname{Im}\zeta + 2\operatorname{Im}\tilde{\gamma} + 4\operatorname{Im}\tilde{\zeta}\right)]\\
&\quad + \frac{3}{16B^2}\left\{f_{1uuu}B^2 - 3f_{1uuv}\left(A - \frac{d_1 n^2}{\ell^2}\right)B\right.\\
&\quad + 2f_{1uvv}\left(A - \frac{d_1 n^2}{\ell^2}\right)^2 + f_{1vvv}\left(A - \frac{d_1 n^2}{\ell^2}\right)C\\
&\quad + \left(A - \frac{d_1 n^2}{\ell^2}\right)\\
&\quad \cdot\left[f_{1uuv}B - 2f_{1uvv}\left(A - \frac{d_1 n^2}{\ell^2}\right) - f_{1vvv}C\right]\\
&\quad \left. + B\left[f_{2uuv}B - 2f_{2uvv}\left(A - \frac{d_1 n^2}{\ell^2}\right) - f_{2vvv}C\right]\right\}.
\end{aligned} \tag{78}$$

Thus we have the following result.

**Theorem 5.** *Assume that condition (18) holds for $n \in \mathbb{N}$. Then the spatially nonhomogeneous Hopf bifurcation of system (1) at $(\lambda_0, 0, 0)$ is supercritical (resp., subcritical) if*

$$\frac{\operatorname{Re}c_1(\lambda_0)}{A'(\lambda_0) + D'(\lambda_0)} < 0 \quad (\text{resp.} > 0); \tag{79}$$

*here $\operatorname{Re}c_1(\lambda_0)$ is given by (78).*

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## References

[1] A. Andronov, E. A. Leontovich, I. I. Gordon, and A. G. Maier, *Theory of Bifurcations of Dynamical Systems on a Plane*, Israel Program for Scientific Translations, Jerusalem, Israel, 1971.

[2] J. Hale, *Theory of Functional Differential Equations*, Springer, New York, NY, USA, 1977.

[3] B. D. Hassard, N. D. Kazarinoff, and Y. H. Wan, *Theory and Applications of Hopf Bifurcation*, Cambridge University Press, Cambridge, UK, 1981.

[4] H. Kielhófer, *Bifurcation Theory: An Introduction with Applications to PDEs*, vol. 156 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 2004.

[5] Y. A. Kuznetsov, *Elements of Applied Bifurcation Theory*, Applied Mathematical Sciences, Springer, New York, NY, USA, 1998.

[6] J.-H. Wu, *Theory and Applications of Partial Functional Differential Equations*, Springer, New York, NY, USA, 1996.

[7] F. Yi, J. Wei, and J. Shi, "Bifurcation and spatiotemporal patterns in a homogeneous diffusive predator-prey system," *Journal of Differential Equations*, vol. 246, no. 5, pp. 1944–1977, 2009.

[8] L.-J. Yang and X.-W. Zeng, "Stability of singular Hopf bifurcations," *Journal of Differential Equations*, vol. 206, no. 1, pp. 30–54, 2004.

[9] J. Y. Jin, J. P. Shi, J. J. Wei, and F. Q. Yi, "Bifurcations of patterned solutions in the diffusive Lengyel-Epstein system of CIMA chemical reactions," *Rocky Mountain Journal of Mathematics*, vol. 43, no. 5, pp. 1637–1674, 2013.

[10] S. G. Ruan, "Diffusion-driven instability in the Gierer-Meinhardt model of morphogenesis," *Natural Resource Modeling*, vol. 11, no. 2, pp. 131–142, 1998.

[11] F. Yi, J. Wei, and J. Shi, "Diffusion-driven instability and bifurcation in the Lengyel-Epstein system," *Nonlinear Analysis: Real World Applications*, vol. 9, no. 3, pp. 1038–1051, 2008.

[12] F. Q. Yi, J. J. Wei, and J. P. Shi, "Global asymptotical behavior of the Lengyel-Epstein reaction-diffusion system," *Applied Mathematics Letters*, vol. 22, no. 1, pp. 52–55, 2009.