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Research Article

A Stability Result for the Solutions of a Certain System of Fourth-Order Delay Differential Equation

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The main purpose of this work is to give sufficient conditions for the uniform stability of the zero solution of a certain fourth-order vector delay differential equation of the following form: $X^{(4)} + F(\dot{X}, \ddot{X})\dddot{X} + \Phi(\ddot{X}) + G(\dot{X}(t-r)) + H(X(t-r)) = 0$. By constructing a Lyapunov functional, we obtained the result of stability.

1. Introduction

As is well known, the stability is a very important problem in the theory and applications of delay differential equations. Therefore, in the literature, some methods have been developed to obtain information on the stability behaviour of the delay differential equations when there is no analytical expression for the solutions. One of these methods is known as Lyapunov's second method; since Lyapunov [1] proposed his famous second method on the stability of motion, the problems related to the investigation of stability of solutions of certain second-, third-, and fourth-order linear and nonlinear, scalar, and vector differential equations have been given great attention in the past five decades due to the importance of the subject.

During this period, stability of solutions for various higher-order linear and nonlinear differential equations has been extensively studied and many results have been obtained in the literature (see, e.g., Krasovskii [2], Yoshizawa [3], Reissig et al. [4], Abou-El-Ela and Sadek [5–7], Bereketoglu and Kart [8], Sadek [9], Tunç [10–13], Abou-El-Ela et al. [14], and the references cited in those works), among which the results performed on asymptotic stability properties of linear and nonlinear scalar and vector differential equations of fourth-order can briefly be summarized as follows.

First in 1990 Abou-El-Ela and Sadek [5] found sufficient conditions for the asymptotic stability of the zero solution of the scalar nonlinear differential equation of the form

$$x^{(4)} + f_1(\dot{x}, \ddot{x})\dddot{x} + f_2(\dot{x}, \ddot{x}) + f_3(x, \dot{x}) + f_4(x) = 0. \quad (1)$$

Later in 2004 Sadek [9] determined sufficient conditions, under which all solutions of the nonhomogeneous vector differential equation

$$X^{(4)} + F(\dot{X}, \ddot{X})\dddot{X} + \Phi(\ddot{X}) + G(\dot{X}) + A_4 X = P(t, X, \dot{X}, \ddot{X}, \dddot{X}) \quad (2)$$

tend to zero as $t \to \infty$.

Recently in 2012 Abou-El-Ela et al. [14] investigated sufficient conditions for the uniform stability of the zero solution of the real fourth-order vector delay differential equation

$$X^{(4)} + A\dddot{X} + \Phi(\ddot{X}) + G(\dot{X}) + H(X(t-r)) = 0. \quad (3)$$

In the present paper, we are concerned with the uniform stability of the zero solution $X = 0$ of real nonlinear autonomous vector delay differential equation of the fourth-order

$$X^{(4)} + F\left(\dot{X}, \ddot{X}\right)\dddot{X} + \Phi\left(\ddot{X}\right) + G\left(\dot{X}(t-r)\right) + H(X(t-r)) = 0, \tag{4}$$

where $X \in R^n$; F is an $n \times n$-symmetric matrix; Φ, G, and H are n-vector continuous functions; $\Phi(0) = G(0) = H(0) = 0$; and r is a bounded delay and positive constant.

Equation (4) represents a system of real fourth-order differential equation with delay

$$\begin{aligned} x_i^{(4)} &+ \sum_{k=1}^{n} f_{ik}\left(\dot{x}_1, \ldots, \dot{x}_n; \ddot{x}_1, \ldots, \ddot{x}_n\right)\dddot{x}_k + \phi_i\left(\ddot{x}_1, \ldots, \ddot{x}_n\right) \\ &+ g_i\left(\dot{x}_1(t-r), \ldots, \dot{x}_n(t-r)\right) \\ &+ h_i\left(x_1(t-r), \ldots, x_n(t-r)\right) = 0, \quad (i = 1, 2, \ldots, n). \end{aligned} \tag{5}$$

The Jacobian matrices $J(F(Y,Z)Y \mid Z)$, $J(F(Y,Z)Z \mid Z)$, $J(F(Y,Z)Y \mid Y)$, $J(F(Y,Z)Z \mid Y)$, $J_\Phi(Z)$, $J_G(Y)$, and $J_H(X)$ are given by

$$\begin{aligned} J(F(Y,Z)Y \mid Z) &= \left(\frac{\partial}{\partial z_j}\sum_{k=1}^{n} f_{ik}y_k\right) = \left(\sum_{k=1}^{n}\frac{\partial f_{ik}}{\partial z_j}y_k\right), \\ J(F(Y,Z)Z \mid Z) &= \left(\frac{\partial}{\partial z_j}\sum_{k=1}^{n} f_{ik}z_k\right) \\ &= F(Y,Z) + \left(\sum_{k=1}^{n}\frac{\partial f_{ik}}{\partial z_j}z_k\right), \\ J(F(Y,Z)Y \mid Y) &= \left(\frac{\partial}{\partial y_j}\sum_{k=1}^{n} f_{ik}y_k\right) \\ &= F(Y,Z) + \left(\sum_{k=1}^{n}\frac{\partial f_{ik}}{\partial y_j}y_k\right), \\ J(F(Y,Z)Z \mid Y) &= \left(\frac{\partial}{\partial y_j}\sum_{k=1}^{n} f_{ik}z_k\right) = \left(\sum_{k=1}^{n}\frac{\partial f_{ik}}{\partial y_j}z_k\right), \\ J_\Phi(Z) &= \left(\frac{\partial \phi_i}{\partial z_j}\right), \qquad J_G(Y) = \left(\frac{\partial g_i}{\partial y_j}\right), \\ J_H(X) &= \left(\frac{\partial h_i}{\partial x_j}\right), \end{aligned} \tag{6}$$

where $(i, j = 1, 2, \ldots, n)$, $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n)$, $(z_1, \ldots, z_n)$, (f_{ik}), $(\phi_1, \ldots, \phi_n)$, $(g_1, \ldots, g_n)$, and $(h_1, \ldots, h_n)$ represent X, Y, Z, F, Φ, G, and H, respectively. It will also be assumed as basic throughout the paper that the Jacobian matrices $J(F(Y,Z)Y \mid Z)$, $J(F(Y,Z)Z \mid Z)$, $J(F(Y,Z)Y \mid Y)$, $J(F(Y,Z)Z \mid Y)$, $J_\Phi(Z)$, $J_G(Y)$, and $J_H(X)$ exist and are continuous. The symbol $\langle X, Y\rangle$ will be used to denote the usual scalar product in R^n for any X, Y in R^n; that is, $\langle X, Y\rangle = \sum_{i=1}^{n} x_i y_i$; thus $\langle X, X\rangle = \|X\|^2$. It is well known that the real symmetric matrix $A = (a_{ij})$, $(i, j = 1, 2, \ldots, n)$ is said to be positive-definite, if and only if the quadratic form $X^T AX$ is positive-definite, where $X \in R^n$ and X^T denotes the transpose of X.

2. Main Result

In order to reach the main result of this paper, we will give some basic information to the stability criteria for a general autonomous delay differential system. We consider

$$\dot{\overline{x}} = \overline{f}\left(\overline{x}_t\right), \quad \overline{x}_t(s) = \overline{x}(t+s), \quad -h \le s \le 0, \; t \ge 0, \tag{7}$$

where $\overline{f} : \mathscr{C}_H \to R^n$ is a continuous mapping, $\overline{f}(0) = 0$, $\mathscr{C}_H := \{\phi \in \mathscr{C}([-h, 0], R^n) : \|\phi\| \le H\}$, and for $H_1 < H$, there exists an $L(H_1) > 0$, with $|\overline{f}(\phi)| \le L(H_1)$ when $\|\phi\| < H_1$.

Theorem 1 (see [15]). *Let $V(\phi) : \mathscr{C}_H \to R$ be a continuous functional satisfying a local Lipschitz condition, $V(0) = 0$, such that*

(i) $W_1(|\phi(0)|) \le V(\phi) \le W_2(\|\phi\|)$, *where W_1, W_2 are wedges;*

(ii) $\dot{V}_{(7)}(\phi) \le 0$, *for $\phi \in \mathscr{C}_H$.*

Then the zero solution of (7) is uniformly stable.

The following theorem will be our main stability result for (4).

Theorem 2. *In addition to the essential assumptions imposed on the functions F, Φ, G, and H, suppose the existence of arbitrary positive constants α_1, α_2, α_3, α_4, α_1', and α_4'. Suppose also for $i = 1, 2, \ldots, n$ the following conditions are satisfied.*

(i) *$F(Y,Z)$, $J(F(Y,Z)Y \mid Z)$, and $J(F(Y,Z)Z \mid Z)$ are symmetric; $\alpha_1' \ge \lambda_i(F(Y,Z)) \ge \alpha_1 > 0$, for all $Y, Z \in R^n$.*

(ii) *$G(0) = 0$, $J_G(Y)$ is symmetric and $\lambda_i(\int_0^1 J_G(\sigma Y)d\sigma) \ge \alpha_3\alpha_4^2/\alpha_4'^2$, for all $Y \in R^n$.*

(iii) *There is a finite constant $\Delta > 0$ such that*

$$\{\alpha_1\alpha_2 - \|J_G(Y)\|\}\,\alpha_3\alpha_4 - \alpha_1\alpha_4'^2\left\|\int_0^1 F(Y,\sigma Z)\,d\sigma\right\| \ge \Delta, \tag{8}$$

for all $Y, Z \in R^n$.

(iv) *One has $0 \le \lambda_i(J_G(Y) - \int_0^1 J_G(\sigma Y)d\sigma) \le \delta_1 < 2\Delta/\alpha_1\alpha_3^2$, for all $Y \in R^n$.*

(v) *One has $\lambda_i(\int_0^1 F(Y,\sigma Z)d\sigma - F(Y,Z)) \le \delta_2 < 2\Delta/\alpha_1^2\alpha_3\alpha_4$, for all $Y, Z \in R^n$.*

(vi) *$J(F(Y,Z)Y \mid Y) - F(Y,Z)$ and $J(F(Y,Z)Z \mid Y)$ are negative-definite.*

(vii) *Also $H(0) = 0$, $J_H(X)$ is symmetric, and $\lambda_i(\int_0^1 J_H(\sigma X)d\sigma) \ge \alpha_4'$, for all $X \in R^n$.*

(viii) *$J_H(X)$ commutes with $J_H(X')$, for all $X, X' \in R^n$ and $0 \le \lambda_i(\alpha_4 I - J_H(X)) \le \varepsilon D_0\alpha_1^2$, for all $X \in R^n$, and $D_0 := \alpha_1\alpha_2 + \alpha_2\alpha_3\alpha_4\alpha_4'^{-2}$.*

(ix) *Also* $\Phi(0) = 0$, $J_{\Phi}(Z)$ *is symmetric, and* $0 \leq \lambda_i(\int_0^1 J_{\Phi}(\sigma Z)d\sigma - \alpha_2 I) \leq \varepsilon_0\alpha_3^3\alpha_4^2/\alpha_4'^4$, *for all* $Z \in R^n$, *where* ε_0 *is a positive constant such that*

$$\varepsilon_0 < \varepsilon = \min\left\{\frac{1}{\alpha_1}, \frac{\alpha_4'^2}{\alpha_3\alpha_4}, \frac{\Delta}{4\alpha_1\alpha_3\alpha_4 D_0}, \frac{\alpha_3\alpha_4}{4\alpha_4'^2 D_0}\left(\frac{2\alpha_4'^2\Delta}{\alpha_1\alpha_3^2\alpha_4^2} - \delta_1\right), \frac{\alpha_1}{4D_0}\left(\frac{2\Delta}{\alpha_1^2\alpha_3\alpha_4} - \delta_2\right)\right\}. \quad (9)$$

Then the zero solution of (4) is uniformly stable, provided that

$$r < \min\left[\frac{\varepsilon}{d_1\sqrt{n}(\alpha_4 + \alpha_1\alpha_2)}, \frac{\Delta}{2\alpha_1\alpha_3\alpha_4\sqrt{n}\{\alpha_4 + \alpha_1\alpha_2(d_1 + d_2 + 2)\}}, \frac{((\alpha_4^2/\alpha_4'^2)\varepsilon - \varepsilon_0)\alpha_3}{\alpha_4\sqrt{n}(d_1 + 2d_2 + 1) + \alpha_1\alpha_2 d_2\sqrt{n}}\right], \quad (10)$$

where

$$d_1 = \varepsilon + \frac{1}{\alpha_1}, \qquad d_2 = \varepsilon + \frac{\alpha_4'^2}{\alpha_3\alpha_4}. \quad (11)$$

The following two lemmas are important for proving Theorem 2.

Lemma 3. *Let A be a real symmetric n × n-matrix and*

$$a' \geq \lambda_i(A) \geq a > 0 \quad (i = 1, 2, \ldots, n), \quad (12)$$

where a', a *are constants. Then*

$$a'\langle X, X\rangle \geq \langle AX, X\rangle \geq a\langle X, X\rangle,$$
$$a'^2\langle X, X\rangle \geq \langle AX, AX\rangle \geq a^2\langle X, X\rangle. \quad (13)$$

For a proof of the above lemma, see Bellman [16].

Lemma 4. *Assume that* $\dot{X} = Y$, $\dot{Y} = Z$, *and* $\dot{Z} = W$. *Then*

(1) $(d/dt)\int_0^1\langle H(\sigma X), X\rangle d\sigma = \langle H(X), Y\rangle$;

(2) $(d/dt)\int_0^1\langle G(\sigma Y), Y\rangle d\sigma = \langle G(Y), Z\rangle$;

(3) $(d/dt)\int_0^1\langle \Phi(\sigma Z), Z\rangle d\sigma = \langle \Phi(Z), W\rangle$;

(4) $(d/dt)\int_0^1\langle \sigma F(Y, \sigma Z)Z, Z\rangle d\sigma \leqq \langle F(Y, Z)Z, W\rangle$;

(5) $(d/dt)\int_0^1\langle F(Y, \sigma Z)Z, Y\rangle d\sigma \leqq \langle F(Y, Z)Y, W\rangle + \|\int_0^1 F(Y, \sigma Z)d\sigma\|\langle Z, Z\rangle$.

Proof. The proof is as follows:

$$\begin{aligned}(1)\ \frac{d}{dt}\int_0^1\langle H(\sigma X), X\rangle d\sigma &= \int_0^1 \sigma\langle J_H(\sigma X)Y, X\rangle d\sigma + \int_0^1\langle H(\sigma X), Y\rangle d\sigma \\ &= \int_0^1 \sigma\langle J_H(\sigma X)X, Y\rangle d\sigma + \int_0^1\langle H(\sigma X), Y\rangle d\sigma \\ &= \int_0^1 \sigma\frac{\partial}{\partial\sigma}\langle H(\sigma X), Y\rangle d\sigma + \int_0^1\langle H(\sigma X), Y\rangle d\sigma \\ &= \sigma\langle H(\sigma X), Y\rangle|_0^1 = \langle H(X), Y\rangle.\end{aligned} \quad (14)$$

The proofs of (2) and (3) are similar to that of (1):

$$\begin{aligned}(4)\ \frac{d}{dt}\int_0^1\langle \sigma F(Y, \sigma Z)Z, Z\rangle d\sigma &= \int_0^1\langle \sigma F(Y, \sigma Z)Z, W\rangle d\sigma \\ &\quad + \int_0^1\langle J(F(Y, \sigma Z)\sigma Z \mid Y)Z, Z\rangle d\sigma \\ &\quad + \int_0^1 \sigma\langle J(F(Y, \sigma Z)\sigma Z \mid \sigma Z)W, Z\rangle d\sigma \\ &\leqq \int_0^1\langle \sigma F(Y, \sigma Z)Z, W\rangle d\sigma \\ &\quad + \int_0^1 \sigma\langle J(F(Y, \sigma Z)\sigma Z \mid \sigma Z)Z, W\rangle d\sigma,\end{aligned} \quad (15)$$

since $J(FZ \mid Y)$ is negative-definite from assumption (vi) and $J(FZ \mid Z)$ is symmetric from assumption (i). Then

$$\begin{aligned}\frac{d}{dt}\int_0^1\langle \sigma F(Y, \sigma Z)Z, Z\rangle d\sigma &\leqq \int_0^1\langle \sigma F(Y, \sigma Z)Z, W\rangle d\sigma \\ &\quad + \int_0^1 \sigma\frac{\partial}{\partial\sigma}\langle \sigma F(Y, \sigma Z)Z, W\rangle d\sigma \\ &= \sigma\langle \sigma F(Y, \sigma Z)Z, W\rangle|_0^1 \\ &= \langle F(Y, Z)Z, W\rangle,\end{aligned}$$

$$(5)\ \frac{d}{dt}\int_0^1 \langle F(Y,\sigma Z)Z,Y\rangle\,d\sigma = \frac{d}{dt}\int_0^1 \langle F(Y,\sigma Z)Y,Z\rangle\,d\sigma = \frac{d}{dt}\int_0^1 \langle F(Y,\sigma Z)Y,W\rangle\,d\sigma + \int_0^1 \langle J(F(Y,\sigma Z)Y \mid Y)Z,Z\rangle\,d\sigma + \int_0^1 \langle \sigma J(F(Y,\sigma Z)Y \mid \sigma Z)W,Z\rangle\,d\sigma. \tag{16}$$

Since $J(FY \mid Z)$ is negative-definite from assumption (i), we have

$$\begin{aligned}\int_0^1 \langle \sigma J(F(Y,\sigma Z)Y \mid \sigma Z)W,Z\rangle\,d\sigma &= \int_0^1 \langle \sigma J(F(Y,\sigma Z)Y \mid \sigma Z)Z,W\rangle\,d\sigma\\ &= \int_0^1 \sigma\frac{\partial}{\partial\sigma}\langle F(Y,\sigma Z)Y,W\rangle\,d\sigma\\ &= \langle F(Y,Z)Y,W\rangle - \int_0^1 \langle F(Y,\sigma Z)Y,W\rangle\,d\sigma,\end{aligned} \tag{17}$$

and then

$$\begin{aligned}\frac{d}{dt}\int_0^1 \langle F(Y,\sigma Z)Z,Y\rangle\,d\sigma &= \langle F(Y,Z)Y,W\rangle + \int_0^1 \langle J(F(Y,\sigma Z)Y \mid Y)Z,Z\rangle\,d\sigma\\ &= \langle F(Y,Z)Y,W\rangle + \int_0^1 \langle F(Y,\sigma Z)Z,Z\rangle\,d\sigma\\ &\quad + \int_0^1 \langle \{J(F(Y,\sigma Z)Y \mid Y) - F(Y,\sigma Z)\}Z,Z\rangle\,d\sigma\\ &\leqq \langle F(Y,Z)Y,W\rangle + \left\|\int_0^1 F(Y,\sigma Z)\,d\sigma\right\|\langle Z,Z\rangle,\end{aligned} \tag{18}$$

since $J(FY \mid Y) - F$ is negative-definite from assumption (vi). □

3. Proof of Theorem 2

For the proof of the main stability theorem, it will be convenient to consider instead of (4) the equivalent system

$$\begin{aligned}&\dot{X} = Y, \qquad \dot{Y} = Z, \qquad \dot{Z} = W,\\ &\dot{W} = -F(Y,Z)W - \Phi(Z) - G(Y) - H(X)\\ &\quad + \int_{t-r}^t J_G(Y(s))Z(s)\,ds + \int_{t-r}^t J_H(X(s))Y(s)\,ds.\end{aligned} \tag{19}$$

The proof of Theorem 2 depends on a scalar differentiable function $V(X_t, Y_t, Z_t, W_t)$; now we define the Lyapunov functional V as

$$\begin{aligned}2V&(X_t,Y_t,Z_t,W_t)\\ &= 2d_2\int_0^1 \langle H(\sigma X),X\rangle\,d\sigma + d_2\langle \alpha_2 Y,Y\rangle\\ &\quad - d_1\langle \alpha_4 Y,Y\rangle + 2\int_0^1 \langle G(\sigma Y),Y\rangle\,d\sigma\\ &\quad + 2d_1\int_0^1 \langle \Phi(\sigma Z),Z\rangle\,d\sigma\\ &\quad - d_2\langle Z,Z\rangle + 2\int_0^1 \langle \sigma F(Y,\sigma Z)Z,Z\rangle\,d\sigma + d_1\langle W,W\rangle\\ &\quad + 2\langle H(X),Y\rangle + 2d_1\langle H(X),Z\rangle\\ &\quad + 2d_2\int_0^1 \langle F(Y,\sigma Z)Z,Y\rangle\,d\sigma\\ &\quad + 2d_1\langle G(Y),Z\rangle + 2d_2\langle Y,W\rangle + 2\langle Z,W\rangle\\ &\quad + 2\mu\int_{-r}^0\int_{t+s}^t \|Y(\theta)\|^2\,d\theta\,ds\\ &\quad + 2\lambda\int_{-r}^0\int_{t+s}^t \|Z(\theta)\|^2\,d\theta\,ds,\end{aligned} \tag{20}$$

where μ and λ are positive constants, which will be determined later. Let

$$F_1(Y,Z) = \int_0^1 F(Y,\sigma Z)\,d\sigma. \tag{21}$$

Since $\lambda_i(F(Y,Z)) \geq \alpha_1 > 0$, for all $Y, Z \in R^n$, it follows that

$$\lambda_i(F_1(Y,Z)) \geq \alpha_1 > 0, \quad \forall Y,Z \in R^n. \tag{22}$$

Further we define

$$\Gamma(Y) = \int_0^1 J_G(\sigma Y)\,d\sigma, \tag{23}$$

and then it follows from (ii) and (iv) that

$$\lambda_i(\Gamma(Y)) \geq \frac{\alpha_3\alpha_4^2}{\alpha_4'^2} > 0, \tag{24}$$

for all $Y \in R^n$, and

$$0 \leq \lambda_i(J_G(Y) - \Gamma(Y)) \leq \delta_1, \quad \forall Y \in R^n. \tag{25}$$

Since

$$\frac{\partial}{\partial\sigma}\Phi(\sigma Z) = J_\Phi(\sigma Z)Z, \qquad \Phi(0) = 0, \tag{26}$$

then

$$\Phi(Z) = \int_0^1 J_\Phi(\sigma Z)Z\,d\sigma. \tag{27}$$

Therefore

$$2d_1 \int_0^1 \langle \Phi(\sigma Z), Z\rangle d\sigma$$
$$= 2d_1 \int_0^1 \int_0^1 \langle J_\Phi(\sigma_1\sigma_2 Z)\sigma_2 Z, Z\rangle d\sigma_1 d\sigma_2$$
$$= 2d_1 \int_0^1 \left[\int_0^1 \left\langle J_\Phi\left(\sigma_1 \widetilde{Z}\right)\widetilde{Z}, Z\right\rangle d\sigma_1\right] d\sigma_2 \quad (28)$$
$$\geq 2d_1 \int_0^1 \alpha_2 \left\langle \widetilde{Z}, Z\right\rangle d\sigma_2, \quad \text{by (ix)}$$
$$= 2d_1 \int_0^1 \alpha_2 \langle Z, Z\rangle \sigma_2 d\sigma_2 = d_1\alpha_2 \langle Z, Z\rangle .$$

Also since

$$2\mu \int_{-r}^0 \int_{t+s}^t \|Y(\theta)\|^2 d\theta\, ds, \qquad 2\lambda \int_{-r}^0 \int_{t+s}^t \|Z(\theta)\|^2 d\theta\, ds \quad (29)$$

are nonnegative, consequently we obtain

$$\begin{aligned} 2V&(X_t, Y_t, Z_t, W_t) \\ &\geq 2d_2 \int_0^1 \langle H(\sigma X), X\rangle d\sigma + d_2 \langle \alpha_2 Y, Y\rangle - d_1 \langle \alpha_4 Y, Y\rangle \\ &+ 2\int_0^1 \langle G(\sigma Y), Y\rangle d\sigma + (\alpha_2 d_1 - d_2)\langle Z, Z\rangle \\ &+ 2\int_0^1 \langle \sigma F(Y, \sigma Z) Z, Z\rangle d\sigma + d_1 \langle W, W\rangle \\ &+ 2\langle H(X), Y\rangle \\ &+ 2d_1 \langle H(X), Z\rangle + 2d_2 \int_0^1 \langle F(Y, \sigma Z) Z, Y\rangle d\sigma \\ &+ 2d_1 \langle G(Y), Z\rangle + 2d_2 \langle Y, W\rangle + 2\langle Z, W\rangle . \end{aligned} \quad (30)$$

Then we can find

$$\begin{aligned} 2V &\geq 2d_2 \int_0^1 \langle H(\sigma X), X\rangle d\sigma - \left\|\Gamma^{-1/2} H(X)\right\|^2 + d_2 \langle \alpha_2 Y, Y\rangle \\ &- d_1 \langle \alpha_4 Y, Y\rangle - d_2^2 \left\|F_1^{1/2} Y\right\|^2 + 2\int_0^1 \langle G(\sigma Y), Y\rangle d\sigma \\ &- \left\|\Gamma^{1/2} Y\right\|^2 + (\alpha_2 d_1 - d_2)\|Z\|^2 - d_1^2 \left\|\Gamma^{1/2} Z\right\|^2 \\ &+ 2\int_0^1 \langle \sigma F(Y, \sigma Z) Z, Z\rangle d\sigma - \left\|F_1^{1/2} Z\right\|^2 + d_1 \|W\|^2 \\ &- \left\|F_1^{-1/2} W\right\|^2 + \left\|F_1^{-1/2} W + F_1^{1/2} Z + d_2 F_1^{1/2} Y\right\|^2 \\ &+ \left\|\Gamma^{-1/2} H(X) + \Gamma^{1/2} Y + d_1 \Gamma^{1/2} Z\right\|^2 . \end{aligned} \quad (31)$$

The matrices F_1 and Γ are symmetric because F and J_G are symmetric. The eigenvalues of F_1 and Γ are positive because of (22) and (24).

Consequently the square roots $F_1^{1/2}$ and $\Gamma^{1/2}$ exist; these are again symmetric and nonsingular for all $Y, Z \in R^n$.

Therefore we get

$$\begin{aligned} 2V &\geq 2d_2 \int_0^1 \langle H(\sigma X), X\rangle d\sigma - \left\langle \Gamma^{-1} H(X), H(X)\right\rangle \\ &+ d_2 \langle \alpha_2 Y, Y\rangle - d_1 \langle \alpha_4 Y, Y\rangle - d_2^2 \langle F_1 Y, Y\rangle \\ &+ 2\int_0^1 \langle G(\sigma Y), Y\rangle d\sigma - \langle \Gamma Y, Y\rangle + (\alpha_2 d_1 - d_2)\|Z\|^2 \\ &- d_1^2 \langle \Gamma Z, Z\rangle + 2\int_0^1 \langle \sigma F(Y, \sigma Z) Z, Z\rangle d\sigma - \langle F_1 Z, Z\rangle \\ &+ d_1 \|W\|^2 - \left\langle F_1^{-1} W, W\right\rangle . \end{aligned} \quad (32)$$

From $\lambda_i(F_1^{-1}) \leq 1/\alpha_1$ and $\lambda_i(\Gamma^{-1}) \leq {\alpha_4'}^2/\alpha_3\alpha_4^2$, because of (22) and (24), we get from Lemma 3 and Cauchy-Schwartz inequality that

$$\begin{aligned} 2V &\geq 2d_2 \int_0^1 \langle H(\sigma X), X\rangle d\sigma - \left\langle \Gamma^{-1} H(X), H(X)\right\rangle \\ &+ 2\int_0^1 \langle G(\sigma Y), Y\rangle d\sigma - \langle \Gamma Y, Y\rangle \\ &+ \left(\alpha_2 d_2 - \alpha_4 d_1 - d_2^2 \|F_1\|\right)\|Y\|^2 \\ &+ \left(\alpha_2 d_1 - d_2 - d_1^2 \|\Gamma\|\right)\|Z\|^2 \\ &+ 2\int_0^1 \langle \sigma F(Y, \sigma Z) Z, Z\rangle d\sigma - \langle F_1 Z, Z\rangle \\ &+ \left(d_1 - \frac{1}{\alpha_1}\right)\|W\|^2 . \end{aligned} \quad (33)$$

From the definitions of d_1, d_2 in (11), it follows that

$$2V(X_t, Y_t, Z_t, W_t) \geq V_1 + V_2 + V_3 + \varepsilon \|W\|^2 , \quad (34)$$

where

$$\begin{aligned} V_1 &:= 2d_2 \int_0^1 \langle H(\sigma X), X\rangle d\sigma - \left\langle \Gamma^{-1} H(X), H(X)\right\rangle , \\ V_2 &:= \left(\alpha_2 d_2 - \alpha_4 d_1 - d_2^2 \|F_1\|\right)\|Y\|^2 \\ &+ 2\int_0^1 \langle G(\sigma Y), Y\rangle d\sigma - \langle \Gamma Y, Y\rangle , \\ V_3 &:= \left(\alpha_2 d_1 - d_2 - d_1^2 \|\Gamma\|\right)\|Z\|^2 \\ &+ 2\int_0^1 \langle \sigma F(Y, \sigma Z) Z, Z\rangle d\sigma - \langle F_1 Z, Z\rangle . \end{aligned} \quad (35)$$

Since

$$\frac{\partial}{\partial\sigma_1}\langle H(\sigma_1 X), H(\sigma_1 X)\rangle = 2\langle J_H(\sigma_1 X) X, H(\sigma_1 X)\rangle, \tag{36}$$

by integrating both sides from $\sigma_1 = 0$ to $\sigma_1 = 1$ and because of $H(0) = 0$, we obtain

$$\langle H(X), H(X)\rangle = 2\int_0^1 \langle J_H(\sigma_1 X) X, H(\sigma_1 X)\rangle\, d\sigma_1. \tag{37}$$

Thus

$$\begin{aligned} V_1 &= 2d_2\int_0^1 \langle H(\sigma X), X\rangle\, d\sigma \\ &\quad - 2\Gamma^{-1}\int_0^1 \langle J_H(\sigma_1 X) X, H(\sigma_1 X)\rangle\, d\sigma_1 \\ &= 2\int_0^1 \left\langle H(\sigma_1 X), \left\{d_2 I - \Gamma^{-1} J_H(\sigma_1 X)\right\} X\right\rangle d\sigma_1. \end{aligned} \tag{38}$$

But from

$$\begin{aligned} &\frac{\partial}{\partial\sigma_2}\left\langle H(\sigma_1\sigma_2 X), \left\{d_2 I - \Gamma^{-1} J_H(\sigma_1 X)\right\} X\right\rangle \\ &\quad = \left\langle \sigma_1 J_H(\sigma_1\sigma_2 X) X, \left\{d_2 I - \Gamma^{-1} J_H(\sigma_1 X)\right\} X\right\rangle, \end{aligned} \tag{39}$$

by integrating both sides from $\sigma_2 = 0$ to $\sigma_2 = 1$ and because of $H(0) = 0$, we find

$$\begin{aligned} &\left\langle H(\sigma_1\sigma_2 X), \left\{d_2 I - \Gamma^{-1} J_H(\sigma_1 X)\right\} X\right\rangle \\ &\quad = \int_0^1 \sigma_1 \left\langle J_H(\sigma_1\sigma_2 X) X, \left\{d_2 I - \Gamma^{-1} J_H(\sigma_1 X)\right\} X\right\rangle d\sigma_2. \end{aligned} \tag{40}$$

Therefore by using (11), (24), (vii), (viii), and Lemma 3, we have

$$\begin{aligned} V_1 &= 2\int_0^1\int_0^1 \sigma_1 \left\langle J_H(\sigma_1\sigma_2 X) X, \right. \\ &\qquad \left. \left\{d_2 I - \Gamma^{-1} J_H(\sigma_1 X)\right\} X\right\rangle d\sigma_2\, d\sigma_1 \\ &= 2\int_0^1\int_0^1 \sigma_1 \left\langle J_H(\sigma_1\sigma_2 X)\left\{d_2 I - \Gamma^{-1} J_H(\sigma_1 X)\right\} X, \right. \\ &\qquad \left. X\right\rangle d\sigma_2\, d\sigma_1 \\ &\geq 2\varepsilon\int_0^1\int_0^1 \langle J_H(\sigma_1\sigma_2 X)\sigma_1 X, X\rangle\, d\sigma_2\, d\sigma_1 \\ &\quad + \frac{2\alpha_4'^2}{\alpha_3\alpha_4^2}\int_0^1\int_0^1 \sigma_1 \langle J_H(\sigma_1\sigma_2 X) X, \\ &\qquad \{\alpha_4 I - J_H(\sigma_1 X)\} X\rangle\, d\sigma_2\, d\sigma_1 \\ &\geq 2\varepsilon\int_0^1\left[\int_0^1 \left\langle J_H\left(\sigma_2\widetilde{X}\right)\widetilde{X}, X\right\rangle d\sigma_2\right] d\sigma_1 \\ &\geq 2\varepsilon\int_0^1 \alpha_4' \left\langle \widetilde{X}, X\right\rangle d\sigma_1 = 2\varepsilon\int_0^1 \alpha_4' \langle X, X\rangle\, \sigma_1 d\sigma_1 \\ &= \varepsilon\alpha_4' \langle X, X\rangle = \varepsilon\alpha_4' \|X\|^2. \end{aligned} \tag{41}$$

To estimate V_2 we need

$$\begin{aligned} &\alpha_2 d_2 - \alpha_4 d_1 - d_2^2\|F_1\| \\ &\quad = d_2\{\alpha_2 - d_1\|J_G(Y)\| - d_2\|F_1\|\} \\ &\qquad + d_1\{d_2\|J_G(Y)\| - \alpha_4\} \\ &\quad \geq d_2\{\alpha_2 - d_1\|J_G(Y)\| - d_2\|F_1\|\}, \end{aligned} \tag{42}$$

since from (11) and (ii) we find that

$$d_2\|J_G(Y)\| - \alpha_4 > \left(\varepsilon + \frac{\alpha_4'^2}{\alpha_3\alpha_4}\right)\frac{\alpha_3\alpha_4^2}{\alpha_4'^2} - \alpha_4 = \varepsilon\frac{\alpha_3\alpha_4^2}{\alpha_4'^2} > 0. \tag{43}$$

Now

$$\begin{aligned} &\alpha_2 - d_1\|J_G(Y)\| - d_2\|F_1\| \\ &\quad = \alpha_2 - \frac{1}{\alpha_1}\|J_G(Y)\| - \frac{\alpha_4'^2}{\alpha_3\alpha_4}\|F_1\| \\ &\qquad - \varepsilon\{\|J_G(Y)\| + \|F_1\|\} \\ &\quad = \frac{1}{\alpha_1\alpha_3\alpha_4}\left[\alpha_3\alpha_4\{\alpha_1\alpha_2 - \|J_G(Y)\|\} - \alpha_1\alpha_4'^2\|F_1\|\right] \\ &\qquad - \varepsilon\{\|J_G(Y)\| + \|F_1\|\} \\ &\quad \geq \frac{\Delta}{\alpha_1\alpha_3\alpha_4} - \varepsilon\left(\alpha_1\alpha_2 + \alpha_2\alpha_3\alpha_4\alpha_4'^{-2}\right), \quad \text{from (iii)}. \end{aligned} \tag{44}$$

Thus we obtain from (viii)

$$\alpha_2 - d_1\|J_G(Y)\| - d_2\|F_1\| \geq \frac{\Delta}{\alpha_1\alpha_3\alpha_4} - \varepsilon D_0. \tag{45}$$

From the identity

$$\int_0^1 \sigma\langle J_G(\sigma Y) Y, Y\rangle\, d\sigma \equiv \langle G(Y), Y\rangle - \int_0^1 \langle G(\sigma Y), Y\rangle\, d\sigma, \tag{46}$$

we get from (25) and by Lemma 3

$$\begin{aligned} 2\int_0^1 \langle G(\sigma Y), Y\rangle\, d\sigma - \langle G(Y), Y\rangle &= \int_0^1 \langle G(\sigma Y), Y\rangle\, d\sigma - \int_0^1 \sigma \left\langle J_G(\sigma Y) Y, Y\right\rangle d\sigma \\ &= \int_0^1 \sigma \langle \Gamma(\sigma Y) Y, Y\rangle\, d\sigma - \int_0^1 \sigma \left\langle J_G(\sigma Y) Y, Y\right\rangle d\sigma \\ &= -\int_0^1 \sigma \left\langle \{J_G(\sigma Y) - \Gamma(\sigma Y)\} Y, Y\right\rangle d\sigma \\ &\geq -\frac{1}{2}\delta_1 \|Y\|^2 . \end{aligned} \tag{47}$$

So we have from (9) and (11)

$$\begin{aligned} V_2 &\geq d_2\left(\frac{\Delta}{\alpha_1\alpha_3\alpha_4} - \varepsilon D_0\right)\|Y\|^2 - \frac{1}{2}\delta_1\|Y\|^2 \\ &\geq \left\{\frac{{\alpha_4'}^2}{\alpha_3\alpha_4}\left(\frac{\Delta}{\alpha_1\alpha_3\alpha_4} - \varepsilon D_0\right) - \frac{1}{2}\delta_1\right\}\|Y\|^2 \\ &\geq \frac{1}{4}\left(\frac{2{\alpha_4'}^2\Delta}{\alpha_1\alpha_3^2\alpha_4^2} - \delta_1\right)\|Y\|^2 , \end{aligned} \tag{48}$$

since $\varepsilon < (\alpha_3\alpha_4/4{\alpha_4'}^2 D_0)(2{\alpha_4'}^2\Delta/\alpha_1\alpha_3^2\alpha_4^2 - \delta_1)$.

To estimate V_3 we need

$$\begin{aligned} \alpha_2 d_1 - d_2 - d_1^2\|\Gamma\| &= d_1\{\alpha_2 - d_1\|\Gamma\| - d_2\|F_1\|\} \\ &\quad + d_2\{d_1\|F_1\| - 1\} \\ &\geq d_1\{\alpha_2 - d_1\|\Gamma\| - d_2\|F_1\|\} \\ &\geq d_1\{\alpha_2 - d_1\|J_G(Y)\| - d_2\|F_1\|\} \\ &\geq \frac{1}{\alpha_1}\left(\frac{\Delta}{\alpha_1\alpha_3\alpha_4} - \varepsilon D_0\right), \end{aligned} \tag{49}$$

by (11), (25), and (45). So from the identity

$$\int_0^1 \sigma\langle F(Y,\sigma Z) Z, Z\rangle\, d\sigma \equiv \int_0^1 \langle F(Y,\sigma Z) Z, Z\rangle\, d\sigma - \int_0^1 \sigma\left\langle F_1(Y,\sigma Z) Z, Z\right\rangle d\sigma, \tag{50}$$

we find

$$\begin{aligned} 2\int_0^1 \sigma\langle F(Y,\sigma Z) Z, Z\rangle\, d\sigma - \left\langle F_1 Z, Z\right\rangle &= \int_0^1 \sigma\langle F(Y,\sigma Z) Z, Z\rangle\, d\sigma - \int_0^1 \sigma\left\langle F_1(Y,\sigma Z) Z, Z\right\rangle d\sigma \\ &= -\int_0^1 \sigma\left\langle \{F(Y,\sigma Z) - F_1(Y,\sigma Z)\} Z, Z\right\rangle d\sigma \\ &\geq -\frac{1}{2}\delta_2\|Z\|^2, \quad \text{by (v)}. \end{aligned} \tag{51}$$

Thus from (9), we obtain

$$\begin{aligned} V_3 &\geq \left\{\frac{1}{\alpha_1}\left(\frac{\Delta}{\alpha_1\alpha_3\alpha_4} - \varepsilon D_0\right) - \frac{1}{2}\delta_2\right\}\|Z\|^2 \\ &\geq \frac{1}{4}\left(\frac{2\Delta}{\alpha_1^2\alpha_3\alpha_4} - \delta_2\right)\|Z\|^2 , \end{aligned} \tag{52}$$

since $\varepsilon < (\alpha_1/4D_0)(2\Delta/\alpha_1^2\alpha_3\alpha_4 - \delta_2)$. Then it follows that

$$\begin{aligned} 2V(X_t, Y_t, Z_t, W_t) &\geq \varepsilon\alpha_4'\|X\|^2 + \frac{1}{4}\left(\frac{2{\alpha_4'}^2\Delta}{\alpha_1\alpha_3^2\alpha_4^2} - \delta_1\right)\|Y\|^2 \\ &\quad + \frac{1}{4}\left(\frac{2\Delta}{\alpha_1^2\alpha_3\alpha_4} - \delta_2\right)\|Z\|^2 + \varepsilon\|W\|^2 . \end{aligned} \tag{53}$$

Since the coefficients are positive constants from the definitions of δ_1, δ_2, and ε in (iv), (v), and (9), then there exists a positive constant D_1 such that

$$V(X_t, Y_t, Z_t, W_t) \geq D_1\left(\|X\|^2 + \|Y\|^2 + \|Z\|^2 + \|W\|^2\right). \tag{54}$$

To prove that

$$V(X_t, Y_t, Z_t, W_t) \leq D_2\left(\|X\|^2 + \|Y\|^2 + \|Z\|^2 + \|W\|^2\right), \tag{55}$$

by using the hypotheses of Theorem 2 we find

$$\|F_1\| \leq \sqrt{n}\alpha_2\alpha_3\alpha_4{\alpha_4'}^{-2}, \quad \text{by (iii)}. \tag{56}$$

Since

$$\frac{\partial\Phi(\sigma Z)}{\partial\sigma} = J_\Phi(\sigma Z) Z, \qquad \Phi(0) = 0, \tag{57}$$

then from (ix) we have

$$\begin{aligned} \|\Phi(Z)\| &= \left\|\int_0^1 J_\Phi(\sigma Z) Z d\sigma\right\| \leq \int_0^1 \|J_\Phi(\sigma Z)\|\|Z\|\, d\sigma \\ &\leq \sqrt{n}\left(\alpha_2 + \frac{\varepsilon\alpha_3^3\alpha_4^2}{{\alpha_4'}^4}\right)\|Z\| , \end{aligned} \tag{58}$$

and also since

$$\frac{\partial G(\sigma Y)}{\partial \sigma} = J_G(\sigma Y) Y, \qquad G(0) = 0, \tag{59}$$

then from (iv) we have

$$\begin{aligned}\|G(Y)\| &= \left\| \int_0^1 J_G(\sigma Y) Y d\sigma \right\| \le \int_0^1 \|J_G(\sigma Y)\| \|Y\| d\sigma \\ &\le \alpha_1 \alpha_2 \sqrt{n} \|Y\|.\end{aligned} \tag{60}$$

Since

$$\frac{\partial H(\sigma X)}{\partial \sigma} = J_H(\sigma X) X, \qquad H(0) = 0, \tag{61}$$

then from (viii) we get

$$\begin{aligned}\|H(X)\| &= \left\| \int_0^1 J_H(\sigma X) X d\sigma \right\| \le \int_0^1 \|J_H(\sigma X)\| \|X\| d\sigma \\ &\le \alpha_4 \sqrt{n} \|X\|.\end{aligned} \tag{62}$$

By using Cauchy-Schwartz inequality $|\langle u, v\rangle| \le (1/2)(\|u\|^2 + \|v\|^2)$ and from

$$\begin{aligned}
&2\mu \int_{-r}^{0} \int_{t+s}^{t} \|Y(\theta)\|^2 d\theta\, ds \\
&\quad = 2\mu \int_{t-r}^{t} (\theta - t + r) \|Y(\theta)\|^2 d\theta \\
&\quad \le 2\mu \|Y\|^2 \int_{t-r}^{t} (\theta - t + r) d\theta \\
&\quad = \mu r^2 \|Y\|^2, \\
&2\lambda \int_{-r}^{0} \int_{t+s}^{t} \|Z(\theta)\|^2 d\theta\, ds \\
&\quad = 2\lambda \int_{t-r}^{t} (\theta - t + r) \|Z(\theta)\|^2 d\theta \\
&\quad \le 2\lambda \|Z\|^2 \int_{t-r}^{t} (\theta - t + r) d\theta \\
&\quad = \lambda r^2 \|Z\|^2.
\end{aligned} \tag{63}$$

Hence there exists a positive constant D_2 satisfying

$$V(X_t, Y_t, Z_t, W_t) \le D_2 \left(\|X\|^2 + \|Y\|^2 + \|Z\|^2 + \|W\|^2 \right). \tag{64}$$

Now from (19), (20), and Lemma 4, we have

$$\begin{aligned}
\frac{dV}{dt} &= d_2 \langle H(X), Y\rangle + d_2 \langle \alpha_2 Y, Z\rangle - d_1 \langle \alpha_4 Y, Z\rangle \\
&\quad + \langle G(Y), Z\rangle + d_1 \langle \Phi(Z), W\rangle \\
&\quad - d_2 \langle Z, W\rangle + \langle F(Y,Z) Z, W\rangle \\
&\quad + d_1 \Big\langle W, -F(Y,Z) W - \Phi(Z) - G(Y) - H(X) \\
&\qquad + \int_{t-r}^{t} J_G(Y(s)) Z(s) ds \\
&\qquad + \int_{t-r}^{t} J_H(X(s)) Y(s) ds \Big\rangle \\
&\quad + \langle J_H(X) Y, Y\rangle + \langle H(X), Z\rangle + d_1 \langle J_H(X) Y, Z\rangle \\
&\quad + d_1 \langle H(X), W\rangle + d_2 \langle F(Y,Z) Y, W\rangle \\
&\quad + d_2 \|F_1\| \langle Z, Z\rangle \\
&\quad + d_1 \langle W, G(Y)\rangle + d_1 \langle J_G(Y) Z, Z\rangle \\
&\quad + d_2 \Big\langle Y, -F(Y,Z) W \\
&\qquad - \Phi(Z) - G(Y) - H(X) \\
&\qquad + \int_{t-r}^{t} J_G(Y(s)) Z(s) ds \\
&\qquad + \int_{t-r}^{t} J_H(X(s)) Y(s) ds \Big\rangle + \langle W, W\rangle \\
&\quad + \Big\langle Z, -F(Y,Z) W \\
&\qquad - \Phi(Z) - G(Y) - H(X) + \int_{t-r}^{t} J_G(Y(s)) Z(s) ds \\
&\qquad + \int_{t-r}^{t} J_H(X(s)) Y(s) ds \Big\rangle + d_2 \langle Z, W\rangle + \mu r \|Y\|^2 \\
&\quad - \mu \int_{t-r}^{t} \|Y(\theta)\|^2 d\theta + \lambda r \|Z\|^2 - \lambda \int_{t-r}^{t} \|Z(\theta)\|^2 d\theta.
\end{aligned} \tag{65}$$

Then we get

$$\begin{aligned}
\frac{dV}{dt} &= d_2 \langle \alpha_2 Y, Z\rangle - d_1 \langle \alpha_4 Y, Z\rangle - d_1 \langle W, F(Y,Z) W\rangle \\
&\quad - d_2 \langle Y, \Phi(Z)\rangle + \langle J_H(X) Y, Y\rangle \\
&\quad + d_2 \|F_1\| \langle Z, Z\rangle + d_1 \langle J_G(Y) Z, Z\rangle + \langle W, W\rangle \\
&\quad + d_1 \langle J_H(X) Y, Z\rangle - \langle Z, \Phi(Z)\rangle - d_2 \langle Y, G(Y)\rangle \\
&\quad + \Big\langle d_1 W + Z + d_2 Y, \int_{t-r}^{t} J_H(X(s)) Y(s) ds \Big\rangle
\end{aligned}$$

$$+\left\langle d_1W+Z+d_2Y,\int_{t-r}^{t}J_G(Y(s))Z(s)\,ds\right\rangle$$
$$+\mu r\|Y\|^2-\mu\int_{t-r}^{t}\|Y(\theta)\|^2\,d\theta+\lambda r\|Z\|^2$$
$$-\lambda\int_{t-r}^{t}\|Z(\theta)\|^2\,d\theta, \tag{66}$$

and it follows that

$$\begin{aligned}\frac{dV}{dt}&=\langle\alpha_4Y,Y\rangle-d_2\langle Y,G(Y)\rangle\\&\quad-\langle\alpha_2Z,Z\rangle+d_1\langle J_G(Y)Z,Z\rangle\\&\quad+d_2\|F_1\|\langle Z,Z\rangle-d_1\langle W,F(Y,Z)W\rangle+\langle W,W\rangle\\&\quad+\left\langle d_1W+Z+d_2Y,\int_{t-r}^{t}J_H(X(s))Y(s)\,ds\right\rangle\\&\quad+\left\langle d_1W+Z+d_2Y,\int_{t-r}^{t}J_G(Y(s))Z(s)\,ds\right\rangle\\&\quad+\mu r\|Y\|^2-\mu\int_{t-r}^{t}\|Y(\theta)\|^2\,d\theta+\lambda r\|Z\|^2\\&\quad-\lambda\int_{t-r}^{t}\|Z(\theta)\|^2\,d\theta+V_4+V_5,\end{aligned} \tag{67}$$

where

$$\begin{aligned}V_4&:=d_2\langle\alpha_2Z,Y\rangle-d_2\langle Y,\Phi(Z)\rangle\\&\quad-\langle Z,\Phi(Z)\rangle+\langle\alpha_2Z,Z\rangle,\\V_5&:=-d_1\langle\alpha_4Z,Y\rangle+d_1\langle J_H(X)Y,Z\rangle\\&\quad+\langle J_H(X)Y,Y\rangle-\langle\alpha_4Y,Y\rangle.\end{aligned} \tag{68}$$

But

$$\begin{aligned}V_4&=-\int_0^1\left[\langle J_\Phi(\sigma Z)Z,Z\rangle-\langle\alpha_2Z,Z\rangle\right.\\&\quad\left.+d_2\{\langle J_\Phi(\sigma Z)Z,Y\rangle-\langle\alpha_2Z,Y\rangle\}\right]d\sigma\\&=-\int_0^1\langle\{J_\Phi(\sigma Z)-\alpha_2I\}Z,Z\rangle\,d\sigma\\&\quad-d_2\int_0^1\langle\{J_\Phi(\sigma Z)-\alpha_2I\}Z,Y\rangle\,d\sigma.\end{aligned} \tag{69}$$

Since $\lambda_i(\int_0^1 J_\Phi(\sigma Z)d\sigma-\alpha_2I)$ is nonnegative by (ix), then from (11) we get

$$\begin{aligned}V_4&\le\frac{d_2^2}{4}\int_0^1\langle\{J_\Phi(\sigma Z)-\alpha_2I\}Y,Y\rangle\,d\sigma\\&\le\frac14\left(\varepsilon+\frac{\alpha_4'^2}{\alpha_3\alpha_4}\right)^2\varepsilon_0\alpha_3^3\alpha_4^2\alpha_4'^{-4}\langle Y,Y\rangle\\&=\frac14\left(\varepsilon\alpha_3\alpha_4\alpha_4'^{-2}+1\right)^2\varepsilon_0\alpha_3\|Y\|^2\\&\le\varepsilon_0\alpha_3\|Y\|^2,\end{aligned} \tag{70}$$

since $\varepsilon<\alpha_4'^2/\alpha_3\alpha_4$ by (9). Also

$$\begin{aligned}V_5&=-\{\langle\alpha_4Y,Y\rangle-\langle J_H(X)Y,Y\rangle+d_1\langle\alpha_4Z,Y\rangle\\&\quad-d_1\langle J_H(X)Y,Z\rangle\}\\&=-\langle\{\alpha_4I-J_H(X)\}Y,Y\rangle-d_1\langle\{\alpha_4I-J_H(X)\}Y,Z\rangle.\end{aligned} \tag{71}$$

But $\lambda_i(\alpha_4I-J_H(X))$ is nonnegative by (viii) and from (11), we get

$$\begin{aligned}V_4&\le\frac{d_1^2}{4}\langle\{\alpha_4I-J_H(X)\}Z,Z\rangle\\&\le\frac14\left(\varepsilon+\frac{1}{\alpha_1}\right)^2\varepsilon D_0\alpha_1^2\langle Z,Z\rangle\\&=\frac14(\varepsilon\alpha_1+1)^2\varepsilon D_0\|Z\|^2\\&\le\varepsilon D_0\|Z\|^2,\quad\text{since }\varepsilon<\frac{1}{\alpha_1}.\end{aligned} \tag{72}$$

Therefore

$$\begin{aligned}\dot V&\le-\{d_2\langle Y,G(Y)\rangle-\langle\alpha_4Y,Y\rangle\}+\varepsilon_0\alpha_3\|Y\|^2\\&\quad-\left(\langle\alpha_2Z,Z\rangle-d_2\|F_1\|\|Z\|^2-d_1\|J_G\|\|Z\|^2\right)\\&\quad-\{d_1\langle W,F(Y,Z)W\rangle-\langle W,W\rangle\}+\varepsilon D_0\|Z\|^2\\&\quad+\left\langle d_1W+Z+d_2Y,\int_{t-r}^{t}J_H(X(s))Y(s)\,ds\right\rangle\\&\quad+\left\langle d_1W+Z+d_2Y,\int_{t-r}^{t}J_G(Y(s))Z(s)\,ds\right\rangle\\&\quad+\mu r\|Y\|^2-\mu\int_{t-r}^{t}\|Y(\theta)\|^2\,d\theta\\&\quad+\lambda r\|Z\|^2-\lambda\int_{t-r}^{t}\|Z(\theta)\|^2\,d\theta.\end{aligned} \tag{73}$$

We know that $\langle Y, G(Y)\rangle = \langle Y, \Gamma(Y)Y\rangle$ and by Lemma 3, we get

$$\begin{aligned}\frac{dV}{dt} \le & -\left(d_2\frac{\alpha_3\alpha_4^2}{\alpha_4'^2} - \alpha_4\right)\|Y\|^2 + \varepsilon_0\alpha_3\|Y\|^2 \\ & - (\alpha_2 - d_1\|J_G\| - d_2\|F_1\|)\|Z\|^2 + \varepsilon D_0\|Z\|^2 \\ & - \{\alpha_1 d_1 - 1\}\|W\|^2 \\ & + \left\langle d_1W + Z + d_2Y, \int_{t-r}^{t} J_H(X(s))Y(s)\,ds\right\rangle \\ & + \left\langle d_1W + Z + d_2Y, \int_{t-r}^{t} J_G(Y(s))Z(s)\,ds\right\rangle \\ & + \mu r\|Y\|^2 - \mu\int_{t-r}^{t}\|Y(\theta)\|^2 d\theta + \lambda r\|Z\|^2 \\ & - \lambda\int_{t-r}^{t}\|Z(\theta)\|^2 d\theta.\end{aligned} \tag{74}$$

Since $\|J_H(X)\| \le \alpha_4\sqrt{n}$ by (viii) and by using Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}&\left|\left\langle d_1W + Z + d_2Y, \int_{t-r}^{t} J_H(X(s))Y(s)\,ds\right\rangle\right| \\ &\quad \le \|d_1W + Z + d_2Y\|\left\|\int_{t-r}^{t} J_H(X(s))Y(s)\,ds\right\| \\ &\quad \le (d_1\|W\| + \|Z\| + d_2\|Y\|)\int_{t-r}^{t}\alpha_4\sqrt{n}\|Y(s)\|\,ds \\ &\quad \le \frac{d_1\alpha_4\sqrt{n}}{2}\left(\|W\|^2 r + \int_{t-r}^{t}\|Y(s)\|^2 ds\right) \\ &\quad + \frac{\alpha_4\sqrt{n}}{2}\left(\|Z\|^2 r + \int_{t-r}^{t}\|Y(s)\|^2 ds\right) \\ &\quad + \frac{d_2\alpha_4\sqrt{n}}{2}\left(\|Y\|^2 r + \int_{t-r}^{t}\|Y(s)\|^2 ds\right).\end{aligned} \tag{75}$$

Also, since $\|J_G(Y)\| \le \alpha_1\alpha_2\sqrt{n}$ by (iii) and by using Cauchy-Schwartz inequality, we find

$$\begin{aligned}&\left|\left\langle d_1W + Z + d_2Y, \int_{t-r}^{t} J_G(Y(s))Z(s)\,ds\right\rangle\right| \\ &\quad \le \|d_1W + Z + d_2Y\|\left\|\int_{t-r}^{t} J_G(Y(s))Z(s)\,ds\right\| \\ &\quad \le (d_1\|W\| + \|Z\| + d_2\|Y\|)\int_{t-r}^{t}\alpha_1\alpha_2\sqrt{n}\|Z(s)\|\,ds \\ &\quad \le \frac{d_1\alpha_1\alpha_2\sqrt{n}}{2}\left(\|W\|^2 r + \int_{t-r}^{t}\|Z(s)\|^2 ds\right) \\ &\quad + \frac{\alpha_1\alpha_2\sqrt{n}}{2}\left(\|Z\|^2 r + \int_{t-r}^{t}\|Z(s)\|^2 ds\right) \\ &\quad + \frac{d_2\alpha_1\alpha_2\sqrt{n}}{2}\left(\|Y\|^2 r + \int_{t-r}^{t}\|Z(s)\|^2 ds\right).\end{aligned} \tag{76}$$

Therefore it follows from (11) and (45) that

$$\begin{aligned}\frac{dV}{dt} \le & -\left\{\left(\frac{\alpha_4^2}{\alpha_4'^2}\varepsilon - \varepsilon_0\right)\alpha_3 - \frac{d_2\alpha_4\sqrt{n}}{2}r - \frac{d_2\alpha_1\alpha_2\sqrt{n}}{2}r - \mu r\right\}\|Y\|^2 \\ & - \left(\frac{\Delta}{2\alpha_1\alpha_3\alpha_4} - \frac{\alpha_4\sqrt{n}}{2}r - \frac{\alpha_1\alpha_2\sqrt{n}}{2}r - \lambda r\right)\|Z\|^2 \\ & - \left(\varepsilon - \frac{d_1\alpha_4\sqrt{n}}{2}r - \frac{d_1\alpha_1\alpha_2\sqrt{n}}{2}r\right)\|W\|^2 \\ & + \left(\frac{d_1\alpha_4\sqrt{n}}{2} + \frac{d_2\alpha_4\sqrt{n}}{2} + \frac{\alpha_4\sqrt{n}}{2} - \mu\right)\int_{t-r}^{t}\|Y(s)\|^2 ds \\ & + \left(\frac{d_1\alpha_1\alpha_2\sqrt{n}}{2} + \frac{d_2\alpha_1\alpha_2\sqrt{n}}{2} + \frac{\alpha_1\alpha_2\sqrt{n}}{2} - \lambda\right)\int_{t-r}^{t}\|Z(s)\|^2 ds,\end{aligned} \tag{77}$$

and if we take

$$\mu = \frac{\alpha_4\sqrt{n}}{2}(d_1 + d_2 + 1), \qquad \lambda = \frac{\alpha_1\alpha_2\sqrt{n}}{2}(d_1 + d_2 + 1), \tag{78}$$

then we have

$$\begin{aligned}\frac{dV}{dt} \le & -\left\{\left(\frac{\alpha_4^2}{\alpha_4'^2}\varepsilon - \varepsilon_0\right)\alpha_3 - \frac{d_2\alpha_4\sqrt{n}}{2}r - \frac{d_2\alpha_1\alpha_2\sqrt{n}}{2}r - \frac{\alpha_4\sqrt{n}}{2}(d_1 + d_2 + 1)r\right\}\|Y\|^2 \\ & - \left\{\frac{\Delta}{2\alpha_1\alpha_3\alpha_4} - \frac{\alpha_4\sqrt{n}}{2}r - \frac{\alpha_1\alpha_2\sqrt{n}}{2}r - \frac{\alpha_1\alpha_2\sqrt{n}}{2}(d_1 + d_2 + 1)r\right\}\|Z\|^2 \\ & - \left(\varepsilon - \frac{d_1\alpha_4\sqrt{n}}{2}r - \frac{d_1\alpha_1\alpha_2\sqrt{n}}{2}r\right)\|W\|^2.\end{aligned} \tag{79}$$

Therefore if

$$r < \min\left[\frac{\varepsilon}{d_1\sqrt{n}(\alpha_4 + \alpha_1\alpha_2)}, \frac{\Delta}{2\alpha_1\alpha_3\alpha_4\sqrt{n}\{\alpha_4 + \alpha_1\alpha_2(d_1 + d_2 + 2)\}}, \frac{\left(\left(\alpha_4^2/\alpha_4'^2\right)\varepsilon - \varepsilon_0\right)\alpha_3}{\alpha_4\sqrt{n}(d_1 + 2d_2 + 1) + \alpha_1\alpha_2 d_2\sqrt{n}}\right], \tag{80}$$

we obtain

$$\frac{dV}{dt}\left(X_t, Y_t, Z_t, W_t\right) \leq -\alpha\left(\|Y\|^2 + \|Z\|^2 + \|W\|^2\right), \quad (81)$$

for some $\alpha > 0$. Therefore from (54), (64), and (81) the functional $V(X_t, Y_t, Z_t, W_t)$ satisfies all the conditions of Theorem 1, so that the zero solution of (4) is uniformly stable.

Thus the proof of Theorem 2 is now complete.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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