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Research Article

Infinitely Many Eigenfunctions for Polynomial Problems: Exact Results

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Let $F(x, y) = a_s(x)y^s + a_{s-1}(x)y^{s-1} + \cdots + a_0(x)$ be a real-valued polynomial function in which the degree s of y in $F(x, y)$ is greater than or equal to 1. For any polynomial $y(x)$, we assume that $T : \mathbb{R}[x] \to \mathbb{R}[x]$ is a nonlinear operator with $T(y(x)) = F(x, y(x))$. In this paper, we will find an eigenfunction $y(x) \in \mathbb{R}[x]$ to satisfy the following equation: $F(x, y(x)) = ay(x)$ for some eigenvalue $a \in \mathbb{R}$ and we call the problem $F(x, y(x)) = ay(x)$ a fixed point like problem. If the number of all eigenfunctions in $F(x, y(x)) = ay(x)$ is infinitely many, we prove that (i) any coefficients of $F(x, y), a_s(x), a_{s-1}(x), \ldots, a_0(x)$, are all constants in $\mathbb{R}$ and (ii) $y(x)$ is an eigenfunction in $F(x, y(x)) = ay(x)$ if and only if $y(x) \in \mathbb{R}$.

1. Introduction and Preliminaries

Lenstra [1] investigated that

$$F(x, y(x)) = 0 \tag{1}$$

in which $F(x, y)$ is a polynomial function over the algebraic rational number field $\mathbb{Q}(\alpha)$ (where α is an algebraic number). He found a polynomial $y = y(x) \in \mathbb{Q}(\alpha)[x]$ satisfying the polynomial equation

$$F(x, y(x)) = x. \tag{2}$$

Further, Tung [2] extended (2) to solve polynomial solutions (near solutions) $y(x) \in \mathbb{K}[x]$ ($\mathbb{K}$ is a field) for the following equation:

$$F(x, y(x)) = ax^m, \tag{3}$$

where $a \in \mathbb{K}$ is a constant depending on the polynomial solution $y(x)$ and $m \in \mathbb{N}$ a given nonnegative integer.

Moreover, Lai and Chen [3–5] extended (3) to solve $y(x) \in \mathbb{R}[x]$ satisfying the polynomial equation as the form

$$F(x, y(x)) = ap^m(x), \quad x \in \mathbb{R}, \tag{4}$$

where $a \in \mathbb{R}$, $m \in \mathbb{N}$, $p(\cdot)$ is an irreducible polynomial in $x \in \mathbb{R}$, and the polynomial function $F(x, y) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is written by

$$F(x, y) = \sum_{i=0}^{s} a_i(x)\, y^i \quad \text{with } s \geq 1, \tag{5}$$

where s denotes the degree $\deg_y F$ of y in $F(x, y)$.

Recently, Chen and Lai [6, 7] research a quasicoincidence problem in which an arbitrary nonzero polynomial function $f(x) \in \mathbb{R}[x]$ is given as follows:

$$F(x, y(x)) = af(x), \tag{6}$$

where a is a constant.

Definition 1 (Chen and Lai, [6]). A polynomial function $y = y(x)$ satisfying (6) is called a quasicoincidence solution corresponding to some real number a. This number a is called a quasicoincidence value corresponding to the polynomial solutions $y = y(x)$.

In this paper, we define a fixed point like problem in which the $y(x) \in \mathbb{R}[x]$ is replaced by the arbitrary polynomial

$f(x) \in \mathbb{R}[x]$ throughout this paper. Then we restate (6) as the following equation:

$$F(x, y(x)) = ay(x). \tag{7}$$

It is a new developed fixed point like problem. We call the polynomial (7) as a *fixed point like* problem. The number of all eigenfunctions in (7) may be infinitely many, or finitely many, or not solvable.

Since there may exist many eigenfunctions corresponding to the eigenvalue a, for convenience, we use the following notations to represent different situations:

(1) $\mathbf{E}_{\text{function}}$, the set of all eigenfunctions "$y(x)$" satisfying (7);

(2) $\mathbf{E}_{\text{value}}$, the set of all eigenvalues "a" satisfying (7);

(3) $\mathbf{E}_{\text{function}}(a)$, the set of all eigenfunctions "$y(x)$" corresponding to a fixed eigenvalue "a".

For each $a \in \mathbb{R}$, the cardinal number of $\mathbf{E}_{\text{function}}(a)$, denoted by $|\mathbf{E}_{\text{function}}(a)|$, satisfies the following condition:

$$\left|\mathbf{E}_{\text{function}}(a)\right| \leq \deg_y F(x, y). \tag{8}$$

In Section 2, we derive some properties of eigenfunctions of $F(x, y)$. If (7) has infinitely many eigenfunctions, the concerned properties are described in Section 3.

Throughout the paper, we denote the polynomial function by

$$\begin{aligned} F(x, y) &= a_s(x) y^s + a_{s-1}(x) y^{s-1} + \cdots + a_1(x) y + a_0(x) \\ &= \sum_{i=0}^{s} a_i(x) y^i \end{aligned} \tag{9}$$

with $s \geq 1$. Moreover, we may assume that $a_0(x)$ is nonzero. Since $a_0(x) = 0$, problem (7) may become

$$a_s(x) y^s(x) + a_{s-1}(x) y^{s-1}(x) + \cdots + a_1(x) y(x) = ay(x). \tag{10}$$

Moreover, if problem (7) has infinitely many eigenfunctions, dividing $y(x)$ by both sides of the above equation, then there may exist infinitely many nonzero eigenfunctions $y(x)$ satisfying

$$a_s(x) y^{s-1}(x) + a_{s-1}(x) y^{s-2}(x) + \cdots + a_1(x) = a \tag{11}$$

for some $a \in \mathbb{R}$. Therefore, this problem becomes a special case of (3).

2. Some Lemmas and a Former Theorem

Throughout this paper, we consider (7) for the polynomial function (9).

Lemma 2. *Let $y(x) \in \mathbf{E}_{\text{function}}$. Then*

$$y(x) = dp(x) \quad \text{for some } d \in \mathbb{R}, \tag{12}$$

and this $p(x)$ is divisible $a_0(x)$ and is denoted by $p(x) \mid a_0(x)$.

Proof. Since $y(x) \in \mathbf{E}_{\text{function}}$, we have $F(x, y(x)) = ay(x)$ for some $a \in \mathbb{R}$. This means

$$\begin{aligned} &a_s(x) y^s(x) + a_{s-1}(x) y^{s-1}(x) + \cdots + a_1(x) y(x) + a_0(x) \\ &\quad = ay(x) \end{aligned} \tag{13}$$

for some $a \in \mathbb{R}$. It leads to

$$\begin{aligned} &y(x)\left(a_s(x) y^{s-1}(x) + a_{s-1}(x) y^{s-2}(x) + \cdots + (a_1(x) - a)\right) \\ &\quad = -a_0(x); \end{aligned} \tag{14}$$

then $y(x)$ is a factor of $a_0(x)$. □

In Lemma 2, any eigenfunction is a factor $p(x)$ of $a_0(x)$. Thus we define a class of this factor as follows.

Notation 1. Let $p(x) \in \mathbb{R}[x]$, and we denote $\Phi(p(x)) = \{\alpha p(x) : \alpha \in \mathbb{R}\}$.

According to Notation 1, it is obvious that for any $p(x)$ in $\mathbb{R}[x]$, we have the cardinal number

$$\left|\Phi(p(x))\right| = \infty. \tag{15}$$

For convenience, we explain the relations of $\mathbf{E}_{\text{function}}$ and $\Phi(p(x))$ in the following lemma.

Lemma 3. *Consider*

$$\mathbf{E}_{\text{function}} = \bigcup_{p(x) \mid a_0(x)} \Phi(p(x)) \cap \mathbf{E}_{\text{function}}. \tag{16}$$

Proof. For any $y(x) \in \mathbf{E}_{\text{function}}$, by Lemma 2, we have $y(x) \mid a_0(x)$. That is,

$$y(x) \in \Phi(p(x)) \tag{17}$$

for some factor $p(x)$ of $a_0(x)$. It follows that

$$\mathbf{E}_{\text{function}} \subseteq \bigcup_{p(x) \mid a_0(x)} \Phi(p(x)) \tag{18}$$

and we obtain

$$\mathbf{E}_{\text{function}} = \bigcup_{p(x) \mid a_0(x)} \Phi(p(x)) \cap \mathbf{E}_{\text{function}}. \tag{19}$$

□

We will use the definitions of "the pigeonhole principle," which concert with Grimaldi [8] and the above relation can be explained as the following lemma.

Lemma 4. *Suppose that the cardinal number $|\mathbf{E}_{\text{function}}| = \infty$; there exists a factor $p(x)$ of $a_0(x)$ such that the cardinal number*

$$\left|\Phi(p(x)) \cap \mathbf{E}_{\text{function}}\right| = \infty. \tag{20}$$

Proof. By Lemma 3, we obtain

$$(\infty =) \left|\mathbf{E}_{\text{function}}\right| = \left|\bigcup_{p(x)|a_0(x)} \Phi\left(p\left(x\right)\right) \cap \mathbf{E}_{\text{function}}\right| \leq \sum_{p(x)|a_0(x)} \left|\Phi\left(p\left(x\right)\right) \cap \mathbf{E}_{\text{function}}\right|. \tag{21}$$

Since the number of all factor $p(x)$ of $a_0(x)$ is at most $2^{\deg a_0(x)}$, by pigeonhole's principle, it yields

$$\left|\Phi\left(p\left(x\right)\right) \cap \mathbf{E}_{\text{function}}\right| = \infty \tag{22}$$

for some factor $p(x)$ of $a_0(x)$. □

In order to solve the problem (7), [6, Lemma 3 and Theorem 11] are needed as follows.

Lemma 5 (see [6, Lemma 3]). *Assume that the number of all quasicoincidence solutions (defined in Definition 1) is infinitely many; then, for any two quasicoincidence solutions $y_1(x)$ and $y_2(x)$, we have*

$$y_1\left(x\right) - y_2\left(x\right) = \lambda g\left(x\right) \tag{23}$$

for some constant $\lambda \in \mathbb{R}$ and some factor $g(x)$ of $f(x)$.

Theorem 6 (see [6, Theorem 11]). *Assume that the number of all quasicoincidence solutions (defined in Definition 1) is infinitely many; then*

$$F\left(x, y\right) = \sum_{i=0}^{s} c_i \frac{f\left(x\right)}{g^i\left(x\right)} \left(y - y_1\left(x\right)\right)^i \tag{24}$$

for some $y_1(x) \in \mathbb{R}[x]$, $g(x)$ is a factor of $f(x)$, and $c_i \in \mathbb{R}$ for $i = 0, 1, \ldots, s$.

3. Main Theorems

In this section, we consider $F(x, y) = ay(x)$ for the polynomial function $F(x, y)$ defined in (9).

We investigate the fixed point like problem of simple polynomial functions $F(x, y)$ with $s = 1$ at first. Theorems 7 and 8 describe the necessary and sufficient results of these simple functions.

Theorem 7. *Let $F(x, y)$ be a polynomial function with $\deg_y F = 1$ as the form $F(x, y) = a_1(x)y + a_0(x)$ for some $a_1(x)$, $a_0(x) \in \mathbb{R}[x]$. If the cardinal number $|\mathbf{E}_{\text{function}}| = \infty$, then*

(i) *$a_1(x) \in \mathbb{R}$;*

(ii) *any eigenfunction $y(x)$ of (7) is of the form*

$$y\left(x\right) = \lambda a_0\left(x\right) \tag{25}$$

for some $\lambda \in \mathbb{R}$.

Proof. Since $|\mathbf{E}_{\text{function}}| = \infty$, by Lemma 4, there exists a factor $p(x)$ of $a_0(x)$ such that

$$\left|\Phi\left(p\left(x\right)\right) \cap \mathbf{E}_{\text{function}}\right| = \infty. \tag{26}$$

There exist two different eigenfunctions $y_1(x), y_2(x) \in \Phi(p(x)) \cap \mathbf{E}_{\text{function}}$ with

$$y_1\left(x\right) = \alpha_1 p\left(x\right), \quad y_2\left(x\right) = \alpha_2 p\left(x\right), \tag{27}$$

for different constants $\alpha_1, \alpha_2 \in \mathbb{R}$. Since $y_1(x), y_2(x) \in \mathbf{E}_{\text{function}}$, we have

$$F\left(x, y_1\left(x\right)\right) = ay_1\left(x\right), \quad F\left(x, y_2\left(x\right)\right) = by_2\left(x\right), \tag{28}$$

where $a, b \in \mathbf{E}_{\text{value}}$. It follows that

$$F\left(x, y_1\left(x\right)\right) = a_1\left(x\right) y_1\left(x\right) + a_0\left(x\right) = ay_1\left(x\right), \quad F\left(x, y_2\left(x\right)\right) = a_1\left(x\right) y_2\left(x\right) + a_0\left(x\right) = by_2\left(x\right). \tag{29}$$

By (27) and (29), we have

$$a_1\left(x\right)\left(\alpha_1 p\left(x\right)\right) + a_0\left(x\right) = a\left(\alpha_1 p\left(x\right)\right), \quad a_1\left(x\right)\left(\alpha_2 p\left(x\right)\right) + a_0\left(x\right) = b\left(\alpha_2 p\left(x\right)\right). \tag{30}$$

By (30), we get

$$a_1\left(x\right)\left(\alpha_1 p\left(x\right) - \alpha_2 p\left(x\right)\right) = a\alpha_1 p\left(x\right) - b\alpha_2 p\left(x\right). \tag{31}$$

Since $\alpha_1 \neq \alpha_2$ and $p(x)$ is nonzero, it follows that

$$a_1\left(x\right) = \frac{a\alpha_1 - b\alpha_2}{\alpha_1 - \alpha_2} \in \mathbb{R}. \tag{32}$$

For any $y(x) \in \mathbf{E}_{\text{function}}$, we have

$$F\left(x, y\left(x\right)\right) = a_1\left(x\right) y\left(x\right) + a_0\left(x\right) = ay\left(x\right). \tag{33}$$

By (32), we let $a_1(x) = \widetilde{a_1} \in \mathbb{R}$, (33) becomes

$$\widetilde{a_1} y\left(x\right) + a_0\left(x\right) = ay\left(x\right), \tag{34}$$

and it follows that

$$a_0\left(x\right) = \left(a - \widetilde{a_1}\right) y\left(x\right). \tag{35}$$

Owing to $a_0(x) \neq 0$, then we obtain

$$y\left(x\right) = \widetilde{\lambda} a_0\left(x\right), \tag{36}$$

where $\widetilde{\lambda} = 1/a - \widetilde{a_1}$. □

The following theorem is the sufficient conditions of Theorem 7.

Theorem 8. *Let $F(x, y)$ be a polynomial function with $\deg_y F = 1$ as the form $F(x, y) = a_1(x)y + a_0(x)$ for some $a_1(x)$, $a_0(x) \in \mathbb{R}[x]$. If*

(i) $a_1(x) \in \mathbb{R}$,

(ii) *any eigenfunction* $y(x)$ *of (7) is of the form*

$$y(x) = \lambda a_0(x), \quad (37)$$

for some $\lambda \in \mathbb{R}$,

then $|\mathbf{E}_{\text{function}}| = \infty$.

Proof. By (i), we let $a_1(x) = a_1 \in \mathbb{R}$, then $F(x, y(x)) = ay(x)$ for some $a \in \mathbb{R}$. This implies

$$a_1 y(x) + a_0(x) = ay(x) \quad (38)$$

and then $y(x) = (1/(a-a_1))a_0(x)$ is an eigenfunction of (7) for any constant $a \neq a_1$. It follows that

$$\infty = \left|\left\{\frac{1}{a-a_1}a_0(x) : a \in \mathbb{R} - \{a_1\}\right\}\right| \leq |\mathbf{E}_{\text{function}}|; \quad (39)$$

then $|\mathbf{E}_{\text{function}}| = \infty$. □

In Theorems 7 and 8, problem (7) with $\deg_y F = 1$ is introduced. In the following theorems, we deal with (7) with $\deg_y F \geq 2$ when the number of all eigenfunctions is infinitely many.

Theorem 9. *Suppose that the cardinal number* $|\mathbf{E}_{\text{function}}| = \infty$ *and* $\deg_y F(x, y) \geq 2$. *Then the polynomial* $F(x, y)$ *can be represented as*

$$F(x, y) = \sum_{i=0}^{s} \beta_i y^i \quad (40)$$

for some constants $\beta_i \in \mathbb{R}$.

Proof. Since $|\mathbf{E}_{\text{function}}| = \infty$, by Lemma 4, there exists a factor $p(x)$ of $a_0(x)$ satisfying

$$|\Phi(p(x)) \cap \mathbf{E}_{\text{function}}| = \infty. \quad (41)$$

Let $y_1(x)$ be an eigenfunction in $\Phi(p(x)) \cap \mathbf{E}_{\text{function}}$ such that

$$F(x, y_1(x)) = a_1 y_1(x) \quad (42)$$

for some eigenvalue $a_1 \in \mathbb{R}$. By Remainder Theorem, we get

$$F(x, y) = (y - y_1(x)) F_1(x, y) + a_1 y_1(x), \quad (43)$$

where $F_1(x, y)$ is the quotient and $a_1 y_1(x)$ is the remainder.

From the above identity and considering any eigenfunction $y(x)$ in $\Phi(p(x)) \cap \mathbf{E}_{\text{function}}/\{y_1(x)\}$ with $F(x, y(x)) = ay(x)$, we substitute (43) by taking $y = y(x)$ above and it becomes

$$\begin{aligned}(ay(x) =)\, & F(x, y(x)) \\ &= (y(x) - y_1(x)) F_1(x, y(x)) + a_1 y_1(x).\end{aligned} \quad (44)$$

Since $y_1(x)$, $y(x) \in \Phi(p(x))$, we have

$$y_1(x) = \lambda_1 p(x), \quad (45)$$

$$y(x) = \lambda p(x) \quad (46)$$

for some different constants λ_1 and λ. Substituting (45) and (46) into (44), it becomes

$$a\lambda p(x) = (\lambda p(x) - \lambda_1 p(x)) F_1(x, y(x)) + a_1 \lambda_1 p(x) \quad (47)$$

and it leads to

$$F_1(x, y(x)) = \frac{a\lambda - a_1\lambda_1}{\lambda - \lambda_1} \in \mathbb{R} \quad (48)$$

for any eigenfunction $y(x) \in \Phi(p(x)) \cap \mathbf{E}_{\text{function}}/\{y_1(x)\}$.

By (48), there exist infinitely many quasicoincidence solutions in $\Phi(p(x)) \cap \mathbf{E}_{\text{function}}/\{y_1(x)\}$ to satisfy

$$F_1(x, y) = af(x) \quad (49)$$

with $f(x) = 1$. This problem is a quasicoincidence problem; then by Theorem 6, we have

$$F_1(x, y) = \sum_{i=0}^{s-1} c_i \frac{f(x)}{g^i(x)} (y - y_1(x))^i. \quad (50)$$

Moreover, since $f(x) = 1$ and $f(x)/g^i(x) \in \mathbb{R}[x]$ for any $i = 0, 1, 2, \ldots, s-1$, it implies that $g(x) \in \mathbb{R}$ and by Lemma 5, any $y_2(x), y_3(x) \in \Phi(p(x)) \cap \mathbf{E}_{\text{function}}/\{y_1(x)\}$, we have

$$y_2(x) - y_3(x) = dg(x) = d' \in \mathbb{R}. \quad (51)$$

By definitions of $\Phi(p(x))$, $y_2(x)$, and $y_3(x)$ can also be represented as

$$\begin{aligned} y_2(x) &= \lambda_2 p(x), \\ y_3(x) &= \lambda_3 p(x) \end{aligned} \quad (52)$$

for some $\lambda_2, \lambda_3 \in \mathbb{R}$. By (51), it follows that

$$y_2(x) - y_3(x) = (\lambda_2 - \lambda_3) p(x) \in \mathbb{R}. \quad (53)$$

Moreover, by (53), this implies that $p(x) \in \mathbb{R}$ and by (45), $y_1(x) \in \mathbb{R}$, say, $y_1(x) = b_1$ and (50) implies that

$$F_1(x, y) = \sum_{i=0}^{s-1} c_i (y - b_1)^i = \sum_{i=0}^{s-1} d_i y^i \quad (54)$$

for some $d_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, s-1$.

By (54), (43) implies that

$$\begin{aligned} F(x, y) &= (y - y_1(x)) F_1(x, y) + a_1 y_1(x) \\ &= (y - b_1) \sum_{i=0}^{s-1} d_i y^i + a_1 b_1 = \sum_{i=0}^{s} \beta_i y^i. \end{aligned} \quad (55)$$

□

Conversely, if $F(x, y)$ can be expressed as in Theorem 9, then the cardinal number $|\mathbf{E}_{\text{function}}| = \infty$; this problem becomes the sufficient conditions of Theorem 9.

Theorem 10. *Assume that*

$$F(x,y)=\sum_{i=0}^{s}\beta_i y^i \tag{56}$$

for some $\beta_i \in \mathbb{R}$ for $i=0,1,\ldots,s$; then

$$|\mathbf{E}_{\text{function}}|=\infty. \tag{57}$$

Proof. For any $y(x)=c\in\mathbb{R}$,

$$\begin{aligned}F(x,y(x))&=\sum_{i=0}^{s}c_i y^i(x)\\&=\sum_{i=0}^{s}c_i c^i\ (\text{this is a constant})=ac,\end{aligned} \tag{58}$$

for some $a=\sum_{i=0}^{s}c_i c^i/c\in\mathbb{R}$. Then $\mathbb{R}\subseteq\mathbf{E}_{\text{function}}$ and then $|\mathbf{E}_{\text{function}}|=\infty$. □

In fact, if $|\mathbf{E}_{\text{function}}|=\infty$, then $\mathbf{E}_{\text{function}}=\mathbb{R}$ and we prove it as follows.

Theorem 11. *If $|\mathbf{E}_{\text{function}}|=\infty$, we have*

$$\mathbf{E}_{\text{function}}=\mathbb{R}. \tag{59}$$

Proof. Since $|\mathbf{E}_{\text{function}}|=\infty$, by the proof of Theorem 10, we have

$$\mathbb{R}\subseteq\mathbf{E}_{\text{function}}. \tag{60}$$

Conversely, considering any $y(x)\in\mathbf{E}_{\text{function}}$, we have

$$F(x,y(x))=\sum_{i=0}^{s}\beta_i y^i(x)=ay(x) \tag{61}$$

for some eigenvalue $a\in\mathbb{R}$. By Lemma 2, we have $y(x)\mid\beta_0$; this implies $y(x)\in\mathbb{R}$ and then $\mathbf{E}_{\text{function}}\subseteq\mathbb{R}$. This proof is completed. □

From Theorems 9 and 11, we can easily obtain the following two corollaries.

Corollary 12. *Let $F(x,y)=\sum_{i=0}^{s}a_i(x)y^i\in\mathbb{R}[x,y]$, $s\geq 2$, with $a_j(x)\notin\mathbb{R}$ for some j, then $|\mathbf{E}_{\text{function}}|<\infty$.*

Proof. This result can be immediately obtained from Theorem 9. □

Corollary 13. *If there exists an eigenfunction $y(x)\in\mathbf{E}_{\text{function}}$ with $y(x)\notin\mathbb{R}$, then $|\mathbf{E}_{\text{function}}|<\infty$.*

Proof. This result can be immediately obtained from Theorem 11. □

From Theorems 7, 8, and 10 and Corollary 12, we provide some examples of fixed point like problem (7) for some $a\in\mathbb{R}$, which have infinitely many eigenfunctions and do not have infinitely many eigenfunctions as follows.

Example 14. In the following examples, we by the form of $F(x,y)$, we can decide whether the number of all eigenfunctions in (7) is infinitely many or not.

(1) If $(x,y)=xy+1$, there do not exist infinitely many eigenfunctions (Theorem 7).

(2) If $F(x,y)=xy+x$, there do not exist infinitely many eigenfunctions (Theorem 7).

(3) If $F(x,y)=y+x$, there exist infinitely many eigenfunctions and

$$\mathbf{E}_{\text{function}}=\{\lambda x:\lambda\in\mathbb{R}\}\ (\text{Theorem 8}). \tag{62}$$

(4) If $F(x,y)=-y^2+7y+1$, there exist infinitely many eigenfunctions and

$$\mathbf{E}_{\text{function}}=\mathbb{R}\ (\text{Theorem 10}). \tag{63}$$

(5) If $F(x,y)=xy^2+xy+1$, there do not exist infinitely many eigenfunctions (Corollary 12).

(6) If $F(x,y)=\sum_{i=0}^{s}c_i y^i+x$, $s\geq 2$, for any constants $c_i\in\mathbb{R}$, there do not exist infinitely many eigenfunctions (Corollary 12).

We would like to provide one open problem as follows.

Further Development. For a real-valued polynomial function $F:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, if $|\mathbf{E}_{\text{function}}|<\infty$, can we find a co-NP complete algorithm to solve all eigenfunctions $y=y(x)$ satisfying (7)?

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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