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Research Article

Applications of Multivalued Contractions on Graphs to Graph-Directed Iterated Function Systems

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We apply a fixed point result for multivalued contractions on complete metric spaces endowed with a graph to graph-directed iterated function systems. More precisely, we construct a suitable metric space endowed with a graph G and a suitable G-contraction such that its fixed points permit us to obtain more information on the attractor of a graph-directed iterated function system.

1. Introduction

Based on the work of Hutchinson [1] and being popularized by Barnsley [2], the method of iterated function systems (IFS) permits us to generate fractals by iterating a collection of transformations $\{T_i : i = 1, \ldots, p\}$. If each T_i is a contraction on a complete metric space M, it was shown in [1] that there exists a unique nonempty compact set $K \subset M$ which is invariant with respect to $\{T_i : i = 1, \ldots, p\}$; that is,

$$K = \bigcup_{i=1}^{p} T_i (K) . \tag{1}$$

This attractor K is such that, for every compact $A \subset M$,

$$g^n (A) \longrightarrow K \quad \text{with respect to the Hausdorff metric,} \tag{2}$$

where

$$g (A) = \bigcup_{i=1}^{p} T_i (A) . \tag{3}$$

The existence of K can be deduced from the Banach fixed point theorem.

A fixed point result which is, in some sense, a combination of the Banach contraction principle and the Knaster-Tarski fixed point theorem in a partially ordered set was obtained by Ran and Reurings [3] in 2004. They considered a monotone, order preserving single-valued map f defined on a complete metric space endowed with a partial ordering. They assumed that f satisfies a contraction condition not necessarily for all x and y, but for those such that $x \leq y$. Subsequently, their result was generalized by many authors, in particular by Nieto, Rodríguez-López, Pouso, Petruşel, and Rus [4–7]. In 2008, Jachymski [8] presented a nice unification of most of the previous results by considering complete metric spaces endowed with a graph G. He introduced the notion of single-valued G-contraction for which he obtained fixed point results.

Using those fixed point results, Gwóźdź-Łukawska and Jachymski [9] developed the Hutchinson-Barnsley theory on complete metric space endowed with a graph G for iterated function systems of single-valued G-contractions.

Different extensions of the concept of single-valued G contractions on complete metric spaces endowed with a graph to multivalued maps were presented by Dinevari and Frigon [10] and by Nicolae et al. [11]. Those extensions led to generalizations of Jachymski's fixed point results and of the Nadler fixed point theorem for multivalued contractions.

In 1988, Mauldin and Williams [12] introduced the notion of geometric graph-directed construction.

Definition 1. A *geometric graph-directed construction* in $\mathbb{R}^m$ consists of

(i) a collection of p nonoverlapping, compact, nonempty subsets of $\mathbb{R}^m$, $J_1, \dots, J_p$ with nonempty interior;

(ii) a directed-graph $H = (V(H), E(H))$ such that $V(H) = \{1, \dots, p\}$ is the set of its vertices, and, for each $i \in V(H)$, there exists some edge $(i, j) \in E(H)$;

(iii) for each $(i, j) \in E(H)$, there is a similarity map $T_{i,j} : \mathbb{R}^m \to \mathbb{R}^m$ with similarity ratios $r_{i,j}$ such that

$$\bigcup_{(i,j)\in E(H)} T_{i,j}\left(J_j\right) \subset J_i; \tag{4}$$

(iv) for each i, $\{T_{i,j}(J_j) : (i, j) \in E(H)\}$ is a nonoverlapping family of sets;

(v) if $[i_1, \dots, i_{q-1}, i_q = i_1]$ is a cycle in H, then

$$\prod_{k=1}^{q} r_{i_{k-1}, i_k} < 1. \tag{5}$$

They showed that a geometric graph-directed construction has an attractor.

Theorem 2 (Mauldin and Williams [12]). *For a geometric graph-directed construction as above, there exists $K_1, \dots, K_p$ a unique collection of nonempty compact sets such that*

$$\forall i \in \{1, \dots, p\}, \ K_i \subset J_i, \quad K_i = \bigcup_{(i,j)\in E(H)} T_{i,j}\left(K_j\right). \tag{6}$$

The set

$$K = \bigcup_{i=1}^{p} K_i \tag{7}$$

is called the attractor of this geometric graph-directed construction.

Geometric graph-directed constructions have been studied and generalized by many authors; see [13–16]. In particular, it was shown in [13] that with an appropriate rescaling, condition (v) can be replaced by

(v)′ for each $(i, j) \in H$, $r_{i,j} < 1$.

Also, in some of those generalizations, similarities on $\mathbb{R}^m$ were replaced by contractions on complete metric spaces and the terminology of graph-directed iterated function system was used. Again, the existence of an attractor K was established.

In this paper, we take into account the graph H to obtain more information on the attractor K of a graph-directed iterated function system. To do so, we apply a fixed point result obtained by the authors [10] for multivalued contractions on complete metric spaces endowed with a graph.

The paper is organized as follows. In Section 2, we present some notations and we recall some results. In Section 3, we consider a space X such that $K \in X$ and on which we define a suitable graph G and a suitable metric. In Section 4, we define an appropriate multivalued G-contraction F. In the last three sections, taking into account the maximal connected component of the graph H, we obtain more information on the attractor K from some fixed points of F.

2. H-Iterated Function System

First of all, we introduce the notion of MW-directed graph and we consider iterated function systems which takes into account the structure of an MW-directed graph.

Definition 3. A directed-graph $H = (V(H), E(H))$ is called an *MW-directed graph* if $V(H) = \{1, \dots, p\}$, H has no parallel edges, and for every $i \in V(H)$, there exists $j \in V(H)$ such that $(i, j) \in E(H)$.

Definition 4. Let $H = (V(H), E(H))$ be an MW-directed graph. A *graph-directed iterated function system over the graph H* (H-IFS) is a collection of p nonempty, bounded, complete metric spaces, $(X_1, d_1), \dots, (X_p, d_p)$, and, for each $(i, j) \in E(H)$, a contraction $T_{i,j} : X_j \to X_i$ with constant of contraction $\lambda_{i,j}$. An H-IFS is denoted $\{T_{i,j}\}_H$.

Definition 5. Let $\{T_{i,j}\}_H$ be an H-IFS. An *attractor* K of the H-IFS is a collection of nonempty compact sets $K = \{K_i\}_H$ such that $K_i \subset X_i$ and

$$K_i = \bigcup_{(i,j)\in E(H)} T_{i,j}\left(K_j\right) \quad \forall i \in \{1, \dots, p\}. \tag{8}$$

The Banach contraction principle insures the existence of an attractor of an H-IFS. We present the proof for sake of completeness. For more information on graph-directed iterated function systems, the reader is referred to [12, 15].

Theorem 6. *An H-IFS, $\{T_{i,j}\}_H$, has a unique attractor K.*

Proof. Consider

$$Y = \left\{ \left(S_1, \dots, S_p\right) \subset \prod_{i=1}^{p} X_i : S_i \text{ is a compact nonempty subset of } X_i \right\} \tag{9}$$

endowed with the metric

$$\rho\left(S, \widehat{S}\right) = \max\left\{D_i\left(S_i, \widehat{S}_i\right) : i = 1, \dots, p\right\}, \tag{10}$$

where D_i is the Hausdorff metric on X_i; that is,

$$D_i\left(S_i, \widehat{S}_i\right) = \inf\left\{\varepsilon > 0 : S_i \subset B\left(\widehat{S}_i, \varepsilon\right),\ \widehat{S}_i \subset B\left(S_i, \varepsilon\right)\right\}, \tag{11}$$

where

$$B\left(S_i, \varepsilon\right) = \left\{y \in X_i : \exists x \in S_i \text{ such that } d_i\left(x, y\right) < \varepsilon\right\}. \tag{12}$$

Let us define $f : Y \to Y$ by

$$f_i\left(S\right) = \bigcup_{(i,j)\in E(H)} T_{i,j}\left(S_j\right). \tag{13}$$

Using the fact that every $T_{i,j}$ is a contraction, one verifies that f is a contraction with constant of contraction

$$\theta = \max\left\{\lambda_{i,j} : (i,j) \in E(H)\right\}. \quad (14)$$

The Banach contraction principle insures the existence of $K \in Y$ a unique fixed point of f. Thus, K is the unique attractor of $\{T_{i,j}\}_H$. □

More information on K will be obtained by applying a fixed point result for multivalued contractions on complete metric spaces endowed with a graph. We recall the notion of G-contraction introduced in [10].

For (X,d) a complete metric space, we consider $G = (V(G), E(G))$ a directed graph such that $X = V(G)$, the diagonal in $X \times X$ is contained in $E(G)$, and G has no parallel edges.

Definition 7. Let $F : X \to X$ be a multivalued map with nonempty values. We say that F is a *G-contraction* if there exists $\alpha \in]0,1[$ such that

(C_G) for all $(x,y) \in E(G)$ and all $u \in F(x)$, there exists $v \in F(y)$ such that $(u,v) \in E(G)$ and $d(u,v) \le \alpha d(x,y)$.

We consider suitable trajectories in X.

Definition 8. Let $F : X \to X$ be a multivalued mapping and $x_0 \in X$. We say that a sequence $\{x_n\}$ is a G_1*-Picard trajectory from* x_0 if $x_n \in F(x_{n-1})$ and $(x_{n-1}, x_n) \in E(G)$ for all $n \in \mathbb{N}$. The set of all such G_1-Picard trajectories from x_0 is denoted by $T_1(F,G,x_0)$.

The reader is referred to [10] for the proof of the following fixed point result for multivalued G-contractions.

Theorem 9. *Let $F : X \to X$ be a multivalued G-contraction such that there exists $(x_0,x_1) \in E(G)$ such that $x_1 \in F(x_0)$. In addition, assume that one of the following conditions holds.*

(i) *F is G_1-Picard continuous from x_0; that is, the limit of any convergent sequence $\{x_n\} \in T_1(F,G,x_0)$ is a fixed point of F.*

(ii) *F has closed values and, for every $\{x_n\}$ in $T_1(F,G,x_0)$ converging to some $x \in X$, there exists a subsequence $\{n_k\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$.*

Then, there exists a G_1-Picard trajectory from x_0, $\{x_n\}$, converging to x a fixed point of F. Moreover, every converging G_1-Picard trajectory from x_0 converges to a fixed point of F.

In what follows, we consider H an MW-directed graph. We will use the following definitions and notations.

A *path from i to j* in H is denoted by $[i_k]_0^N = [i_0,\dots,i_N]$, where $i = i_0$, $j = i_N$, and $(i_{k-1}, i_k) \in E(H)$ for every $k = 1,\dots,N$.

We say that a subgraph $C = (V(C), E(C))$ of H is *connected* if for every $i, j \in V(C)$ there exists a path from i to j in C. A *connected component* of H is a maximal connected subgraph of H. We denote

$$C(H) = \{C : C \text{ is a connected component of } H\}. \quad (15)$$

It follows from the definition of MW-directed graph that

$$\emptyset \ne C(H) = \{C_\alpha : \alpha \in \Lambda\}, \quad \text{where } \Lambda \text{ has finite cardinality.} \quad (16)$$

We can define a partial order on $C(H)$ as follows:

$$C_\alpha \preceq C_\beta \Longleftrightarrow \exists\, [i_k]_0^N \text{ a path in } H \text{ such that } i_0 \in C_\alpha,\ i_N \in C_\beta. \quad (17)$$

We write $C_\alpha \prec C_\beta$ to mean $C_\alpha \preceq C_\beta$ and $C_\alpha \ne C_\beta$. We say that C_α and C_β are *incomparable* if $C_\alpha \npreceq C_\beta$ and $C_\beta \npreceq C_\alpha$.

We denote the set of vertices from which there is a path in H reaching $i \in H$ by

$$[i]_\leftarrow = \{j \in V(H) : \text{there is a path from } j \text{ to } i \text{ in } H\}. \quad (18)$$

Similarly, for $C \in C(H)$, we denote the set of vertices from which there is a path in H reaching $V(C)$ by

$$[C]_\leftarrow = \bigcup_{i \in V(C)} [i]_\leftarrow. \quad (19)$$

3. A Suitable Metric Space Endowed with a Directed Graph

Let H be an MW-directed graph with $V(H) = \{1,\dots,p\}$. For $i \in V(H)$, let (X_i, d_i) be a bounded complete metric space.

In this section, using H and the spaces X_i, we define a complete metric space endowed with a suitable directed graph. Let us recall that

$$C(H) = \{C : C \text{ is a connected component of } H\}. \quad (20)$$

We consider the space X of p-tuples $A = (A_1,\dots,A_p)$ satisfying the following properties:

(Xi) $A_i \subset X_i$ is compact for every $i = 1,\dots,p$;

(Xii) if $A_i \ne \emptyset$ for some $i \in V(C)$ and $C \in C(H)$, then $A_j \ne \emptyset$ for all $j \in V(C)$;

(Xiii) there exists $C \in C(H)$ and $i \in V(C)$ such that $A_i \ne \emptyset$.

It is important to point out that, for $A = (A_1,\dots,A_p) \in X$, some A_i can be empty.

We endow X with the metric

$$d(A,B) = \max_{i\in\{1,\dots,p\}} \overline{D}_i(A_i, B_i), \quad (21)$$

where

$$\overline{D}_i(A_i,B_i) = \begin{cases} D_i(A_i,B_i), & \text{if } A_i \ne \emptyset,\ B_i \ne \emptyset, \\ 0, & \text{if } A_i = \emptyset = B_i, \\ R_i, & \text{otherwise,} \end{cases} \quad (22)$$

where D_i is the Hausdorff metric in X_i and $R_i > R$ is a constant which will be fixed later, with

$$R = \max\{\operatorname{diam}(X_i) : i = 1, \dots, p\}. \tag{23}$$

It is clear that (X, d) is a complete metric space.

Taking into account the graph H, we want to endow X with a directed graph. To do so, we distinguish vertices of H which are in a connected component from the others. We set

$$V^c = \bigcup_{C \in C(H)} V(C), \tag{24}$$

$$V^e = V(H) \setminus V^c. \tag{25}$$

We define the graph G as follows: $V(G) = X$, and for $A, B \in X$, $(A, B) \in E(G)$ if and only if

(G) for every $i \in \{1, \dots, p\}$, one of the following properties holds:

(i) $A_i = B_i = \emptyset$, or $A_i \neq \emptyset$ and $B_i \neq \emptyset$;

(ii) $A_i = \emptyset$, $B_i \neq \emptyset$, and one of the following statements is true:

(a) $i \in V^e$ and there exists $j \in V(H)$ such that $(i, j) \in E(H)$ and $A_j \neq \emptyset$;

(b) $i \in V(C)$ for some $C \in C(H)$ and there exist $k \in V(C)$ and $j \in V(H)$ such that $(k, j) \in E(H)$ and $A_j \neq \emptyset$;

(iii) $A_i \neq \emptyset$, $B_i = \emptyset$, $i \in V^e$, and one of the following properties is satisfied:

(a) there is no $j \in V(H)$ such that $(j, i) \in E(H)$;

(b) for every $j \in V(H)$ such that $(j, i) \in E(H)$, one has $B_j \neq \emptyset$.

Example 10. Let H be the MW-graph of Figure 1. We consider X the associated metric space satisfying (Xi)–(Xiii) endowed with the graph G satisfying the condition (G). Let A_i^k be nonempty compact subsets of X_i for all $i \in \{1, \dots, 9\}$ and $k \in \{1, \dots, 7\}$. We consider the following elements of X:

$$\begin{aligned}
A^1 &= \left(\emptyset, \emptyset, A_3^1, A_4^1, A_5^1, \emptyset, \emptyset, \emptyset, \emptyset\right), \\
A^2 &= \left(\emptyset, \emptyset, A_3^2, A_4^2, A_5^2, A_6^2, \emptyset, \emptyset, \emptyset\right), \\
A^3 &= \left(\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, A_7^3, A_8^3, A_9^3\right), \\
A^4 &= \left(\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, A_6^4, A_7^4, A_8^4, A_9^4\right), \\
A^5 &= \left(\emptyset, \emptyset, A_3^5, A_4^5, A_5^5, \emptyset, A_7^5, A_8^5, A_9^5\right), \\
A^6 &= \left(A_1^6, \emptyset, A_3^6, A_4^6, A_5^6, \emptyset, \emptyset, \emptyset, \emptyset\right), \\
A^7 &= \left(\emptyset, A_2^7, A_3^7, A_4^7, A_5^7, \emptyset, \emptyset, \emptyset, \emptyset\right).
\end{aligned} \tag{26}$$

Here is the list of all edges of G between them:

$$\begin{aligned}
&\left\{\left(A^1, A^7\right), \left(A^2, A^1\right), \left(A^2, A^7\right), \left(A^3, A^4\right), \left(A^3, A^4\right), \right. \\
&\left. \left(A^4, A^5\right), \left(A^6, A^1\right), \left(A^6, A^7\right), \left(A^7, A^6\right)\right\} \subset E(G).
\end{aligned} \tag{27}$$

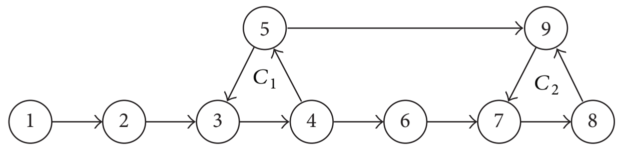

Figure 1: An MW-graph H.

Now, we want to fix R_i in (22) in such a way that we will be able to define a suitable multivalued G-contraction on X in the next section. To this aim, we decompose $V(H)$ in appropriate subsets V_μ with $\mu \in I$ a totally ordered set.

Lemma 11. *Let H be an MW-directed graph. Then there exist I a totally ordered set and $\{V_\mu : \mu \in I\}$ a family of nonempty disjoint subsets, and, for every $i \in \{1, \dots, p\}$, there exists $R_i > R$ such that*

(1) *$V(H) = \cup_{\mu \in I} V_\mu$;*

(2) *if $V(C) \cap V_\mu \neq \emptyset$ for some $\mu \in I$ and some $C \in C(H)$, then $V(C) \subset V_\mu$;*

(3) *if $\mu < \nu$ in I, for all $i \in V_\mu$, and $j \in V_\nu$, then $j \notin [i]_{\leftarrow}$;*

(4) *for every $\mu \in I$, one has $R_i = R_j$ for every $i, j \in V_\mu$;*

(5) *for every $\mu < \nu \in I$, one has $R_i < R_j$ for every $i \in V_\mu$, $j \in V_\nu$.*

Proof. We want to separate vertices of H in suitable subsets. Let us recall that some vertices are in a connected component, and some others are not:

$$V(H) = V^c \cup V^e, \tag{28}$$

where V^c and V^e are defined in (24) and (25), respectively.

First of all, we examine vertices in V^c. Let

$$\begin{aligned}
L = \max \{ & n \in \mathbb{N} : \text{there exists a chain} \\
& C_{\alpha_1} \prec \cdots \prec C_{\alpha_n} \text{ in } C(H) \}.
\end{aligned} \tag{29}$$

We denote

$$\begin{aligned}
C(H)_1 &= \left\{C \in C(H) : \nexists \widehat{C} \in C(H) \text{ such that } \widehat{C} \prec C\right\}, \\
C(H)_2 &= \left\{C \in C(H) \setminus C(H)_1 : \nexists \widehat{C} \in C(H) \setminus C(H)_1 \right. \\
&\qquad \left. \text{such that } \widehat{C} \prec C\right\}, \\
&\ \ \vdots \\
C(H)_L &= \left\{C \in C(H) \setminus \bigcup_{k=1}^{L-1} C(H)_k : \nexists \widehat{C} \in C(H) \setminus \bigcup_{k=1}^{L-1} C(H)_k \right. \\
&\qquad \left. \text{such that } \widehat{C} \prec C\right\}.
\end{aligned} \tag{30}$$

We define

$$V_{k,0} = \bigcup_{C \in C(H)_k} V(C) \quad \text{for } k = 1, \dots, L. \tag{31}$$

Observe that

$$V^c = \bigcup_{k=1}^{L} V_{k,0}, \quad V_{k,0} \cap V_{j,0} = \emptyset \quad \text{if } k \neq j. \tag{32}$$

Now, we separate vertices in V^e in suitable subsets. We first separate them in two sets: those which can be reached by a path starting from a vertex in a connected component, and those which cannot. This last set is denoted:

$$V^0 = \{j \in V^e : V^c \cap [j]_{\leftarrow} = \emptyset\}. \tag{33}$$

If $V^0 \neq \emptyset$, let

$$N_0 = \max\left\{n : \text{there is a path } [i_k]_1^n \text{ such that } i_k \in V^0 \text{ for every } k = 1, \dots, N_0\right\}. \tag{34}$$

We define

$$\begin{aligned} V_{0,1} &= \left\{i \in V^0 : \nexists j \in V^0 \text{ such that } (j,i) \in E(H)\right\}, \\ V_{0,2} &= \left\{i \in V^0 \setminus V_{0,1} : \nexists j \in V^0 \setminus V_{0,1} \text{ such that } (j,i) \in E(H)\right\}, \\ &\vdots \\ V_{0,N_0} &= \left\{i \in V^0 \setminus \bigcup_{k=1}^{N_0-1} V_{0,k} : \nexists j \in V^0 \setminus \bigcup_{k=1}^{N_0-1} V_{0,k} \text{ such that } (j,i) \in E(H)\right\}. \end{aligned} \tag{35}$$

Observe that

$$V^0 = \bigcup_{k=1}^{N_0} V_{0,k}, \quad V_{0,k} \cap V_{0,j} = \emptyset \quad \text{if } k \neq j. \tag{36}$$

If $V^e \setminus V^0 \neq \emptyset$, it follows from Definition 3 that, for every $j \in V^e \setminus V^0$, there exist $C_\alpha, C_\beta \in C(H)$ such that

$$C_\alpha \prec C_\beta, \quad V(C_\alpha) \subset [j]_{\leftarrow}, \quad j \in [C_\beta]_{\leftarrow}. \tag{37}$$

In other words, j is on a path from C_α to C_β. Hence, $L > 1$, where L is defined in (29).

If $L \geq 2$, we first examine vertices on a path from some $i \in V_{1,0}$ to some $j \in V_{2,0}$. Let

$$N_1 = \max\left\{n : \text{there is a path } [i_k]_0^{1+n} \text{ such that } i_0 \in V_{1,0},\ i_{1+n} \in V_{2,0} \text{ and } i_k \in V^e\ \forall k = 1, \dots, n\right\}. \tag{38}$$

If $N_1 \geq 1$, we define

$$V_{1,1} = \left\{i \in V^e : \exists [i_k]_0^{1+N_1} \text{ with } i = i_1,\ i_0 \in V_{1,0},\ i_{1+N_1} \in V_{2,0},\ i_k \in V^e \text{ for } k = 1, \dots, N_1\right\}. \tag{39}$$

If $N_1 \geq 2$, we define

$$V_{1,2} = \left\{i \in V^e \setminus V_{1,1} : \exists [i_k]_1^{1+N_1} \text{ with } i = i_2,\ i_1 \in V_{1,0} \cup V_{1,1},\ i_{1+N_1} \in V_{2,0},\ i_k \in V^e \setminus V_{1,1} \text{ for } k = 2, \dots, N_1\right\}. \tag{40}$$

We define inductively $V_{1,1}, \dots, V_{1,N_1}$.

We denote the set of vertices on a path from $V_{1,0}$ to $V_{2,0}$ by

$$V^1 = V_{1,0} \cup V_{2,0} \cup \bigcup_{k=1}^{N_1} V_{1,k}. \tag{41}$$

If $L \geq 3$, we examine vertices on a path from some $i \in V^1$ to some $j \in V_{3,0}$. Let

$$N_2 = \max\left\{n : \text{there is a path } [i_k]_0^{1+n} \text{ such that } i_0 \in V^1,\ i_{1+n} \in V_{3,0} \text{ and } i_k \in V^e \setminus V^1\ \forall k = 1, \dots, n\right\}. \tag{42}$$

If $N_2 \geq 1$, we define

$$V_{2,1} = \left\{i \in V^e \setminus V^1 : \exists [i_k]_0^{1+N_2} \text{ with } i = i_1, i_0 \in V^1,\ i_{1+N_2} \in V_{3,0},\ i_k \in V^e \setminus V^1 \text{ for } k = 1, \dots, N_2\right\}. \tag{43}$$

If $N_2 \geq 2$, we define

$$V_{2,2} = \left\{i \in V^e \setminus \left(V^1 \cup V_{2,1}\right) : \exists [i_k]_1^{1+N_2} \text{ with } i = i_2,\ i_1 \in V^1 \cup V_{2,1},\ i_{1+N_2} \in V_{3,0}, \text{ and } i_k \in V^e \setminus \left(V^1 \cup V_{2,1}\right) \text{ for } k = 2, \dots, N_2\right\}. \tag{44}$$

Similarly, we define $V_{2,j}$ for $j \leq N_2$.

So, inductively, we define the following subsets of $V^e \setminus V^0$:

$$V_{1,1}, \dots, V_{1,N_1}, V_{2,1}, \dots, V_{2,N_2}, \dots, V_{L-1,1}, \dots, V_{L-1,N_{L-1}}. \tag{45}$$

Each vertex in one of those sets is on a path from one connected component to another.

We have decomposed $V(H)$ in a collection of disjoint sets:

$$V_{0,1}, \dots, V_{0,N_0}, V_{1,0}, V_{1,1}, \dots, V_{1,N_1}, V_{2,0}, \dots, V_{L-1,N_{L-1}}, V_{L,0}. \tag{46}$$

We denote

$$I = \{(k,0) : 1 \leq k \leq L\} \cup \left\{(k,l) : 0 \leq k \leq L-1, 1 \leq l \leq N_k\right\}. \tag{47}$$

We endow I with the order

$$(k_1, l_1) \leq (k_2, l_2) \Longleftrightarrow k_1 < k_2, \text{ or } k_1 = k_2,\ l_1 \leq l_2. \tag{48}$$

By construction,

$$V(H) = \bigcup_{\mu \in I} V_\mu, \quad V_\mu \cap V_\nu = \emptyset \quad \text{if } \mu \neq \nu. \tag{49}$$

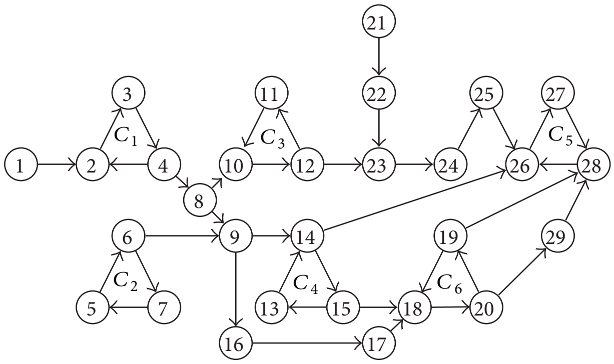

Figure 2: An MW-graph H with $C(H) = \{C_1, \ldots, C_6\}$.

Also, for every $C \in C(H)$, there exists $\mu \in I$ such that $V(C) \subset V_\mu$. Moreover, for $\mu, \nu \in I$ such that $\mu < \nu$, one has $j \notin [i]_\leftarrow$ for every $i \in V_\mu$, and $j \in V_\nu$.

Finally, we choose $\sigma : I \to]1, \infty[$ a strictly increasing map. We define

$$R_i = \sigma(\mu) R \quad \text{for } i \in \{1, \ldots, p\} \cap V_\mu,\ \mu \in I. \tag{50}$$

By construction, statements (4) and (5) are satisfied. □

Remark 12. Let $(A, B) \in E(G)$. From the definition of the graph G and Lemma 11, we can make the following observations.

(1) If for some $i \in V(H)$, (G)(ii) holds with some $j \in V(H)$ such that $A_j \neq \emptyset$; let $\mu_i, \mu_j \in I$ be such that $i \in V_{\mu_i}$ and $j \in V_{\mu_j}$. Then, $\mu_i < \mu_j$.

(2) If for some $i \in V(H)$, (G)(iii)(b) holds, let $\mu_i \in I$ be such that $i \in V_{\mu_i}$. Then, for all $j \in V(H)$ such that $(j, i) \in E(H)$, there is $\mu_j \in I$ such that $j \in V_{\mu_j}$ and one has $\mu_j < \mu_i$.

Example 13. We consider H the MW-graph of Figure 2 for which we describe the collection of subsets V_μ constructed as in the proof of Lemma 11. In this graph,

$$\begin{aligned} C(H) &= \{C_1, \ldots, C_6\}, \\ V^c &= \{2, 3, 4, 5, 6, 7, 10, 11, 12, 13, 14, 15, \\ &\quad 18, 19, 20, 26, 27, 28\}, \\ V^e &= \{1, 8, 9, 16, 17, 21, 22, 23, 24, 25, 29\}. \end{aligned} \tag{51}$$

Since $C_2 \prec C_4 \prec C_6 \prec C_5$, one has $L = 4$, and

$$\begin{aligned} V_{1,0} &= \{2, 3, 4, 5, 6, 7\}, \\ V_{2,0} &= \{10, 11, 12, 13, 14, 15\}, \\ V_{3,0} &= \{18, 19, 20\}, \\ V_{4,0} &= \{26, 27, 28\}. \end{aligned} \tag{52}$$

By considering the paths from $V_{1,0}$ to $V_{2,0}$, one sees that $N_1 = 2$, and

$$V_{1,1} = \{8\}, \qquad V_{1,2} = \{9\}. \tag{53}$$

By considering the paths from $V_{1,0} \cup V_{1,1} \cup V_{1,2} \cup V_{2,0}$ to $V_{3,0}$, one sees that $N_2 = 2$, and

$$V_{2,1} = \{16\}, \qquad V_{2,2} = \{17\}. \tag{54}$$

Similarly, one has $N_3 = 3$, and

$$V_{3,1} = \{23\}, \qquad V_{3,2} = \{24\}, \qquad V_{3,3} = \{25, 29\}. \tag{55}$$

So, the vertices which are not in one of the previous sets are in $V^0 = \{1, 21, 22\}$. Similarly, $N_0 = 2$, and

$$V_{0,1} = \{1, 21\}, \qquad V_{0,2} = \{22\}. \tag{56}$$

So, I is the totally ordered set:

$$\begin{aligned} &(0,1) < (0,2) < (1,0) < (1,1) < (1,2) < (2,0) < (2,1) \\ &\quad < (2,2) < (3,0) < (3,1) < (3,2) < (3,3) < (4,0), \\ &V(H) = \{1, \ldots, 29\} = \bigcup_{\mu \in I} V_\mu. \end{aligned} \tag{57}$$

4. A G-Contraction

In this section, we consider a graph-directed iterated function system over the graph H, $\{T_{i,j}\}_H$. We will define an appropriate multivalued G-contraction on X, where G and X are, respectively, the graph and the metric space endowed with this graph and defined in the previous section. This G-contraction will be used to get more information on the attractor of this H-IFS.

Let $A \in X$. For each j such that $A_j \neq \emptyset$, $T_{i,j}(A_j) \neq \emptyset$ for all i such that $(i, j) \in E(H)$. So, it is important to distinguish all those edges. To this aim, we introduce the following notations.

Let V^e be the subset of vertices in $V(H)$ which are not in connected components of H and defined in (25). So, for $i \in V^e$, we denote

$$E_i(A) = \{(i, j) \in E(H) : A_j \neq \emptyset\}. \tag{58}$$

For $\emptyset \neq P \subset E_i(A)$, we define

$$U_i^e(A, P) = \bigcup_{(i,j) \in P} T_{i,j}(A_j). \tag{59}$$

Let V^c be the subset of vertices in $V(H)$ which are in connected components of H and defined in (24). So, for $i \in V^c$, there exists $C \in C(H)$ such that $i \in V(C)$. We consider the set of edges from a vertex of C to a vertex outside of C for which the component of A is nonempty:

$$E_C(A) = \{(k, j) \in E(H) : k \in V(C),\ j \notin V(C),\ A_j \neq \emptyset\}. \tag{60}$$

For $k \in V(C)$, we denote

$$\begin{aligned} \left\{i \xrightarrow{C} k\right\} = \{[i_k]_0^N \ &\text{which is a path in } C \text{ from } i = i_0 \\ &\text{to } k = i_N \text{ and containing no cycle}\}, \end{aligned} \tag{61}$$

and we define $T_{i\to k}: X_k \to X_i$ by

$$T_{i\to k}(x) = \bigcup_{[i_k]_0^N \in \{i \xrightarrow{C} k\}} T_{i_0,i_1} \circ \cdots \circ T_{i_{N-1},i_N}(x). \tag{62}$$

We define

$$U_i^c(A,P) = \begin{cases} \emptyset, & \text{if } P = \emptyset, \\ \bigcup_{(k,j)\in P} T_{i\to k} \circ T_{k,j}(A_j), & \text{if } \emptyset \neq P \subset E_C(A). \end{cases} \tag{63}$$

We also define

$$W_i(A) = \begin{cases} \emptyset, & \text{if } A_i = \emptyset, \\ \bigcup_{(i,j)\in E(C)} T_{i,j}(A_j), & \text{if } A_i \neq \emptyset, \end{cases} \tag{64}$$

where $E(C) = \{(k,j) \in E(H) : k, j \in V(C)\}$.

We have all the ingredients to define the multivalued map $F : X \to X$. For $A \in X$,

$$U = (U_1, \ldots, U_p) \in F(A) \Longleftrightarrow U_i \in F_i(A), \tag{65}$$

where $F_i(A)$ is defined as follows.

For $i \in V^e$,

$$F_i(A) = \begin{cases} \emptyset, & \text{if } E_i(A) = \emptyset, \\ \{U_i^e(A,P) : \emptyset \neq P \subset E_i(A)\}, & \text{if } E_i(A) \neq \emptyset. \end{cases} \tag{66}$$

For $i \in V(C)$ for some $C \in C(H)$,

$$F_i(A) = \begin{cases} \emptyset, & \text{if } A_i = \emptyset, \\ & E_C(A) = \emptyset, \\ \{U_i^c(A,P) : \emptyset \neq P \subset E_C(A)\}, & \text{if } A_i = \emptyset, \\ & E_C(A) \neq \emptyset, \\ \{W_i(A) \cup U_i^c(A,P) : P \subset E_C(A)\}, & \text{if } A_i \neq \emptyset. \end{cases} \tag{67}$$

Observe that F is well defined. Indeed, if $U \in F(A)$ is such that $U_i \neq \emptyset$ for i in some $V(C)$, then $U_j \neq \emptyset$ for all $j \in V(C)$. Also, there exists $C \in C(H)$ such that $U_i \neq \emptyset$ for all $i \in V(C)$. Moreover, the values of F are finite and hence closed.

We show that F is a multivalued G-contraction.

Proposition 14. *Let $F : X \to X$ be the multivalued map defined above. Then F is a G-contraction.*

Proof. We want to show that F is a G-contraction with constant of contraction:

$$\begin{aligned} \lambda = \max\Bigg\{ & \max\{\lambda_{i,j} : (i,j) \in E(H)\}, \\ & \max\left\{\frac{R}{R_i} : i \in \{1,\ldots,p\}\right\}, \\ & \max\Bigg\{\frac{R_i}{R_j} : i \in V_{\mu_i}, j \in V_{\mu_j} \text{ for } \mu_i, \mu_j \in I \\ & \qquad \text{such that } \mu_i < \mu_j\Bigg\}\Bigg\}, \end{aligned} \tag{68}$$

where R_i, I, and V_μ for $\mu \in I$ are given in Lemma 11. For $i, k \in V(C)$ for some $C \in C(H)$, we denote

$$\lambda_{i\to k} = \max\left\{\lambda_{i_0,i_1} \cdots \lambda_{i_{N-1},i_N} : [i_k]_0^N \in \left\{i \xrightarrow{C} k\right\}\right\}, \tag{69}$$

where $\{i \xrightarrow{C} k\}$ is given in (61). Observe that $\lambda_{i\to k} \leq \lambda$.

Let $A, B \in X$ be such that $(A,B) \in E(G)$ and $U \in F(A)$. We look for $\widetilde{U} \in F(B)$ such that $(U,\widetilde{U}) \in E(G)$ and $d(U,\widetilde{U}) \leq \lambda d(A,B)$.

Step 1 ($i \in V^e$). Let $\mu \in I$ be such that $i \in V_\mu$.

Case 1 ($U_i = \emptyset$ and $\widetilde{U}_i \neq \emptyset$ for every $\widetilde{U} \in F(B)$). In this case, $E_i(A) = \emptyset$ and $E_i(B) \neq \emptyset$ by (66). Choose some $(i,j) \in E_i(B)$. Therefore, $A_j = \emptyset$, $B_j \neq \emptyset$, and for $\nu \in I$ such that $j \in V_\nu$, one has $\mu < \nu$.

By condition (G)(ii)(a), if $j \in V^e$, there exists $l \in V(H)$ such that $(j,l) \in E(H)$ and $A_l \neq \emptyset$. So, $(j,l) \in E_j(A)$.

On the other hand, if $j \in V(C)$ for some $C \in C(H)$, by condition (G)(ii)(b), there exist $k \in V(C)$ and $l \in V(H)$ such that $(k,l) \in E(H)$ and $A_l \neq \emptyset$. So, $(k,l) \in E_C(A)$ and $j,k \in V(C)$.

So, for the case $j \in V^e$ and the case $j \in V^c$, we obtain by (66) and (67),

$$U_i = \emptyset, \quad \widetilde{U}_i \neq \emptyset, \quad U_j \neq \emptyset \quad \text{for some } (i,j) \in V(H). \tag{70}$$

Moreover, by (21), (22), and (68),

$$\begin{aligned} & \overline{D}_i(U_i,\widetilde{U}_i) \\ & = R_i = \frac{R_i}{R_j}\overline{D}_j(A_j,B_j) \leq \lambda d(A,B) \quad \forall \widetilde{U} \in F(B). \end{aligned} \tag{71}$$

Case 2 ($U_i \neq \emptyset$ and $\widetilde{U}_i = \emptyset$ for every $\widetilde{U} \in F(B)$). In this case, $E_i(A) \neq \emptyset$ and $E_i(B) = \emptyset$ by (66). Choose some $(i,j) \in E_i(A)$. Therefore, $A_j \neq \emptyset$, $B_j = \emptyset$, and for $\nu \in I$ such that $j \in V_\nu$, one has $\mu < \nu$. By conditions (G)(i) and (G)(iii), one has $j \in V^e$ and $B_i \neq \emptyset$. By (66), (67) and since $B_i \neq \emptyset$, one has

$$\begin{aligned} & U_i \neq \emptyset, \quad \widetilde{U}_i = \emptyset \text{ and one of the following situations hold:} \\ & \quad \text{(i) there is no } k \in V(H) \text{ such that } (k,i) \in E(H); \\ & \quad \text{(ii) } \forall k \in V(H) \text{ such that } (k,i) \in E(H), \\ & \qquad \widetilde{U}_k \neq \emptyset \quad \forall \widetilde{U} \in F(B). \end{aligned} \tag{72}$$

Also, by (21), (22), and (68),

$$\begin{aligned} & \overline{D}_i(U_i,\widetilde{U}_i) \\ & = R_i = \frac{R_i}{R_j}\overline{D}_j(A_j,B_j) \leq \lambda d(A,B) \quad \forall \widetilde{U} \in F(B). \end{aligned} \tag{73}$$

Case 3 ($U_i \neq \emptyset$ and $\widetilde{U}_i \neq \emptyset$ for every $\widetilde{U} \in F(B)$). In this case, $U_i = U_i^e(A,P)$ for some $\emptyset \neq P \subset E_i(A)$, and $E_i(B) \neq \emptyset$ by (66).

If $P \subset E_i(B)$, one has by (21), (59), and (68),

$$\begin{aligned} &D_i\left(U_i^e(A,P), U_i^e(B,P)\right) \\ &\quad = D_i\left(\bigcup_{(i,j)\in P} T_{i,j}(A_j), \bigcup_{(i,j)\in P} T_{i,j}(B_j)\right) \\ &\quad \leq \max_{(i,j)\in P} D_i\left(T_{i,j}(A_j), T_{i,j}(B_j)\right) \\ &\quad \leq \max_{(i,j)\in P} \lambda_{i,j} D_j\left(A_j, B_j\right) \leq \lambda d(A,B). \end{aligned} \tag{74}$$

If $P \not\subset E_i(B)$, choose some $(i,j) \in P \setminus E_i(B)$. So, $A_j \neq \emptyset$, $B_j = \emptyset$, and, for $\nu \in I$ such that $j \in V_\nu$, one has $\mu < \nu$. Thus, by (21), (22), and (68),

$$\overline{D}_i\left(U_i, \widetilde{U}_i\right) \leq R = \frac{R}{R_j}\overline{D}_j\left(A_j, B_j\right) \leq \lambda d(A,B) \quad \forall \widetilde{U} \in F(B). \tag{75}$$

Combining (74) and (75), for $U_i = U_i^e(A,P)$ for some $P \subset E_i(A)$, we choose $\widetilde{U}_i \in F_i(B)$ such that

$$\widetilde{U}_i = \begin{cases} U_i^e(B,P), & \text{if } P \subset E_C(A) \cap E_C(B), \\ \widetilde{U}_i, & \text{otherwise, with some } \widetilde{U}_i \in F_i(B); \end{cases} \tag{76}$$

and we get

$$\overline{D}_i\left(U_i, \widetilde{U}_i\right) \leq \lambda d(A,B). \tag{77}$$

Step 2 ($i \in V(C)$ for some $C \in C(H)$). Let $\mu \in I$ be such that $i \in V_\mu$.

Case 4 ($U_i = \emptyset$ and $\widetilde{U}_i \neq \emptyset$ for every $\widetilde{U} \in F(B)$). In this case, $A_i = E_C(A) = \emptyset$ and $B_i \cup E_C(B) \neq \emptyset$ by (67).

If $B_i \neq \emptyset$, by condition (G)(ii)(b), there exist $k \in V(C)$ and $j \in V(H)$ such that $(k,j) \in E(H)$ and $A_j \neq \emptyset$. So, $(k,j) \in E_C(A)$. This contradicts the fact that $E_C(A) = \emptyset$.

If $E_C(B) \neq \emptyset$, by (60), there exist $k \in V(C)$ and $j \in V(H) \setminus V(C)$ such that $(k,j) \in E(H)$ and $B_j \neq \emptyset$ and, for $\nu \in I$ such that $j \in V_\nu$, one has $\mu < \nu$. Since $E_C(A) = \emptyset$, $A_j = \emptyset$. If $j \in V^e$, by condition (G)(ii)(a), there exists $l \in V(H)$ such that $(j,l) \in E(H)$ and $A_l \neq \emptyset$. So, $E_j(A) \neq \emptyset$, and $U_j \neq \emptyset$ by (66). On the other hand, if $j \in V(\widehat{C})$ for some $\widehat{C} \in C(H)$, by condition (G)(ii)(b), there exist $m \in V(\widehat{C})$, $l \in V(H)$ such that $(m,l) \in E(H)$ and $A_l \neq \emptyset$. So, $E_{\widehat{C}}(A) \neq \emptyset$ and $U_j \neq \emptyset$ by (67). Thus, for the case $j \in V^e$ and the case $j \in V^c$, we obtain

$$U_i = \emptyset, \quad \widetilde{U}_i \neq \emptyset, \quad U_j \neq \emptyset \quad \text{for some } (k,j) \in E_C(B). \tag{78}$$

Moreover, by (21), (22), and (68),

$$\begin{aligned} &\overline{D}_i\left(U_i, \widetilde{U}_i\right) \\ &\quad = R_i = \frac{R_i}{R_j}\overline{D}_j\left(A_j, B_j\right) \leq \lambda d(A,B) \quad \forall \widetilde{U} \in F(B). \end{aligned} \tag{79}$$

Case 5 ($U_i \neq \emptyset$ and $\widetilde{U}_i = \emptyset$ for every $\widetilde{U} \in F(B)$). In this case, $A_i \cup E_C(A) \neq \emptyset$ and $B_i \cup E_C(B) = \emptyset$ by (67). From condition (G)(iii), we deduce that $A_i = B_i = \emptyset$. Let $(k,j) \in E_C(A)$. One has $A_j \neq \emptyset$ and $B_j = \emptyset$ since $(k,j) \notin E_C(B)$. By condition (G)(iii), $j \in V^e$ and $B_k \neq \emptyset$ since $(k,j) \in E(H)$. This implies that $B_i \neq \emptyset$ by condition (Xii) since $i,k \in V(C)$. This is a contradiction. Thus,

$$U_i \neq \emptyset, \quad \widetilde{U}_i = \emptyset \quad \forall \widetilde{U} \in F(B) \ \text{ is impossible.} \tag{80}$$

Case 6 ($U_i \neq \emptyset$ and $\widetilde{U}_i \neq \emptyset$ for every $\widetilde{U} \in F(B)$). In this case, $A_i \cup E_C(A) \neq \emptyset$ and $B_i \cup E_C(B) \neq \emptyset$ by (67).

If $A_i \neq \emptyset$, by condition (G)(iii), $B_i \neq \emptyset$. So $W_i(A) \neq \emptyset$, $W_i(B) \neq \emptyset$, and, by (21), (64), and (68),

$$\begin{aligned} &D_i\left(W_i(A), W_i(B)\right) \\ &\quad = D_i\left(\bigcup_{(i,j)\in E(C)} T_{i,j}(A_j), \bigcup_{(i,j)\in E(C)} T_{i,j}(B_j)\right) \\ &\quad \leq \max_{(i,j)\in E(C)} D_i\left(T_{i,j}(A_j), T_{i,j}(B_j)\right) \\ &\quad \leq \max_{(i,j)\in E(C)} \lambda_{i,j} D_j\left(A_j, B_j\right) \\ &\quad \leq \lambda \max_{(i,j)\in E(C)} D_j\left(A_j, B_j\right) \leq \lambda d(A,B). \end{aligned} \tag{81}$$

If $E_C(A) \neq \emptyset$, for $\emptyset \neq P \subset E_C(A)$ such that $P \subset E_C(B)$, one has by (21), (62), (63), (68), and (69),

$$\begin{aligned} &D_i\left(U_i^c(A,P), U_i^c(B,P)\right) \\ &\quad = D_i\left(\bigcup_{(k,j)\in P} T_{i\to k}\circ T_{k,j}(A_j), \bigcup_{(k,j)\in P} T_{i\to k}\circ T_{k,j}(B_j)\right) \\ &\quad \leq \max_{(k,j)\in P} \lambda_{i\to k} D_k\left(T_{k,j}(A_j), T_{k,j}(B_j)\right) \\ &\quad \leq \max_{(k,j)\in P} \lambda_{i\to k}\lambda_{k,j} D_j\left(A_j, B_j\right) \\ &\quad \leq \lambda \max_{(k,j)\in P} D_j\left(A_j, B_j\right) \leq \lambda d(A,B). \end{aligned} \tag{82}$$

If $P \subset E_C(A)$ and $P \not\subset E_C(B)$, there exists $(k,j) \in P$ such that $A_j \neq \emptyset$, $B_j = \emptyset$ and, for $\nu \in I$ such that $j \in V_\nu$, one has $\mu < \nu$. Hence, by (21), (22), and (68),

$$\overline{D}_i\left(U_i, \widetilde{U}_i\right) \leq R = \frac{R}{R_j}\overline{D}_j\left(A_j, B_j\right) \leq \lambda d(A,B) \quad \forall \widetilde{U} \in F(B). \tag{83}$$

Combining (67), (81), (82), and (83), we choose $\widetilde{U}_i \in F_i(B)$ such that

$$\widetilde{U}_i = \begin{cases} W_i(B), & \text{if } U_i = W_i(A), \\ U_i^c(B,P), & \text{if } U_i = U_i^c(A,P) \\ & \text{for } \emptyset \neq P \subset E_C(A) \cap E_C(B), \\ W_i(B) \cup U_i^c(B,P), & \text{if } U_i = W_i(A) \cup U_i^c(A,P) \\ & \text{for } \emptyset \neq P \subset E_C(A) \cap E_C(B), \\ \widetilde{U}_i, & \text{otherwise}, \\ & \text{with some } \widetilde{U}_i \in F_i(B); \end{cases} \tag{84}$$

and we get

$$\overline{D}_i\left(U_i, \widetilde{U}_i\right) \leq \lambda d(A,B). \tag{85}$$

Step 3 (choice of an appropriate $\widetilde{U} \in F(B)$). Finally, we choose $\widetilde{U} \in F(B)$ as follows:

$$\widetilde{U}_i = \begin{cases} \emptyset, & \text{if } i \in V^e, E_i(B) = \emptyset, \\ \text{some } \widetilde{U}_i \in F_i(B), & \text{if } i \in V^e, U_i = \emptyset, \\ & E_i(B) \neq \emptyset, \\ \widetilde{U}_i \text{ given by (76)}, & \text{if } i \in V^e, U_i \neq \emptyset, \\ & E_i(B) \neq \emptyset, \\ \emptyset, & \text{if } i \in V(C), \\ & B_i \cup E_C(B) = \emptyset, \\ \text{some } \widetilde{U}_i \in F_i(B), & \text{if } i \in V(C), U_i = \emptyset, \\ \widetilde{U}_i \text{ given by (84)}, & \text{if } i \in V(C), U_i \neq \emptyset, \\ & B_i \cup E_C(B) \neq \emptyset. \end{cases} \tag{86}$$

It follows from (70), (72), (78), and (80) that

$$\left(U, \widetilde{U}\right) \in E(G). \tag{87}$$

Finally, from (71), (73), (77), (79), and (85), we deduce that

$$d\left(U, \widetilde{U}\right) \leq \lambda d(A,B). \tag{88}$$

Therefore, F is a G-contraction. □

Here is another property satisfied by the multivalued map F.

Lemma 15. *Let $F : X \to X$ be the multivalued map defined above. Then, for every $A^0 \in X$ and every $\{A^n\}$ G_1-Picard trajectory from A^0 converging to some $A \in X$, there exists $N \in \mathbb{N}$ such that $(A^n, A) \in E(G)$ for all $n > N$.*

Proof. Let $A^0 \in X$ and $\{A^n\}$ a G_1-Picard trajectory from A^0 such that $A^n \to A$. Thus, there exists $N \in \mathbb{N}$ such that $d(A^n, A) < R$ for all $n > N$. So, by (21) and (22), $A^n = (A_1^n, \ldots, A_p^n)$ and $A = (A_1, \ldots, A_p)$ are such that, for all $n > N$ and all $i \in V(H)$, $A_i^n = \emptyset$ if and only if $A_i = \emptyset$. Thus, (G)(i) is satisfied and $(A^n, A) \in E(G)$ for all $n > N$. □

5. Attractor of an H-IFS and Elements of $C(H)$

For $H = (V(H), E(H))$ an MW-directed graph, and $\{T_{i,j}\}_H$ a graph-directed iterated function system over the graph H, we consider K the attractor of this H-IFS insured by Theorem 6. We want to get more information on K by taking into account the connected components of H.

Theorem 16. *Let $H = (V(H), E(H))$ be an MW-directed graph. Let $\{T_{i,j}\}_H$ be an H-IFS and K its attractor. Then the following statements hold.*

(1) *For every $C \in C(H)$, there exists $K^+(C) \subset K$ such that*

  (a) *$K_i^+(C) \neq \emptyset$ for every $i \in V(C)$;*
  (b) *$K_i^+(C) \neq \emptyset$ for every $i \in [C]_\leftarrow$, where $[C]_\leftarrow$ is defined in (19).*
  (c) *$K_i^+(C) = \emptyset$ for every $i \notin [C]_\leftarrow$.*

(2) *If $C_1, C_2 \in C(H)$ are such that $C_1 \preceq C_2$, then $K^+(C_1) \subset K^+(C_2)$.*

(3) *If $C_1, C_2 \in C(H)$ are incomparable, then*

$$K_i^+(C_1) \cap K_i^+(C_2) = \emptyset \quad \forall i \notin \left([C_1]_\leftarrow\right) \cap \left([C_2]_\leftarrow\right). \tag{89}$$

(4) *There exists $K^- \in X$ such that $K^- \subset K$ and*

  (a) *for every $C \in C(H)$, $K_i^- = K_i^+(C)$ for every $i \in V(C)$ and $K_i^- \subset K_i^+(C)$ for every $i \in [C]_\leftarrow$;*
  (b) *if $C_1, C_2 \in C(H)$ are such that $C_1 \preceq C_2$, then*

$$K_i^- \subset K_i^+(C_1) \subset K_i^+(C_2) \quad \forall i \in [C_1]_\leftarrow; \tag{90}$$

  (c) *if $C_1, C_2 \in C(H)$ are incomparable, then,*

$$K_i^- \subset K_i^+(C_1) \cap K_i^+(C_2) \quad \forall i \in \left([C_1]_\leftarrow\right) \cap \left([C_2]_\leftarrow\right). \tag{91}$$

Proof. (1) Let $F : X \to X$ be the multivalued map defined in (65), (66), and (67). We know that F is a G-contraction by Proposition 14. Also, it follows from Lemma 15 that F satisfies condition (ii) of Theorem 9.

Theorem 6 and the definition of F imply that fixed points of F are included in K.

Let $C \in C(H)$. We want to show that there exists $K^+(C)$ a fixed point of F satisfying the required properties. Fix

$$A^0 = \left(A_1^0, \ldots, A_p^0\right) \in X \quad \text{such that } A_i^0 \neq \emptyset \Longleftrightarrow i \in V(C). \tag{92}$$

For $n \in \mathbb{N} \cup \{0\}$, we choose inductively

$$A^{n+1} \in F\left(A^n\right) \text{ the biggest element of } F\left(A^n\right). \tag{93}$$

That is, by (66) and (67), $A^{n+1} = (A_1^{n+1}, \ldots, A_p^{n+1}) \in F(A^n)$ is chosen as follows.

For $i \in V^e$,

$$A_i^{n+1} = \begin{cases} \emptyset, & \text{if } E_i(A^n) = \emptyset; \\ U_i^e(A^n, E_i(A^n)), & \text{if } E_i(A^n) \neq \emptyset, \end{cases} \tag{94}$$

where E_i^e and U_i^e are defined in (58) and (59), respectively.

For $i \in V(\widehat{C})$ for some $\widehat{C} \in C(H)$,

$$A_i^{n+1} = \begin{cases} \emptyset, & \text{if } A_i^n = E_{\widehat{C}}(A^n) = \emptyset; \\ U_i^c(A^n, E_{\widehat{C}}(A^n)), & \text{if } A_i^n = \emptyset, E_{\widehat{C}}(A^n) \neq \emptyset; \\ W_i(A^n) \cup U_i^c(A^n, E_{\widehat{C}}(A^n)), & \text{if } A_i^n \neq \emptyset, \end{cases} \tag{95}$$

where $E_{\widehat{C}}$, U_i^c, and W_i are defined in (60), (63), and (64), respectively.

Arguing as in the proof of Proposition 14, one has that $(A^{n-1}, A^n) \in E(G)$ and

$$d(A^n, A^{n+1}) \leq \lambda d(A^{n-1}, A^n) \quad \forall n \in \mathbb{N}. \tag{96}$$

By Theorem 9, $\{A^n\}$ is a G_1-Picard trajectory converging to some $K^+(C) \in X$ a fixed point of F.

Observe that, for every $n \in \mathbb{N}$ and every $i \in V(C)$, $A_i^n \neq \emptyset$. Therefore,

$$K_i^+(C) \neq \emptyset \quad \forall i \in V(C). \tag{97}$$

Similarly, observe that, by construction, $A_i^n = \emptyset$ for every $i \notin [C]_{\leftarrow}$. Indeed, for such i, $E_i(A^{n-1}) = \emptyset$ if $i \in V^e$, and $A_i^{n-1} = E_{\widehat{C}}(A^{n-1}) = \emptyset$ if $i \in V(\widehat{C})$ for some $\widehat{C} \in V(C)$. Thus,

$$K_i^+(C) = \emptyset \quad \forall i \notin [C]_{\leftarrow}. \tag{98}$$

On the other hand, let

$$N_C = \max_{i \in [C]_{\leftarrow}} \left\{ \min \left\{ N : i = i_0, i_N \in V(C), \ [i_k]_0^N \text{ is a path in } H \text{ from } i \text{ to } i_N \right\} \right\}. \tag{99}$$

Again by construction, $A_i^n \neq \emptyset$ for all $n > N_C$, for all $i \in [C]_{\leftarrow}$. So,

$$K_i^+(C) \neq \emptyset \quad \forall i \in [C]_{\leftarrow}. \tag{100}$$

Finally, observe that $K^+(C)$ is independent of $A^0 \subset X$ chosen as in (92). Indeed, for

$$\widetilde{A}^0 = (\widetilde{A}_1^0, \dots, \widetilde{A}_p^0) \in X \quad \text{such that } \widetilde{A}_i^0 \neq \emptyset \Longleftrightarrow i \in V(C), \tag{101}$$

we define inductively $\widetilde{A}^{n+1} \in F(\widetilde{A}^n)$ as in (93). Observe that $(A^n, \widetilde{A}^n) \in E(G)$ for all $n \in \mathbb{N} \cup \{0\}$. Arguing as in Proposition 14, one has

$$d(A^{n+1}, \widetilde{A}^{n+1}) \leq \lambda d(A^n, \widetilde{A}^n) \quad \forall n \in \mathbb{N}. \tag{102}$$

This inequality combined with the fact that $A^n \to K^+(C)$ implies that $\widetilde{A}^n \to K^+(C)$.

(2) Let $C_1, C_2 \in C(H)$ be such that $C_1 \preceq C_2$. One has

$$\{i \in [C_1]_{\leftarrow}\} \subset \{i \in [C_2]_{\leftarrow}\}. \tag{103}$$

Let $B^0 = (B_1^0, \dots, B_p^0) \in X$ be such that

$$B_j^0 = \begin{cases} K_j^+(C_2), & \text{if } j \in [C_1]_{\leftarrow}, \\ \emptyset, & \text{if } j \notin [C_1]_{\leftarrow}. \end{cases} \tag{104}$$

By (1) and (G)(i), one has $(K^+(C_1), B^0) \in E(G)$ and $K^+(C_1) \in F(K^+(C_1))$. Let B^1 be the biggest element in $F(B^0)$; that is, B^1 is chosen similarly to (94) and (95). Observe that $B^1 \subset K^+(C_2)$, since $B^0 \subset K^+(C_2)$, $K^+(C_2) \in F(K^+(C_2))$, and by the definitions of F and $K^+(C_2)$. Arguing as in the proof of Proposition 14, one has $(K^+(C_1), B^1) \in E(G)$ and

$$d(K^+(C_1), B^1) \leq \lambda d(K^+(C_1), B^0). \tag{105}$$

Repeating this argument, we obtain $\{B^n\}$ a G_1-Picard trajectory from B^0 such that

$$B^n \subset K^+(C_2), \qquad d(K^+(C_1), B^n) \leq \lambda^n d(K^+(C_1), B^0) \quad \forall n \in \mathbb{N}. \tag{106}$$

Therefore, $B^n \to K^+(C_1)$ and

$$K^+(C_1) \subset K^+(C_2). \tag{107}$$

(3) If $C_1, C_2 \in C(H)$ are incomparable, it follows directly from (1)(c) that

$$K_i^+(C_1) \cap K_i^+(C_2) = \emptyset \quad \forall i \notin ([C_1]_{\leftarrow}) \cap ([C_2]_{\leftarrow}). \tag{108}$$

(4) For every $C \in C(H)$, $C = (V(C), E(C))$ is an MW-directed graph and

$$\{T_{i,j} : (i,j) \in E(C)\} \tag{109}$$

is a graph-directed iterated function system over the graph C. Let

$$K^-(C) = (K_i^-)_{i \in V(C)} \tag{110}$$

be the attractor of this graph-directed iterated system insured by Theorem 6.

We define $K^- \in X$ by

$$K^- = (K_1^-, \dots, K_p^-), \text{ where}$$
$$K_i^- = \begin{cases} K_i^-(C), & \text{if } i \in V(C) \text{ for some } C \in C(H), \\ \emptyset, & \text{if } i \in V^e. \end{cases} \tag{111}$$

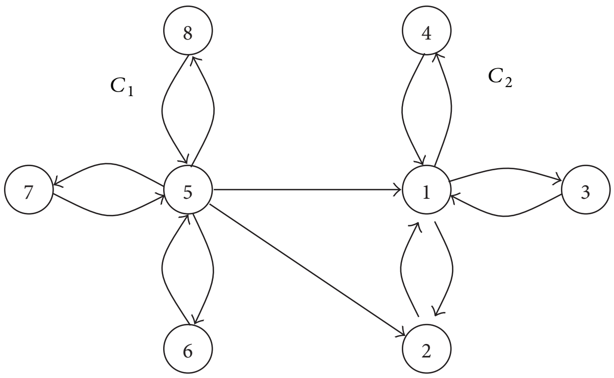

FIGURE 3: An MW-graph H with $C(H) = \{C_1, C_2\}$.

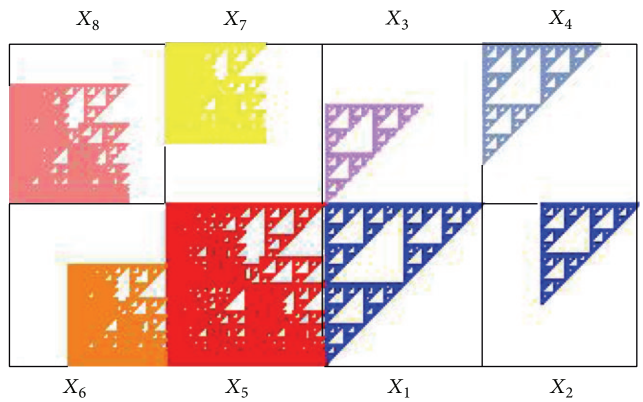

FIGURE 4: The set $K^+(C_2)$.

Let $C \in C(H)$ and $\{A^n\}$ the G_1-Picard trajectory from A^0 defined in (92) and (93). By (95), for all $n \in \mathbb{N}$, $E_C(A^{n-1}) = \emptyset$ and $A_i^n = W_i(A^{n-1})$ for all $i \in V(C)$. So, using (64) and (67) and the fact that $A_i^n \to K_i^+(C) \in F_i(K^+(C))$ for every $i \in V(C)$, we deduce that

$$K_i^+(C) = \bigcup_{(i,j)\in E(C)} T_{i,j}\left(K_j^+(C)\right) \quad \forall i \in V(C). \tag{112}$$

By definition of K^-,

$$K_i^- = \bigcup_{(i,j)\in E(C)} T_{i,j}\left(K_j^-\right) \quad \forall i \in V(C). \tag{113}$$

The uniqueness of the fixed point of this operator implies that

$$K_i^+(C) = K_i^- \quad \forall i \in V(C). \tag{114}$$

On the other hand, if $i \in V^e \cap [C]_{\leftarrow}$, one has $\emptyset = K_i^- \subset K_i^+(C)$. If $i \in V(\widehat{C}) \cap [C]_{\leftarrow}$ for some $C \neq \widehat{C} \in C(H)$, then $\widehat{C} \preceq C$. It follows from (114) and (2) that $K_i^- = K_i^+(\widehat{C}) \subset K_i^+(C)$.

The properties (4)(b) and (4)(c) follow directly from (2) and (4)(a). □

Example 17. Let H be the MW-graph of Figure 3.

We consider the H-IFS, $\{T_{i,j}\}_H$, with the metric spaces:

$$\begin{aligned} X_1 &= [1,2]\times[0,1], & X_2 &= [2,3]\times[0,1], \\ X_3 &= [1,2]\times[1,2], & X_4 &= [2,3]\times[1,2], \\ X_5 &= [0,1]\times[0,1], & X_6 &= [-1,0]\times[0,1], \\ X_7 &= [0,1]\times[1,2], & X_8 &= [-1,0]\times[1,2], \end{aligned} \tag{115}$$

and the contractions:

$$\begin{aligned} T_{1,2}(x) &= M_1 x + \left(\frac{-2}{5}, \frac{1}{5}\right), & T_{1,3}(x) &= M_1 x + \left(\frac{1}{5}, \frac{-4}{5}\right), \\ T_{1,4}(x) &= M_3 x + \left(\frac{-1}{3}, \frac{-1}{3}\right), & T_{2,1}(x) &= M_2 x + \left(\frac{14}{8}, \frac{3}{8}\right), \\ T_{3,1}(x) &= M_2 x + \left(\frac{3}{8}, 1\right), & T_{4,1}(x) &= M_4 x + \left(\frac{5}{4}, \frac{5}{4}\right), \\ T_{5,1}(x) &= M_4 x + \left(\frac{-2}{4}, \frac{1}{4}\right), & T_{5,2}(x) &= M_3 x + (-1, 0), \\ T_{5,6}(x) &= M_1 x + (1, 0), & T_{5,7}(x) &= M_1 x + \left(0, \frac{-3}{5}\right), \\ T_{5,8}(x) &= M_3 x + \left(\frac{2}{3}, \frac{-2}{3}\right), & T_{6,5}(x) &= M_2 x + \left(\frac{-5}{8}, 0\right), \\ T_{7,5}(x) &= M_2 x + \left(0, \frac{11}{8}\right), & T_{8,5}(x) &= M_4 x + (-1, 1), \end{aligned} \tag{116}$$

where

$$\begin{aligned} M_1 &= \begin{pmatrix} \frac{4}{5} & 0 \\ 0 & \frac{4}{5} \end{pmatrix}, & M_2 &= \begin{pmatrix} \frac{5}{8} & 0 \\ 0 & \frac{5}{8} \end{pmatrix}, \\ M_3 &= \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix}, & M_4 &= \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix}. \end{aligned} \tag{117}$$

Figures 4 and 5 present $K^+(C_2)$ and K^-, respectively.

6. Attractor of an H-IFS and Subsets of $C(H)$

We obtain other pieces of information on the attractor of the graph-directed iterated function system by considering subsets of $C(H)$.

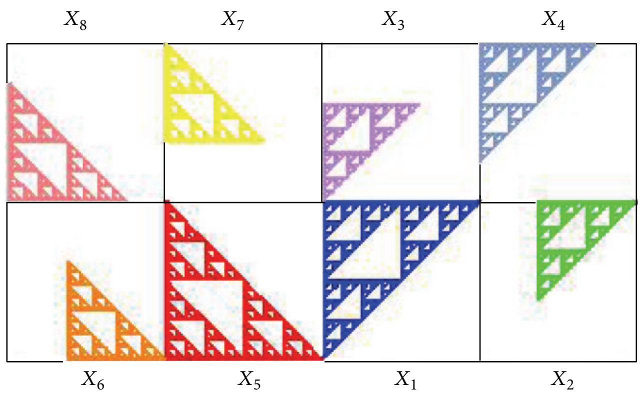

FIGURE 5: The set K^-.

Theorem 18. *Let $H = (V(H), E(H))$ be an MW-directed graph. Let $\{T_{i,j}\}_H$ be an H-IFS and K its attractor. Then the following statements hold:*

(1) *for every $\mathcal{S} \subset C(H)$, there exists $K^+(\mathcal{S}) \subset K$ such that*

  (a) *$K^+(C) \subset K^+(\mathcal{S})$ for every $C \in \mathcal{S}$;*
  (b) *$K_i^+(C) = K_i^+(\mathcal{S})$ for every $i \in V(C)$ and every maximal element $C \in \mathcal{S}$;*
  (c) *$K_i^+(\mathcal{S}) \neq \emptyset$ if and only if $i \in \bigcup_{C\in\mathcal{S}}[C]_{\leftarrow}$*

(2) *if $\mathcal{S}_1, \mathcal{S}_2 \subset C(H)$ are such that, for every $C_1 \in \mathcal{S}_1$, there exists $C_2 \in \mathcal{S}_2$ such that $C_1 \preceq C_2$, then $K^+(\mathcal{S}_1) \subset K^+(\mathcal{S}_2)$,*

(3) *for $\mathcal{S}_1, \mathcal{S}_2 \subset C(H)$, one has*

$$K^+(\mathcal{S}_1) \cap K^+(\mathcal{S}_2) = \emptyset$$
$$\text{if } \left(\bigcup_{C\in\mathcal{S}_1}[C]_{\leftarrow}\right) \cap \left(\bigcup_{C\in\mathcal{S}_2}[C]_{\leftarrow}\right) = \emptyset \tag{118}$$

(4) *the attractor K is such that $K = K^+(C(H))$.*

Proof. (1) By Proposition 14 and Lemma 15, the map $F : X \to X$ defined in (65), (66), and (67) is a G-contraction satisfying condition (ii) of Theorem 9. Also, from the proof of Theorem 16, we know that fixed points of F are included in K.

Let $\mathcal{S} \subset C(H)$. We want to show that there exists $K^+(\mathcal{S})$ a fixed point of F satisfying the required properties. Fix

$$\widehat{A}^0 = \left(\widehat{A}_1^0, \ldots, \widehat{A}_p^0\right) \in X \text{ such that } \widehat{A}_i^0 \neq \emptyset \Longleftrightarrow i \in \bigcup_{C\in\mathcal{S}} V(C),$$
$$\widehat{A}_i^0 = A_i^0 \text{ if } i \in V(C) \text{ for } C \in \mathcal{S}, \text{ where}$$
$$A^0 \text{ is defined in (92)}. \tag{119}$$

For $n \in \mathbb{N} \cup \{0\}$, we choose inductively

$$\widehat{A}^{n+1} \in F\left(\widehat{A}^n\right) \quad \text{the biggest element of } F\left(\widehat{A}^n\right). \tag{120}$$

Arguing as in the proof of Theorem 16, one deduces that $\{\widehat{A}^n\}$ is a G_1-Picard trajectory converging to some $K^+(\mathcal{S}) \in X$ a fixed point of F. Also, $K^+(\mathcal{S})$ is independent of $\widehat{A}^0$ chosen as in (119).

For $C \in \mathcal{S}$, observe that $A^n \subset \widehat{A}^n$ for all $n \in \mathbb{N} \cup \{0\}$, where A^n is defined in (92) and (93). Since $\widehat{A}^n \to K^+(\mathcal{S})$ and $A^n \to K^+(C)$, we deduce that

$$K^+(C) \subset K^+(\mathcal{S}). \tag{121}$$

It follows from this inclusion and Theorem 16(1)(b) that

$$K_i^+(\mathcal{S}) \neq \emptyset \quad \forall i \in \bigcup_{C\in\mathcal{S}}[C]_{\leftarrow}. \tag{122}$$

On the other hand,

$$\widehat{A}_i^n = \emptyset \quad \forall i \notin \bigcup_{C\in\mathcal{S}}[C]_{\leftarrow}, \ \forall n \in \mathbb{N}. \tag{123}$$

Thus, (1)(c) holds.

In the particular case where $C \in \mathcal{S}$ is maximal, one has

$$A_i^n = \widehat{A}_i^n \quad \forall i \in V(C), \ \forall n \in \mathbb{N} \cup \{0\}, \tag{124}$$

where A^n is defined in (93). Since

$$A_i^n \longrightarrow K_i^+(C), \qquad \widehat{A}_i^n \longrightarrow K_i^+(\mathcal{S}), \tag{125}$$

one has

$$K_i^+(C) = K_i^+(\mathcal{S}) \quad \forall i \in V(C). \tag{126}$$

((2) and (3)) The proofs are, respectively, analogous to those of (2) and (3) in Theorem 16.

(4) Let $\mathcal{S} = C(H)$. Since $K^+(C(H))$ is independent of the choice of $\widehat{A}^0$ in (119), we can fix

$$\widehat{A}^0 = \left(\widehat{A}_1^0, \ldots, \widehat{A}_p^0\right) \in X \text{ such that}$$
$$\widehat{A}_i^0 = \begin{cases} K_i, & \text{if } i \in V^c, \\ \emptyset, & \text{if } i \in V^e, \end{cases} \tag{127}$$

where V^c and V^e are defined in (24) and (25), respectively. Let $\widehat{A}^n$ be defined as in (120). We know that $\widehat{A}^n \to K^+(C(H))$. On the other hand, since K is the unique attractor of this H-IFS obtained in Theorem 6, we deduce that $K = K^+(C(H))$. □

In the following result, we see that the maximal elements of $C(H)$ play a key role.

Corollary 19. *Let $H = (V(H), E(H))$ be an MW-directed graph and $\{T_{i,j}\}_H$ an H-IFS. Then, for every $\mathcal{S}_1, \mathcal{S}_2 \subset C(H)$ such that*

$$\{C \in \mathcal{S}_1 : C \text{ is a maximal element of } \mathcal{S}_1\}$$
$$= \{C \in \mathcal{S}_2 : C \text{ is a maximal element of } \mathcal{S}_2\}, \tag{128}$$

one has

$$K^+(\mathcal{S}_1) = K^+(\mathcal{S}_2). \tag{129}$$

Proof. Let $\mathcal{S} \subset C(H)$ and let

$$\mathcal{S}_m = \{C \in \mathcal{S} : C \text{ is a maximal element of } \mathcal{S}\}. \tag{130}$$

To conclude, it is sufficient to show that

$$K^+(\mathcal{S}) = K^+(\mathcal{S}_m). \tag{131}$$

It follows from Theorem 18(2) that

$$K^+(\mathcal{S}) \subset K^+(\mathcal{S}_m), \qquad K^+(\mathcal{S}_m) \subset K^+(\mathcal{S}). \tag{132}$$

□

7. Other Fixed Points of Our G-Contraction

In the proofs of Theorems 16 and 18, $K^+(C)$ and $K^+(\mathcal{S})$ were obtained as fixed points of the multivalued G-contraction F. In fact, much more fixed points of F can be obtained in order to get more information on the attractor K.

Let $\mathcal{S} \subset C(H)$. For a vertex $i \in V^e$, we consider the set of edges from i on a path to some vertex in $\mathcal{S}$:

$$\mathscr{E}_i(\mathcal{S}) = \begin{cases} \emptyset, & \text{if } i \notin \bigcup_{C\in\mathcal{S}} [C]_{\leftarrow}, \\ \left\{(i,j) \in E(H) : i, j \in \bigcup_{C\in\mathcal{S}} [C]_{\leftarrow}\right\}, & \text{otherwise.} \end{cases} \tag{133}$$

Similarly, for $\widehat{C} \in C(H)$, we consider

$$\mathscr{E}_{\widehat{C}}(\mathcal{S}) = \begin{cases} \emptyset, & \text{if } V(\widehat{C}) \not\subset \bigcup_{C\in\mathcal{S}} [C]_{\leftarrow}, \\ \left\{(i,j) \in E(H) : i \in V(\widehat{C}), \right. \\ \left. \quad j \notin V(\widehat{C}), j \in \bigcup_{C\in\mathcal{S}} [C]_{\leftarrow}\right\}, & \text{otherwise.} \end{cases} \tag{134}$$

Finally, we consider suitable subsets of edges on paths in H reaching $\mathcal{S}$, that is, subsets of $\mathscr{E}_i(\mathcal{S})$ and $\mathscr{E}_{\widehat{C}}(\mathcal{S})$,

$$\begin{aligned} \mathcal{Q}(\mathcal{S}) = \Big\{ & Q = (Q_i)_{i\in V^e} \times (Q_{\widehat{C}})_{\widehat{C}\in C(H)} : Q_{\widehat{C}} \subset \mathscr{E}_{\widehat{C}}(\mathcal{S}) \\ & \forall \widehat{C} \in C(H),\ \forall i \in V^e, \\ & Q_i \subset \mathscr{E}_i(\mathcal{S}),\ Q_i \neq \emptyset \text{ if } \mathscr{E}_i(\mathcal{S}) \neq \emptyset \Big\}. \end{aligned} \tag{135}$$

Using $\mathcal{Q}(\mathcal{S})$, we can obtain more information on $K^+(\mathcal{S})$.

Theorem 20. *Let $H = (V(H), E(H))$ be an MW-directed graph and $\{T_{i,j}\}_H$ an H-IFS. Then, the following statements hold.*

(1) *For every $\mathcal{S} \subset C(H)$ and every $Q \in \mathcal{Q}(\mathcal{S})$, there exists $K(\mathcal{S}, Q) \in X$ such that*

   (a) *$K(\mathcal{S}, Q) \subset K^+(\mathcal{S})$;*

   (b) *$K_i(\mathcal{S}, Q) \neq \emptyset$ if and only if $i \in \bigcup_{C\in\mathcal{S}} [C]_{\leftarrow}$;*

   (c) *$K_i(\mathcal{S}, Q) = K_i^+(\mathcal{S})$ for every $i \in V(C)$ and every $C \in \mathcal{S}$ maximal element in $\mathcal{S}$.*

(2) *For every $\mathcal{S} \subset C(H)$, if $Q, \widehat{Q} \in \mathcal{Q}(\mathcal{S})$ are such that $Q \subset \widehat{Q}$, then $K(\mathcal{S}, Q) \subset K(\mathcal{S}, \widehat{Q})$.*

(3) *Let $\mathcal{S}_1, \mathcal{S}_2 \subset C(H)$ be such that $\mathcal{S}_1 \subset \mathcal{S}_2$. If $Q \in \mathcal{Q}(\mathcal{S}_1) \cap \mathcal{Q}(\mathcal{S}_2)$, then $K(\mathcal{S}_1, Q) \subset K(\mathcal{S}_2, Q)$.*

(4) *Let $\mathcal{S}_1, \mathcal{S}_2 \subset C(H)$ be such that, for every $C_1 \in \mathcal{S}_1$, there exists $C_2 \in \mathcal{S}_2$ such that $C_1 \preceq C_2$. If $Q^1 \in \mathcal{Q}(\mathcal{S}_1)$ and $Q^2 \in \mathcal{Q}(\mathcal{S}_2)$ are such that $Q^1 \subset Q^2$, then $K(\mathcal{S}_1, Q^1) \subset K(\mathcal{S}_2, Q^2)$.*

(5) *For every $\mathcal{S} \subset C(H)$ and every $Q \in \mathcal{Q}(\mathcal{S})$, $K_i^- \subset K_i(\mathcal{S}, Q)$ for every $i \in V(\widehat{C})$ and every $\widehat{C} \in C(H)$ such that $V(\widehat{C}) \subset \bigcup_{C\in\mathcal{S}} [C]_{\leftarrow}$.*

Proof. (1) Let $Q \in \mathcal{Q}(\mathcal{S})$. From Proposition 14 and Lemma 15, $F : X \to X$ the multivalued map defined in (65), (66), and (67) is a G-contraction satisfying condition (ii) of Theorem 9. We want to show that there exists $K(\mathcal{S}, Q)$ a fixed point of F satisfying the required properties.

Fix

$$A^n(\mathcal{S}, Q) = \widehat{A}^n \in X \quad \forall n = 0, \dots, p, \tag{136}$$

where $\widehat{A}^n$ is defined in (119) and (120). From the definition of F, we can observe that

$$A^p(\mathcal{S}, Q) = \left(A_1^p(\mathcal{S}, Q), \dots, A_p^p(\mathcal{S}, Q)\right) \in X \tag{137}$$

is such that

$$A_i^p(\mathcal{S}, Q) \neq \emptyset \Longleftrightarrow i \in \bigcup_{C\in\mathcal{S}} [C]_{\leftarrow}. \tag{138}$$

Moreover, for every $i \in V^e$,

$$\begin{aligned} Q_i &\subset E_i(A^p(\mathcal{S}, Q)) \quad \forall i \in \bigcup_{C\in\mathcal{S}} [C]_{\leftarrow}, \\ Q_i &= E_i(A^p(\mathcal{S}, Q)) = \emptyset \quad \forall i \notin \bigcup_{C\in\mathcal{S}} [C]_{\leftarrow}, \end{aligned} \tag{139}$$

where $E_i(A^p(\mathcal{S}, Q))$ is defined in (58). Similarly, for every $\widehat{C} \in C(\mathrm{H})$,

$$\begin{aligned} Q_{\widehat{C}} &\subset E_{\widehat{C}}(A^p(\mathcal{S}, Q)) \quad \text{if } V(\widehat{C}) \subset \bigcup_{C\in\mathcal{S}} [C]_{\leftarrow}, \\ Q_{\widehat{C}} &= E_{\widehat{C}}(A^p(\mathcal{S}, Q)) = \emptyset \quad \text{if } V(\widehat{C}) \not\subset \bigcup_{C\in\mathcal{S}} [C]_{\leftarrow}, \end{aligned} \tag{140}$$

where $E_{\widehat{C}}(A^p(\mathcal{S}, Q))$ is defined in (60).

For $n > p$, we choose inductively

$$A^n(\mathcal{S}, Q) = \left(A_1^n(\mathcal{S}, Q), \dots, A_p^n(\mathcal{S}, Q)\right) \in F\left(A^{n-1}(\mathcal{S}, Q)\right) \tag{141}$$

with

$$A_i^n(\mathcal{S},Q) = \begin{cases} \emptyset, & \text{if } i \notin \bigcup_{C\in\mathcal{S}} [C]_{\leftarrow}, \\ U_i^e\left(A^{n-1}(\mathcal{S},Q),Q_i\right), & \text{if } i \in V^e \cap \bigcup_{C\in\mathcal{S}} [C]_{\leftarrow}, \\ W_i\left(A^{n-1}(\mathcal{S},Q)\right) & \\ \cup U_i^c\left(A^{n-1}(\mathcal{S},Q),Q_{\widehat{C}}\right), & \text{if } \widehat{C} \in C(H), \\ & i \in V(\widehat{C}) \cap \bigcup_{C\in\mathcal{S}} [C]_{\leftarrow}, \end{cases} \tag{142}$$

where U_i^e, U_i^c, and W_i are defined in (59), (63), and (64), respectively.

Arguing as in the proof of Theorem 16, one deduces that $\{A^n(\mathcal{S},Q)\}$ is a G_1-Picard trajectory converging to some $K(\mathcal{S},Q) \in X$ a fixed point of F. So, $K(\mathcal{S},Q)$ satisfies (1)(b). Again, it can be shown that $K(\mathcal{S},Q)$ is independent of $A^0(\mathcal{S},Q)$ chosen as in (136).

Observe that

$$A^n(\mathcal{S},Q) = \left(A_1^n(\mathcal{S},Q),\ldots,A_p^n(\mathcal{S},Q)\right) \subset \widehat{A}^n = \left(\widehat{A}_1^n,\ldots,\widehat{A}_p^n\right) \quad \forall n, \tag{143}$$

where $\widehat{A}^n$ is defined in (120) and $\widehat{A}^n \to K^+(\mathcal{S})$. Moreover, for every C maximal element in $\mathcal{S}$, $\mathscr{E}_C(\mathcal{S}) = \emptyset$ and

$$A_i^n(\mathcal{S},Q) = \widehat{A}_i^n \quad \forall i \in V(C). \tag{144}$$

Therefore, $K(\mathcal{S},Q)$ satisfies (1)(a),(c).

(2) Let $Q,\widehat{Q} \in \mathcal{Q}(\mathcal{S})$ be such that $Q \subset \widehat{Q}$. From (141) and (142), one sees that

$$A^n(\mathcal{S},Q) \subset A^n\left(\mathcal{S},\widehat{Q}\right) \quad \forall n \in \mathbb{N}. \tag{145}$$

Since $A^n(\mathcal{S},Q) \to K(\mathcal{S},Q)$ and $A^n(\mathcal{S},\widehat{Q}) \to K(\mathcal{S},\widehat{Q})$, one has that

$$K(\mathcal{S},Q) \subset K\left(\mathcal{S},\widehat{Q}\right). \tag{146}$$

(3) Let $\mathcal{S}_1,\mathcal{S}_2 \subset C(H)$ be such that $\mathcal{S}_1 \subset \mathcal{S}_2$ and let $Q \in \mathcal{Q}(\mathcal{S}_1) \cap \mathcal{Q}(\mathcal{S}_2)$. From (141) and (142), one sees that

$$A^n(\mathcal{S}_1,Q) \subset A^n(\mathcal{S}_2,Q) \quad \forall n \in \mathbb{N}. \tag{147}$$

Since $A^n(\mathcal{S}_1,Q) \to K(\mathcal{S}_1,Q)$ and $A^n(\mathcal{S}_2,Q) \to K(\mathcal{S}_2,Q)$, one has that

$$K(\mathcal{S}_1,Q) \subset K(\mathcal{S}_2,Q). \tag{148}$$

(4) Let $\mathcal{S}_1,\mathcal{S}_2 \subset C(H)$ be such that, for every $C_1 \in \mathcal{S}_1$, there exists $C_2 \in \mathcal{S}_2$ such that $C_1 \preceq C_2$. One has

$$\left\{i \in \bigcup_{C_1\in\mathcal{S}_1} [C_1]_{\leftarrow}\right\} \subset \left\{i \in \bigcup_{C_2\in\mathcal{S}_2} [C_2]_{\leftarrow}\right\}. \tag{149}$$

Let $Q^1 \in \mathcal{Q}(\mathcal{S}_1)$ and $Q^2 \in \mathcal{Q}(\mathcal{S}_2)$ be such that $Q^1 \subset Q^2$. Fix

$$B^p\left(\mathcal{S}_1,Q^1\right) = \left(B_1^p\left(\mathcal{S}_1,Q^1\right),\ldots,B_p^p\left(\mathcal{S}_1,Q^1\right)\right) \in X \tag{150}$$

to be such that

$$B_j^p\left(\mathcal{S}_1,Q^1\right) = \begin{cases} K_j\left(\mathcal{S}_2,Q^2\right), & \text{if } j \in \bigcup_{C_1\in\mathcal{S}_1} [C_1]_{\leftarrow}, \\ \emptyset, & \text{if } j \notin \bigcup_{C_1\in\mathcal{S}_1} [C_1]_{\leftarrow}. \end{cases} \tag{151}$$

One has $(K(\mathcal{S}_1,Q^1),B^p(\mathcal{S}_1,Q^1)) \in E(G)$ and $K(\mathcal{S}_1,Q^1) \in F(K(\mathcal{S}_1,Q^1))$. For $n = p+1$, we define

$$B^n\left(\mathcal{S}_1,Q^1\right) = \left(B_1^n\left(\mathcal{S}_1,Q^1\right),\ldots,B_p^n\left(\mathcal{S}_1,Q^1\right)\right) \in F\left(B^p\left(\mathcal{S}_1,Q^1\right)\right) \tag{152}$$

by

$$B_i^n\left(\mathcal{S}_1,Q^1\right) = \begin{cases} \emptyset, & \text{if } i \notin \bigcup_{C\in\mathcal{S}_1} [C]_{\leftarrow}, \\ U_i^e\left(B^p\left(\mathcal{S}_1,Q^1\right),Q_i^1\right), & \text{if } i \in V^e \cap \bigcup_{C\in\mathcal{S}_1} [C]_{\leftarrow}, \\ W_i\left(B^p\left(\mathcal{S}_1,Q^1\right)\right) & \\ \cup U_i^c\left(B^p\left(\mathcal{S}_1,Q^1\right),Q_{\widehat{C}}^1\right), & \text{if } \widehat{C} \in C(H), \\ & i \in V(\widehat{C}) \cap \bigcup_{C\in\mathcal{S}_1} [C]_{\leftarrow}. \end{cases} \tag{153}$$

Since $B^p(\mathcal{S}_1,Q^1) \subset K(\mathcal{S}_2,Q^2)$, $K(\mathcal{S}_2,Q^2) \in F(K(\mathcal{S}_2,Q^2))$, $Q^1 \subset Q^2$ and using the definitions of F and $K(\mathcal{S}_2,Q^2)$, we deduce that $B^{p+1}(\mathcal{S}_1,Q^1) \subset K(\mathcal{S}_2,Q^2)$. Also, $(K(\mathcal{S}_1,Q^1), B^{p+1}(\mathcal{S}_1,Q^1)) \in E(G)$. Arguing as in the proof of Proposition 14, one has

$$\begin{aligned} d&\left(K\left(\mathcal{S}_1,Q^1\right),B^{p+1}\left(\mathcal{S}_1,Q^1\right)\right) \\ &\leq \lambda d\left(K\left(\mathcal{S}_1,Q^1\right),B^p\left(\mathcal{S}_1,Q^1\right)\right). \end{aligned} \tag{154}$$

Repeating this argument, we obtain for every $n \geq p$, $B^n(\mathcal{S}_1,Q^1) \in K(\mathcal{S}_2,Q^2)$ such that $B^n(\mathcal{S}_1,Q^1) \to K(\mathcal{S}_1,Q^1)$. Therefore,

$$K\left(\mathcal{S}_1,Q^1\right) \subset K\left(\mathcal{S}_2,Q^2\right). \tag{155}$$

(5) Let $\mathcal{S} \subset C(H)$ and $\widehat{C} \in C(H)$ be such that $V(\widehat{C}) \subset \bigcup_{C\in\mathcal{S}}[C]_{\leftarrow}$. Let

$$Q = (Q_i)_{i\in V^e} \times (Q_C)_{C\in C(H)} \in \mathcal{Q}(\mathcal{S}). \tag{156}$$

We define

$$\widehat{Q} = \left(\widehat{Q}_i\right)_{i\in V^e} \times \left(\widehat{Q}_C\right)_{C\in C(H)} \tag{157}$$

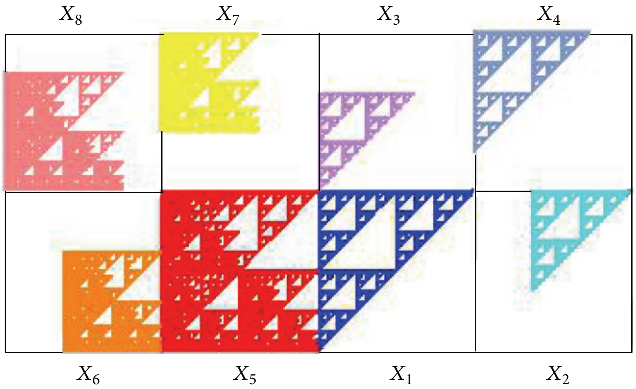

Figure 6: The set $K(C_2, Q^1)$.

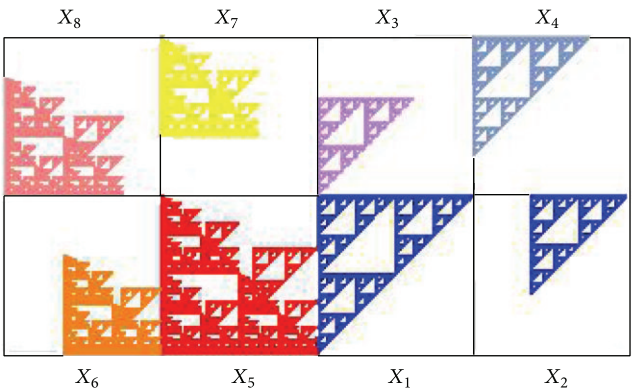

Figure 7: The set $K(C_2, Q^2)$.

by

$$\widehat{Q}_i = \begin{cases} Q_i, & \text{if } i \in V^e, \mathscr{E}_i\left(\widehat{C}\right) \neq \emptyset, \\ \emptyset, & \text{if } i \in V^e, \mathscr{E}_i\left(\widehat{C}\right) = \emptyset; \end{cases}$$
$$\widehat{Q}_C = \emptyset, \quad \text{for } C \in C(H). \tag{158}$$

Clearly, $\widehat{Q} \in \mathscr{Q}(\widehat{C})$ and $\widehat{Q} \subset Q$. It follows from (2), (4), and Theorem 16(4) that

$$K\left(\widehat{C}, \widehat{Q}\right) \subset K(\mathscr{S}, Q),$$
$$K_i\left(\widehat{C}, \widehat{Q}\right) = K_i^+\left(\widehat{C}\right) = K_i^- \quad \forall i \in V\left(\widehat{C}\right). \tag{159}$$

□

Example 21. Let $\{T_{i,j}\}_H$ be the H-IFS considered in Example 17. One has $C(H) = \{C_1, C_2\}$, $V^e = \emptyset$, $\mathscr{E}_{C_2}(C_2) = \emptyset$, and $\mathscr{E}_{C_1}(C_2) = \{(5,1),(5,2)\}$. For $k = 1, 2$ let $Q^k = Q^k_{C_1} \times Q^k_{C_2} \in \mathscr{Q}(C_2)$ be given by

$$Q^1_{C_1} = \{(5,1)\}, \qquad Q^2_{C_1} = \{(5,2)\}, \qquad Q^1_{C_2} = Q^2_{C_2} = \emptyset. \tag{160}$$

Figures 6 and 7 present $K(C_2, Q^1)$ and $K(C_2, Q^2)$, respectively. Observe that

$$K\left(C_2, Q^1\right) \neq K\left(C_2, Q^2\right), \qquad K\left(C_2, Q^1\right) \subsetneq K^+\left(C_2\right),$$
$$K\left(C_2, Q^2\right) \subsetneq K^+\left(C_2\right), \tag{161}$$

where $K^+(C_2)$ is presented in Figure 4.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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