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Research Article

Boundedness for a Class of Singular Integral Operators on Both Classical and Product Hardy Spaces

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We found that the classical Calderón-Zygmund singular integral operators are bounded on both the classical Hardy spaces and the product Hardy spaces. The purpose of this paper is to extend this result to a more general class. More precisely, we introduce a class of singular integral operators including the classical Calderón-Zygmund singular integral operators and show that they are bounded on both the classical Hardy spaces and the product Hardy spaces.

1. Introduction

The classical Hardy spaces and the product Hardy spaces play important roles in Harmonic analysis, which are due to the original work of Fefferman and Stein [1] and Gundy and Stein [2], respectively. It is well known that these two Hardy spaces are essentially different. For instance, see [3–5]. It has been known that the classical Calderón-Zygmund singular integral operators are bounded on the classical Hardy spaces and the product singular integral operators are bounded on the product Hardy spaces. Surprisingly, in [6], we found that the classical Calderón-Zygmund singular integral operators are also bounded on the product Hardy spaces. More precisely, if T is a bounded operator on $L^2(\mathbb{R}^2)$ with $Tf(x) = p\cdot v\cdot \mathscr{K} * f(x)$, where the kernel $\mathscr{K}\in C^2(\mathbb{R}^2\setminus\{0\})$ and satisfies $|\partial_x^\alpha \mathscr{K}(x)|\le C/|x|^{2+\alpha}$ for $0\le|\alpha|\le 2$ and $x\in\mathbb{R}^2\setminus\{0\}$, then T is bounded on both the classical Hardy spaces $H^p(\mathbb{R}^2)$ and the product Hardy space $H^p(\mathbb{R}\times\mathbb{R})$.

A natural question arises: weather there exist a more general class of operators that are bounded both on the classical Hardy spaces and the product Hardy spaces. The purpose of this paper is to answer this question. Now we first recall the definitions of the classical Hardy spaces $H^p(\mathbb{R}^2)$ (see [1] for more details) and the product Hardy spaces $H^p(\mathbb{R}\times\mathbb{R})$(see [2, 7] for more details).

We let $\mathcal{S}_1$ be the set including all $\psi\in\mathcal{S}(\mathbb{R}^2)$ that satisfy $\int_{\mathbb{R}^2}\psi(x)dx = 0$ and $\sum_{j\in\mathbb{Z}}|\widehat{\psi}(2^{-j}\xi)|^2 = 1$ for all $\xi\in\mathbb{R}^2\setminus\{0\}$. And let $\mathcal{S}_2$ be the set including all $\widetilde{\psi}\in\mathcal{S}(\mathbb{R}^2)$ that satisfy $\int_{\mathbb{R}}\widetilde{\psi}(x_1,x_2)dx_1 = \int_{\mathbb{R}}\widetilde{\psi}(x_1,x_2)dx_2 = 0$ and $\sum_{j,k\in\mathbb{Z}}|\widehat{\widetilde{\psi}}(2^{-j}\xi_1,2^{-k}\xi_2)|^2 = 1$ for all $\xi=(\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\}$.

Given an $\psi\in\mathcal{S}_1$, the Littlewood-Paley-Stein square function of $f\in\mathcal{S}'(\mathbb{R}^2)$ is defined by $g_\psi(f)(x) = \{\sum_{j\in\mathbb{Z}}|\psi_j * f(x)|^2\}^{1/2}$, where $\psi_j(x) = 2^{2j}\psi(2^jx_1,2^jx_2)$, $x=(x_1,x_2)$. And the discrete square function is defined by $g^d_\psi(f)(x) = \{\sum_{j\in\mathbb{Z}}\sum_Q|\psi_j * f(c_Q)|^2\chi_Q(x)\}^{1/2}$, where Q are dyadic cubes in $\mathbb{R}^2$ with the side length $l(Q) = 2^{-j}$ and the center c_Q and χ_Q is the characteristic function. It is well known that if $0<p<\infty$, then $\|g_\psi(f)\|_{L^p(\mathbb{R}^2)}\approx\|g^d_\psi(f)\|_{L^p(\mathbb{R}^2)}$ and, for different $\psi,\phi\in\mathcal{S}_1$, $\|g_\psi(f)\|_{L^p(\mathbb{R}^2)}\approx\|g_\phi(f)\|_{L^p(\mathbb{R}^2)}$.

The classical Hardy space $H^p(\mathbb{R}^2)$ is then defined by

$$H^p\left(\mathbb{R}^2\right) = \left\{f\in\mathcal{S}'\setminus\mathscr{P}\left(\mathbb{R}^2\right) : g_\psi\left(f\right)\left(x\right)\in L^p\left(\mathbb{R}^2\right)\right\}, \quad (1)$$

where $\mathcal{S}'\setminus\mathscr{P}$ denotes the space of distributions modulo polynomials. The $H^p(\mathbb{R}^2)$ norm is defined by $\|f\|_{H^p(\mathbb{R}^2)} = \|g_\psi(f)\|_{L^p(\mathbb{R}^2)}$.

Similarly, given a $\widetilde{\psi}\in\mathcal{S}_2$, the product square function of $f\in\mathcal{S}'(\mathbb{R}^2)$ is defined by $g_{\widetilde{\psi}}(f)_P(x) = \{\sum_{j,k\in\mathbb{Z}}|\widetilde{\psi}_{j,k} * f(x)|^2\}^{1/2}$, where $\widetilde{\psi}_{j,k}(x) = 2^{j+k}\widetilde{\psi}(2^jx_1,2^kx_2)$, $x = (x_1,x_2)$. And the discrete square function is defined by $g^d_{\tilde\psi}(f)_P(x_1,x_2) = \{\sum_{j,k\in\mathbb{Z}}\sum_{I,J}|\tilde\psi_{j,k} * f(c_I,c_J)|^2\chi_I(x_1)\chi_J(x_2)\}^{1/2}$, where I, J are dyadic intervals in $\mathbb{R}$ with the side length $l(I) = 2^{-j}$, $l(J) = 2^{-k}$ and the center c_I, c_J, respectively. Also, for $0 < p < \infty$, $\|g_{\tilde\psi}(f)_P\|_{L^p(\mathbb{R}^2)} \approx \|g^d_{\tilde\psi}(f)_P\|_{L^p(\mathbb{R}^2)}$ and, for different $\tilde\psi, \tilde\phi \in \mathcal{S}_2$, $\|g_{\tilde\psi}(f)_P\|_{L^p(\mathbb{R}^2)} \approx \|g_{\tilde\phi}(f)_P\|_{L^p(\mathbb{R}^2)}$.

The product Hardy space $H^p(\mathbb{R}\times\mathbb{R})$ is then defined by

$$H^p(\mathbb{R}\times\mathbb{R}) = \left\{f \in \mathcal{S}' \setminus \mathcal{P}\left(\mathbb{R}^2\right) : g_{\tilde\psi}(f)_P(x) \in L^p\left(\mathbb{R}^2\right)\right\}. \tag{2}$$

The $H^p(\mathbb{R}\times\mathbb{R})$ norm is defined by $\|f\|_{H^p(\mathbb{R}\times\mathbb{R})} = \|g_{\tilde\psi}(f)_P\|_{L^p(\mathbb{R}^2)}$.

The following theorem is our main result.

Theorem 1. *If $2/3 < p \le 1$, $\delta > 0$ and T is an operator bounded on $L^2(\mathbb{R}^2)$ with $Tf(x) = p\cdot v\cdot \mathcal{K} * f(x)$, where the kernel $\mathcal{K} \in C^2(\mathbb{R}^2 \setminus \{(0,\mathbb{R})\cup(\mathbb{R},0)\})$ and satisfies $|\partial^\alpha_{x_1}\partial^\beta_{x_2}\mathcal{K}(x_1,x_2)| \le C(1/|x_1|^{1+\alpha})(1/|x_2|^{1+\beta})((|x_1|/|x_2|) + (|x_2|/|x_1|))^{-\delta}$ for all $|\alpha|, |\beta| \le 1$, then T is bounded on both $H^p(\mathbb{R}^2)$ and $H^p(\mathbb{R}\times\mathbb{R})$.*

Remark 2. (I) In [8], we have shown that the operator T is bounded on $L^p(\mathbb{R}^2)$ for all $1 < p < \infty$.

(II) It is easy to verify that the classical Calderón-Zygmund singular integral operators are contained in our class. Moreover, some more operators will be in our class. For example, the operator $T = p\cdot v\cdot\mathcal{K} * f(x)$ with $\mathcal{K}(x_1,x_2) = sgn(x_1+x_2)/(|x_1||x_2|)^{1/2}(|x_1|+|x_2|)$.

Throughout this paper, we do the following conventions.

(a) The notation $A \approx B$ means that $C_1A \le B \le C_2A$ for some positive constants C_1, C_2.

(b) If Q is a cube or interval, then we denote by c_Q its center and by $l(Q)$ its side length.

(c) For $j \in \mathbb{Z}$ and a large positive integer N, we denote the set $\mathcal{D}_j(\mathbb{R}^2) = \{Q$, where Q are dyadic cubes in $\mathbb{R}^2$ with side length $l(Q) = 2^{-j}\}$ and $\mathcal{D}^N_j(\mathbb{R}^2) = \mathcal{D}_{j+N}(\mathbb{R}^2)$.

(d) For j, $k \in \mathbb{Z}$ and a large positive integer N, we denote the set $\mathcal{D}_{j,k}(\mathbb{R}^2) = \{R = I\times J$, where I and J are dyadic intervals in $\mathbb{R}$ with side length $l(I) = 2^{-j}$ and $l(J) = 2^{-k}$, respectively$\}$ and $\mathcal{D}^N_{j,k}(\mathbb{R}^2) = \mathcal{D}_{j+N,k+N}(\mathbb{R}^2)$.

(e) $j\wedge j'$ means the minimum of j and j'.

2. Proof of Theorem 1

The first crucial tool in the proof of Theorem 1 is to apply the following discrete Calderón identity (see [9] for more details).

Lemma 3. *If $0 < p \le 1$, then consider the following.*

(a) Suppose that $\phi \in \mathcal{S}_1 \cap C^\infty_0(\mathbb{R}^2)$ with $\operatorname{supp}(\phi) \subset \{x : |x| \le 1\}$. Then for all $f \in L^2(\mathbb{R}^2)\cap H^p(\mathbb{R}^2)$, there exist a large positive integer N (depending only on p) and a function $h \in L^2(\mathbb{R}^2)\cap H^p(\mathbb{R}^2)$ such that $f(x) = \sum_{j\in\mathbb{Z}}\sum_{Q\in\mathcal{D}^N_j(\mathbb{R}^2)}|Q|\phi_j(x-c_Q)(\phi_j * h)(c_Q)$, where the series converges in $L^2(\mathbb{R}^2)$. Moreover, $\|f\|_{L^2(\mathbb{R}^2)} \approx \|h\|_{L^2(\mathbb{R}^2)}$ and $\|f\|_{H^p(\mathbb{R}^2)} \approx \|h\|_{L^2(\mathbb{R}^2)}$.

(b) Suppose that $\tilde\phi \in \mathcal{S}_2 \cap C^\infty_0(\mathbb{R}^2)$ with $\operatorname{supp}(\tilde\phi) \subset \{x : |x| \le 1\}$. Then for all $f \in L^2(\mathbb{R}^2)\cap H^p(\mathbb{R}\times\mathbb{R})$, there exist a large positive integer N (depending only on p) and a function $h \in L^2(\mathbb{R}^2)\cap H^p(\mathbb{R}\times\mathbb{R})$ such that $f(x) = \sum_{j,k\in\mathbb{Z}}\sum_{R\in\mathcal{D}^N_{j,k}(\mathbb{R}^2)}|R|\phi_{j,k}(x-c_R)(\phi_{j,k}*h)(c_R)$, where the series converges in $L^2(\mathbb{R}^2)$. Moreover, $\|f\|_{H^p(\mathbb{R}^2)} \approx \|h\|_{L^2(\mathbb{R}^2)}$ and $\|f\|_{H^p(\mathbb{R}\times\mathbb{R})} \approx \|h\|_{H^p(\mathbb{R}\times\mathbb{R})_P}$.

For the proof, we refer readers to [9].

The second crucial tool in the proof of Theorem 1 is the following orthogonal estimates.

Lemma 4. *Suppose that $0 < \lambda \le \min(\delta, 1/2)$ and $\mathcal{K}$ is the kernel as in Theorem 1; then*

(a) for $\phi \in C^\infty_0(\mathbb{R}^2)$ with $\int_{\mathbb{R}^2}\phi(x)dx = 0$, one has $|\mathcal{K} * \phi_j(x)| \le C2^{2j}(1/(1+|2^jx_1|^{1+\lambda}))(1/(1+|2^jx_2|^{1+\lambda}))$, for all $x = (x_1,x_2) \in \mathbb{R}^2$ and $j \in \mathbb{Z}$, where C is a constant independent of j and x;

(b) for $\tilde\phi \in C^\infty_0(\mathbb{R}^2)$ with $\int_{\mathbb{R}}\tilde\phi(x_1,x_2)dx_1 = \int_{\mathbb{R}}\tilde\phi(x_1,x_2)dx_2 = 0$, one has $|\mathcal{K} * \tilde\phi_{j,k}(x)| \le C2^{j+k}(1/(1+|2^jx_1|^{1+\lambda}))(1/(1+|2^kx_2|^{1+\lambda}))$, for all $x = (x_1,x_2) \in \mathbb{R}^2$ and $j, k \in \mathbb{Z}$, where C is a constant independent of j, k, and x.

Proof. (a) Since $\mathcal{K}$ is single-parameter dilation invariant, that is, for each $\delta > 0$, $\delta^2\mathcal{K}(\delta x_1,\delta x_2)$ satisfies the same hypotheses, with the same bounds as $\mathcal{K}$. We just need to show that $|\mathcal{K} * \phi(x)| \le C((1/(1+|x_1|^{1+\lambda})))(1/(1+|x_2|^{1+\lambda}))$, for all $x = (x_1,x_2) \in \mathbb{R}^2$. Without loss of generality, we may assume that $\operatorname{supp}(\phi) \subset \{x : |x| \le 1\}$. To get the required estimate, we will discuss it in the following three cases: (I) $|x_1| \ge 2, |x_2| \ge 2$; (II) $|x_1| \ge 2, |x_2| < 2$ or $|x_1| < 2, |x_2| \ge 2$; (III) $|x_1| < 2, |x_2| < 2$.

For case (I), $|x_1| \ge 2, |x_2| \ge 2$, by the moment condition of ϕ, we have

$$\begin{aligned}
&|\mathcal{K} * \phi(x)| \\
&\quad = \left|\iint_{\mathbb{R}^2}\mathcal{K}(x_1-y_1,x_2-y_2)\phi(y_1,y_2)\,dy_1dy_2\right| \\
&\quad = \left|\iint_{\mathbb{R}^2}(\mathcal{K}(x_1-y_1,x_2-y_2)-\mathcal{K}(x_1,x_2))\right. \\
&\qquad\qquad \left.\times\phi(y_1,y_2)\,dy_1dy_2\right| \\
&\quad \le C\left|\int_{\mathbb{R}^2}\left(\frac{1}{|x_1|^2}\frac{1}{|x_2|}+\frac{1}{|x_1|}\frac{1}{|x_2|^2}\right)\left(\frac{|x_1|}{|x_2|}+\frac{|x_2|}{|x_1|}\right)^{-\delta}\right. \\
&\qquad\qquad \left.\times|\phi(y_1,y_2)|\,dy_1dy_2\right|
\end{aligned}$$

$$\leq C\left(\frac{1}{|x_1|^2}\frac{1}{|x_2|}+\frac{1}{|x_1|}\frac{1}{|x_2|^2}\right)\left(\frac{|x_1|}{|x_2|}+\frac{|x_2|}{|x_1|}\right)^{-\delta} \leq \frac{C}{1+|x_1|^{1+\lambda}}\frac{1}{1+|x_2|^{1+\lambda}}. \tag{3}$$

For case (II), $|x_1| \geq 2, |x_2| < 2$ or $|x_1| < 2, |x_2| \geq 2$, we have

$$\begin{aligned}|\mathscr{K} * \phi(x)| &\leq \int_{\mathbb{R}^2} |\mathscr{K}(x_1-y_1, x_2-y_2)\phi(y_1,y_2)|\,dy_1dy_2 \\ &\leq C\int_{\substack{|y_1|\leq 1\\ |y_2|\leq 1}} \frac{1}{|x_1-y_1|}\frac{1}{|x_2-y_2|} \times\left(\frac{|x_1-y_1|}{|x_2-y_2|}+\frac{|x_2-y_2|}{|x_1-x_2|}\right)^{-\delta} dy_1dy_2 \\ &\leq \frac{C}{1+|x_1|^{1+\delta}}\frac{1}{1+|x_2|^{1+\delta}} \leq \frac{C}{1+|x_1|^{1+\lambda}}\frac{1}{1+|x_2|^{1+\lambda}}.\end{aligned} \tag{4}$$

For case (III), $|x_1| < 2, |x_2| < 2$, we let $\eta \in C_0^{\infty}(\mathbb{R}^2)$ with $0 \leq \eta(x) \leq 1$ and $\eta(x) = 1$ when $|x| \leq 4$ and $\eta(x) = 0$ when $|x| \geq 8$. We have

$$\begin{aligned}|\mathscr{K} * \phi(x)| &= \left|\iint_{\mathbb{R}^2} \mathscr{K}(y_1,y_2)\phi(x_1-y_1,x_2-y_2)\eta(y_1,y_2)\,dy_1dy_2\right| \\ &\leq \left|\iint_{\mathbb{R}^2} \mathscr{K}(y_1,y_2)(\phi(x_1-y_1,x_2-y_2)-\phi(x_1,x_2)) \times\eta(y_1,y_2)\,dy_1dy_2\right| \\ &\quad+\left|\iint_{\mathbb{R}^2} \mathscr{K}(y_1,y_2)\phi(x_1,x_2)\eta(y_1,y_2)\,dy_1dy_2\right| \\ &\leq C\left|\iint_{\substack{|y_1|\leq 8\\ |y_2|\leq 8}} \frac{1}{|y_1|}\frac{1}{|y_2|}\left(\frac{|y_2|}{|y_1|}+\frac{|y_1|}{|y_2|}\right)^{-\delta} \times(|y_1|+|y_2|)\,dy_1dy_2\right| \\ &\quad+C\left|\iint_{\mathbb{R}^2} \widetilde{\mathscr{K}}(\xi_1,\xi_2)\widetilde{\eta}(\xi_1,\xi_2)\,dy_1dy_2\right| \\ &\leq C \leq \frac{C}{1+|x_1|^{1+\lambda}}\frac{1}{1+|x_2|^{1+\lambda}}.\end{aligned} \tag{5}$$

(b) Without loss of generality, we may assume that $\mathrm{supp}(\widetilde{\phi}) \subset \{x : |x| \leq 1\}$. The required estimate will be discussed in the following four cases: (I) $|x_1| \geq 2^{-j+1}, |x_2| \geq 2^{-k+1}$; (II) $|x_1| \geq 2^{-j+1}, |x_2| < 2^{-k+1}$; (III) $|x_1| < 2^{-j+1}, |x_2| \geq 2^{-k+1}$; (IV) $|x_1| < 2^{-j+1}, |x_2| < 2^{-k+1}$.

For case (I), $|x_1| \geq 2^{-j+1}, |x_2| \geq 2^{-k+1}$, by the moment condition of $\widetilde{\phi}$, we have

$$\begin{aligned}|\mathscr{K} * \widetilde{\phi}_{j,k}(x)| &= 2^{j+k}\left|\iint_{\mathbb{R}^2} \{(\mathscr{K}(x_1-y_1,x_2-y_2)-\mathscr{K}(x_1,x_2-y_2)) - (\mathscr{K}(x_1-y_1,x_2)-\mathscr{K}(x_1,x_2))\} \times\widetilde{\phi}(2^jy_1,2^ky_2)\,dy_1dy_2\right| \\ &= 2^{j+k}\left|\iint_{\mathbb{R}^2}\int_{x_2}^{x_2-y_2}\int_{x_1}^{x_1-y_1}\partial_{z_1}^1\partial_{z_2}^1\mathscr{K}(z_1,z_2)\,dz_1dz_2 \times\widetilde{\phi}(2^jy_1,2^ky_2)\,dy_1dy_2\right| \\ &\leq C2^{j+k}\int_{\mathbb{R}^2}\frac{1}{|x_1|^2}\frac{1}{|x_2|^2}\left(\frac{|x_1|}{|x_2|}+\frac{|x_2|}{|x_1|}\right)^{-\delta} \times|y_1||y_2|\left|\widetilde{\phi}(2^jy_1,2^ky_2)\right|dy_1dy_2 \\ &\leq C2^{-j-k}\frac{1}{|x_1|^2}\frac{1}{|x_2|^2} \leq C2^{j+k}\frac{1}{1+|2^jx_1|^{1+\lambda}}\frac{1}{1+|2^kx_2|^{1+\lambda}}.\end{aligned} \tag{6}$$

For case (II), $|x_1| \geq 2^{-j+1}, |x_2| < 2^{-k+1}$, similarly, we have

$$\begin{aligned}|\mathscr{K} * \widetilde{\phi}_{j,k}(x)| &= 2^{j+k}\left|\iint_{\mathbb{R}^2}(\mathscr{K}(x_1-y_1,x_2-y_2)-\mathscr{K}(x_1,x_2-y_2)) \times\widetilde{\phi}(2^jy_1,2^ky_2)\,dy_1dy_2\right| \\ &= 2^{j+k}\left|\iint_{\mathbb{R}^2}\int_{x_1}^{x_1-y_1}\partial_{z_1}^1\mathscr{K}(z_1,x_2-y_2)\,dz_1 \times\widetilde{\phi}(2^jy_1,2^ky_2)\,dy_1dy_2\right| \\ &\leq C2^{j+k}\int_{\substack{|y_1|\leq 2^{-j}\\ |y_2|\leq 2^{-k}}}\frac{1}{|x_1|^2}\frac{1}{|x_2-y_2|} \times\left(\frac{|x_1|}{|x_2-y_2|}+\frac{|x_2-y_2|}{|x_1|}\right)^{-\delta}|y_1|\,dy_1dy_2 \\ &\leq C2^{-j+k}\frac{1}{|x_1|^2} \leq C2^{j+k}\frac{1}{1+|2^jx_1|^{1+\lambda}}\frac{1}{1+|2^kx_2|^{1+\lambda}}.\end{aligned} \tag{7}$$

The cases (II) and (III) are symmetric, so case (III) follows.

For case (IV), $|x_1| < 2^{-j+1}$, $|x_2| < 2^{-k+1}$, we let $\theta \in C_0^\infty(\mathbb{R})$ with $0 \le \theta(x) \le 1$ and $\theta(x) = 1$ when $|x| \le 4$ and $\theta(x) = 0$ when $|x| \ge 8$. Then

$$\begin{aligned}
&\left|\mathscr{K} * \widetilde{\phi}_{j,k}(x)\right| \\
&= 2^{j+k}\left|\iint_{\mathbb{R}^2} \mathscr{K}(y_1, y_2)\widetilde{\phi}\left(2^j(x_1 - y_1), 2^k(x_2 - y_2)\right) \times \theta\left(2^j y_1\right)\theta\left(2^k y_2\right) dy_1 dy_2\right| \\
&\le 2^{j+k}\left|\iint_{\mathbb{R}^2} \mathscr{K}(y_1, y_2) \times \left(\widetilde{\phi}\left(2^j(x_1 - y_1), 2^k(x_2 - y_2)\right) - \widetilde{\phi} \times \left(2^j x_1, 2^k x_2\right)\right)\theta\left(2^j y_1\right)\theta\left(2^k y_2\right) dy_1 dy_2\right| \\
&\quad + 2^{j+k}\left|\iint_{\mathbb{R}^2} \mathscr{K}(y_1, y_2)\widetilde{\phi}\left(2^j x_1, 2^k x_2\right) \times \theta\left(2^j y_1\right)\theta\left(2^k y_2\right) dy_1 dy_2\right| \\
&\le C2^{j+k}\int_{\substack{|y_1| \le 2^{-j+3} \\ |y_2| \le 2^{-k+3}}} \frac{1}{|y_1|}\frac{1}{|y_2|}\left(\frac{|y_1|}{|y_2|} + \frac{|y_2|}{|y_1|}\right)^{-\delta} \times \left(\left|2^j y_1\right| + \left|2^k y_2\right|\right) dy_1 dy_2 \\
&\quad + C\left|\iint_{\mathbb{R}^2} \widehat{\mathscr{K}}(\xi_1, \xi_2)\widehat{\theta}\left(2^{-j}\xi_1\right)\widehat{\theta}\left(2^{-k}\xi_2\right) d\xi_1 d\xi_2\right| \\
&\le C2^{j+k} \le C2^{j+k}\frac{1}{1 + \left|2^j x_1\right|^{1+\lambda}}\frac{1}{1 + \left|2^k x_2\right|^{1+\lambda}}.
\end{aligned} \tag{8}$$

This completes the proof of Lemma 4. □

As a consequence of Lemma 4, we have the following.

Lemma 5. *(a) Under hypothesis (a) of Lemma 4, one has*

$$\left|\phi_j * \mathscr{K} * \phi_{j'}(x)\right| \le C2^{-|j-j'|}\frac{2^{j\wedge j'}}{1 + \left|2^{j\wedge j'} x_1\right|^{1+\lambda}} \times \frac{2^{j\wedge j'}}{1 + \left|2^{j\wedge j'} x_2\right|^{1+\lambda}}, \tag{9}$$

for all $j, j' \in \mathbb{Z}$ and $x = (x_1, x_2) \in \mathbb{R}^2$.

(b) Under hypothesis (b) of Lemma 4, one has

$$\begin{aligned}
&\left|\widetilde{\phi}_{j,k} * \mathscr{K} * \widetilde{\phi}_{j',k'}(x)\right| \\
&\le C2^{-|j-j'|}2^{-|k-k'|}\frac{2^{j\wedge j'}}{1 + \left|2^{j\wedge j'} x_1\right|^{1+\lambda}} \times \frac{2^{k\wedge k'}}{1 + \left|2^{k\wedge k'} x_2\right|^{1+\lambda}},
\end{aligned} \tag{10}$$

for all j, j', k and $k' \in \mathbb{Z}$ and $x = (x_1, x_2) \in \mathbb{R}^2$.

The proof of Lemma 5 is based on the following two observations: (1) convolution operation is commutative; that is, $\phi_j * \mathscr{K} * \phi_{j'}(x) = \mathscr{K} * (\phi_j * \phi_{j'})(x)$ (or $\widetilde{\phi}_{j,k} * \mathscr{K} * \widetilde{\phi}_{j',k'}(x) = \mathscr{K} * (\widetilde{\phi}_{j,k} * \widetilde{\phi}_{j',k'})(x)$); (2) $\phi_j * \phi_{j'}$ (or $\widetilde{\phi}_{j,k} * \widetilde{\phi}_{j',k'}$) satisfies the same estimate as $\phi_{j\wedge j'}$ (or $\widetilde{\phi}_{j\wedge j',k\wedge k'}$) in Lemma 4 with the bound $C2^{-|j-j'|}$ (or $C2^{-|j-j'|}2^{-|k-k'|}$). The details are left to the readers.

The last crucial tool in the proof of Theorem 1 is the following strongly maximal function estimates.

Lemma 6. *Suppose that $\lambda > 0$, $2/3 < q \le 1$, N, j, j', k and $k' \in \mathbb{Z}$, and $F \in L^2(\mathbb{R}^2)$. Then consider the following.*

(a) If $Q' \in \mathscr{D}_{j'}(\mathbb{R}^2)$, $u = (u_1, u_2)$, and $v = (v_1, v_2) \in Q'$, then one has

$$\begin{aligned}
&\sum_{Q = I\times J \in \mathscr{D}_j^N(\mathbb{R}^2)} \frac{2^{j\wedge j'}}{\left(1 + 2^{j\wedge j'}\left|u_1 - c_I\right|\right)^{1+\lambda}}\frac{2^{j\wedge j'}}{\left(1 + 2^{j\wedge j'}\left|u_2 - c_J\right|\right)^{1+\lambda}} \times \left|F(c_Q)\right| \\
&\le C2^{2(j\wedge j')(1-1/q)+2j/q} \times \left\{M_S\left[\left(\sum_{Q\in\mathscr{D}_j^N(\mathbb{R}^2)}\left|F(c_Q)\right|^2\chi_Q\right)^{q/2}\right]\right\}^{1/q}(v),
\end{aligned} \tag{11}$$

where M_S is the strongly maximal operator.

(b) If $R' = I' \times J' \in \mathscr{D}_{j',k'}(\mathbb{R}^2)$, $u = (u_1, u_2)$, and $v = (v_1, v_2) \in R'$, then one has

$$\begin{aligned}
&\sum_{R = I\times J \in \mathscr{D}_{j,k}^N(\mathbb{R}^2)} \frac{2^{j\wedge j'}}{\left(1 + 2^{j\wedge j'}\left|u_1 - c_I\right|\right)^{1+\lambda}}\frac{2^{k\wedge k'}}{\left(1 + 2^{k\wedge k'}\left|u_2 - c_J\right|\right)^{1+\lambda}} \times \left|F(c_I, c_J)\right| \\
&\le C2^{(j\wedge j')(1-1/q)+j/q}2^{(k\wedge k')(1-1/q)+k/q} \times \left\{M_S\left[\left(\sum_{R\in\mathscr{D}_{j,k}^N(\mathbb{R}^2)}\left|F(c_R)\right|^2\chi_R\right)^{q/2}\right]\right\}^{1/q}(v).
\end{aligned} \tag{12}$$

For the proof, we refer readers to [10].

Proof of Theorem 1. Firstly we show that T is bounded on the classical Hardy space $H^p(\mathbb{R}^2)$. Since $L^2(\mathbb{R}^2) \cap H^p(\mathbb{R}^2)$ is dense in $H^p(\mathbb{R}^2)$, we just need to show that, for all $f \in L^2(\mathbb{R}^2) \cap H^p(\mathbb{R}^2)$, we have $\|Tf\|_{H^p(\mathbb{R}^2)} \le C\|f\|_{H^p(\mathbb{R}^2)}$; that is, for a fixed $\psi \in \mathcal{S}_1$,

$$\left\| g_\psi(Tf) \right\|_{L^p(\mathbb{R}^2)} \leq C \|f\|_{H^p(\mathbb{R}^2)}. \tag{13}$$

Note that

$$g_\psi(Tf)(x) = \left\{ \sum_{j' \in \mathbb{Z}} \sum_{Q' = I' \times J' \in \mathcal{D}_{j'}(\mathbb{R}^2)} \left| \psi_{j'} * \mathcal{K} * f(c_{Q'}) \right|^2 \chi_{Q'}(x) \right\}^{1/2}. \tag{14}$$

For $j' \in \mathbb{Z}$, $Q' = I' \times J' \in \mathcal{D}_{j'}(\mathbb{R}^2)$, and $x \in Q'$, applying (a) of Lemma 3, we have

$$\left| \psi_{j'} * \mathcal{K} * f(c_{Q'}) \right| = \left| \sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_j^N(\mathbb{R}^2)} |Q| \psi_{j'} * \mathcal{K} * \psi_j(u - c_Q)(\psi_j * h)(c_Q) \right|. \tag{15}$$

By (a) of Lemma 5, we have

$$\begin{aligned} \left| \psi_{j'} * \mathcal{K} * f(c_{Q'}) \right| &\leq C \sum_{j \in \mathbb{Z}} \sum_{Q = I \times J \in \mathcal{D}_{j'}(\mathbb{R}^2)} 2^{-2j} 2^{-|j-j'|} \\ &\quad \times \frac{2^{j \wedge j'}}{1 + \left| 2^{j \wedge j'}(u_1 - c_I) \right|^{1+\lambda}} \\ &\quad \times \frac{2^{j \wedge j'}}{1 + \left| 2^{j \wedge j'}(u_2 - c_J) \right|^{1+\lambda}} (\psi_j * h)(c_Q). \end{aligned} \tag{16}$$

Applying (a) of Lemma 6 with $F = \psi_j * h$ and $2/3 < q < p$, we have

$$\begin{aligned} \left| \psi_{j'} * \mathcal{K} * f(c_{Q'}) \right| &\leq C \sum_{j \in \mathbb{Z}} 2^{-2j} 2^{-|j-j'|} 2^{2(j \wedge j')(1-1/q)+2j/q} \\ &\quad \times \left\{ M_S \left[\left(\sum_{Q \in \mathcal{D}_j^N(\mathbb{R}^2)} \left| \psi_j * h(c_Q) \right|^2 \chi_Q \right)^{q/2} \right] \right\}^{1/q} (x). \end{aligned} \tag{17}$$

Therefore,

$$\begin{aligned} \left| g_\psi(Tf)(x) \right|^2 &= \sum_{j' \in \mathbb{Z}} \sum_{Q' \in \mathcal{D}_{j'}(\mathbb{R}^2)} \left| \psi_{j'} * \mathcal{K} * f(c_{Q'}) \right|^2 \chi_{Q'}(x) \\ &\leq C \sum_{j' \in \mathbb{Z}} \sum_{Q' \in \mathcal{D}_{j'}(\mathbb{R}^2)} \left| \sum_{j \in \mathbb{Z}} 2^{-2j} 2^{-|j-j'|} 2^{2(j \wedge j')(1-1/q)+2j/q} \right. \\ &\quad \left. \times \left\{ \left[\left(\sum_{Q \in \mathcal{D}_j^N(\mathbb{R}^2)} \left| \psi_j * h(c_Q) \right|^2 \chi_Q \right)^{q/2} \right] \right\}^{1/q} (x) \right|^2 \chi_{Q'}(x) \\ &\leq C \sum_{j' \in \mathbb{Z}} \left(\sum_{j \in \mathbb{Z}} 2^{-2j} 2^{-|j-j'|} 2^{2(j \wedge j')(1-1/q)+2j/q} \right) \\ &\quad \times \left(\sum_{j \in \mathbb{Z}} 2^{-2j} 2^{-|j-j'|} 2^{2(j \wedge j')(1-1/q)+2j/q} \left\{ M_S \left[\left(\sum_{Q \in \mathcal{D}_j^N(\mathbb{R}^2)} \left| \psi_j * h(c_Q) \right|^2 \chi_Q \right)^{q/2} \right] \right\}^{2/q} (x) \right) \end{aligned}$$

$$\leq C \sum_{j' \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} 2^{-2j} 2^{-|j-j'|} 2^{2(j\wedge j')(1-1/q)+2j/q} \left\{ M_S \left[\left(\sum_{Q \in \mathcal{D}_j^N(\mathbb{R}^2)} \left|\psi_j * h(c_Q)\right|^2 \chi_Q \right)^{q/2} \right] \right\}^{2/q} (x)$$

$$\leq C \sum_{j \in \mathbb{Z}} \left\{ M_S \left[\left(\sum_{Q \in \mathcal{D}_j^N(\mathbb{R}^2)} \left|\psi_j * h(c_Q)\right|^2 \chi_Q \right)^{q/2} \right] \right\}^{2/q} (x). \quad (18)$$

Applying Fefferman-Stein's vector-valued strong maximal inequality (see [11] for more details) on $L^{p/q}(\ell^{2/q})$, we have

$$\begin{aligned} \|T(f)\|_{H^p(\mathbb{R}^2)} &= \left\|g_\psi(Tf)\right\|_{L^p(\mathbb{R}^2)} \\ &\leq C \left\| \left\{ \sum_{j \in \mathbb{Z}} \left\{ M_S \left[\left(\sum_{Q \in \mathcal{D}_j^N(\mathbb{R}^2)} \left|\psi_j * h(c_Q)\right|^2 \chi_Q \right)^{q/2} \right] \right\}^{2/q} (x) \right\}^{1/2} \right\|_{L^p(\mathbb{R}^2)} \\ &\leq C \left\| \left\{ \sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_j^N(\mathbb{R}^2)} \left|\psi_j * h(c_Q)\right|^2 \chi_Q(x) \right\}^{1/2} \right\|_{L^p(\mathbb{R}^2)} \\ &\leq C\|h\|_{H^p(\mathbb{R}^2)} \leq C\|f\|_{H^p(\mathbb{R}^2)}. \end{aligned} \quad (19)$$

The proof of T's boundedness on $H^p(\mathbb{R} \times \mathbb{R})$ is almost the same as above; that is, we just need to replace (a) of the required lemmas to (b). Here we omit the details. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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