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*Research Article*

# Discrete Weighted Pseudo-Almost Automorphy and Applications

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We deal with discrete weighted pseudo almost automorphy which extends some classical concepts and systematically explore its properties in Banach space including a composition result. As an application, we establish some sufficient criteria for the existence and uniqueness of the discrete weighted pseudo almost automorphic solutions to the Volterra difference equations of convolution type and also to nonautonomous semilinear difference equations. Some examples are presented to illustrate the main findings.

## 1. Introduction

The notation of (continuous) almost automorphy, introduced by Bochner [1], is related to and more general than (continuous) almost periodicity. Since then, this concept has been attracting the attention of many researchers and the interest in this topic still increases. There is a lager literature on this topic. We refer to the recent books [2, 3], where the authors gave an important overview about the theory of (continuous) almost automorphic functions and their applications to differential equations. Recently, a new more general type of almost automorphy called (continuous) weighted pseudo almost automorphy is proposed by Blot et al. [4], which generalizes various extensions of (continuous) almost automorphy; one can refer to [5] for more details.

Discrete almost automorphic functions, a class of functions which are more general than discrete almost periodic ones, were considered in [6] in connection with the study of (continuous) almost automorphic bounded mild solutions of differential equations; see also [7–9]. Similar to (continuous) almost automorphic functions, discrete almost automorphic functions have made important applications to differential equations in Banach space. The range of applications of discrete almost automorphic functions include first order nonlinear difference equations [10], Volterra difference equations [11–13], nonautonomous difference equations [14, 15], and nonlinear stochastic difference equations [16]. On the other hand, recently, the concept of discrete weighted pseudo almost automorphic functions, which generalizes the notion of discrete almost automorphic functions, is introduced by Abbas in [17] and some basic properties of these functions are explored.

In this paper, we conduct further studies on discrete weighted pseudo almost automorphic functions; the main idea consists of enlarging the weighted ergodic space, with the help of two weighted functions, extending some results of [17]. We systematically explore its properties in Banach space including completeness, translation invariance, and composition results. As an application, the existence and uniqueness of the discrete weighted pseudo almost automorphic solutions to the Volterra difference equations of convolution type and nonautonomous semilinear difference equations are investigated. To the best of our knowledge, discrete weighted pseudo almost automorphy of Volterra difference equations and nonautonomous semilinear difference equations are an untreated topic and this is the main motivation of this paper.

The paper is organized as follows. In Section 2, some notations and preliminary results are presented. In Section 3, we propose a new class of functions called discrete weighted pseudo almost automorphic functions with the help of two weighted functions, explore its properties, and establish the composition theorem. Section 4 is divided into the existence and uniqueness of discrete weighted pseudo almost automorphic solutions to the Volterra difference equations of convolution type and nonautonomous semilinear difference equations, respectively. In Section 5, we provide some examples to illustrate our main results.

## 2. Preliminaries and Basic Results

Let $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ be two Banach spaces and $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}^+\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{C}$ stand for the set of natural numbers, integers, nonnegative integers, real numbers, nonnegative real numbers, and complex numbers, respectively. Let $A$ be bounded linear operator; $\sigma_p(A)$ denotes the point spectrum of $A$. $B(x,\delta)$ stands for open balls with center $x$ and radius less than $\delta$. Let $\nu : \mathbb{Z}^+ \to \mathbb{C}$; if $\sum_{k=0}^{\infty} |\nu(k)| < \infty$, we call that $\nu$ is a summable function.

In order to facilitate the discussion below, we further introduce the following notations.

(i) $S^1 = \{\lambda \in \mathbb{C}, |\lambda| = 1\}$.

(ii) $l^\infty(\mathbb{Z}, X) = \{x : \mathbb{Z} \to X : \|x\|_d = \sup_{n\in\mathbb{Z}}\|x(n)\| < \infty\}$.

(iii) $L(X,Y)$: the Banach space of bounded linear operators from $X$ to $Y$ endowed with the operator topology. In particular, we write $L(X)$ when $X = Y$.

(iv) $\mathscr{UC}(\mathbb{Z}\times X, X)$: the set of all functions $f : \mathbb{Z}\times X \to X$ satisfying that $\forall\varepsilon > 0$, $\exists\delta > 0$ such that

$$\left\| f(k,x) - f(k,y)\right\| \le \varepsilon \tag{1}$$

for all $k \in \mathbb{Z}$ and $x, y \in X$ with $\|x - y\| \le \delta$.

Next, we recall the so-called Matkowski's fixed point theorem [18] and exponential dichotomy on $\mathbb{Z}$ [19, 20] which will be used in the sequel.

**Theorem 1** (Matkowski's fixed point theorem [18]). *Let $(X, d)$ be a complete metric space and let $\mathscr{F} : X \to X$ be a map such that*

$$d(\mathscr{F}x, \mathscr{F}y) \le \Phi(d(x,y)), \quad \forall x, y \in X, \tag{2}$$

*where $\Phi : [0,+\infty) \to [0,+\infty)$ is a nondecreasing function such that $\lim_{n\to+\infty}\Phi^n(t) = 0$ for all $t > 0$. Then $\mathscr{F}$ has a unique fixed point $z \in X$.*

Given a sequence $\{A(n)\}_{n\in\mathbb{Z}} \subset L(X)$ of invertible operators, define

$$\mathscr{A}(m,n) = \begin{cases} A(m-1)\cdots A(n), & \text{if } m > n, \\ \mathrm{Id}, & \text{if } m = n, \\ A^{-1}(m)\cdots A^{-1}(n-1), & \text{if } m < n, \end{cases} \tag{3}$$

where Id is the identity operator in $X$.

For the first order difference equation

$$x(n+1) = A(n)x(n), \quad n \in \mathbb{Z}. \tag{4}$$

*Definition 2* (see [19]). Equation (4) is said to have an exponential dichotomy if there exist projections $P(n) \in L(X)$ for all $n \in \mathbb{Z}$ and positive constants $\eta$, $\nu$, $\alpha$, $\beta$ such that

(i) $P(m)\mathscr{A}(m,n) = \mathscr{A}(m,n)P(n)$, $m, n \in \mathbb{N}$,

(ii) $\|\mathscr{A}(m,n)P(n)\| \le \eta e^{-\alpha(m-n)}$, $m \ge n$,

(iii) $\|\mathscr{A}(m,n)Q(n)\| \le \nu e^{-\beta(n-m)}$, $n \ge m$,

where $Q(n) = \mathrm{Id} - P(n)$ is the complementary projection of $P(n)$.

Finally, we recall the concept of discrete almost automorphic function.

*Definition 3* (see [6]). A function $f \in l^\infty(\mathbb{Z}, X)$ is said to be discrete almost automorphic if for every integer sequence $(k_n')$, there exists a subsequence $(k_n)$ such that $\lim_{n\to\infty} f(k + k_n) := g(k)$ is well defined for each $k \in \mathbb{Z}$, and $\lim_{n\to\infty} g(k - k_n) = f(k)$ for each $k \in \mathbb{Z}$.

*Remark 4.* (i) If $f$ is (continuous) almost automorphic function in $\mathbb{R}$, then $f|_{\mathbb{Z}}$ is discrete almost automorphic.(ii) If the convergence in Definition 3 is uniform on $\mathbb{Z}$, then we get discrete almost periodicity, so discrete almost automorphy is more general than discrete almost periodicity.(iii) Example of discrete almost automorphic functions which are not discrete almost periodicity was first constructed by Veech [21]. Note that the function

$$f(k) = \frac{1}{2 + \cos k + \cos\sqrt{2}k}, \quad k \in \mathbb{Z} \tag{5}$$

is discrete almost automorphic function but not discrete almost periodic (see [10] for more details).

Throughout the paper, we denote $AA_d(\mathbb{Z}, X)$ the set of discrete almost automorphic functions. Note that if $f \in AA_d(\mathbb{Z}, X)$, then $K := \{f(k) : k \in \mathbb{Z}\}$ is relative compact in $X$ and $f$ is a bounded function.

*Definition 5* (see [6]). A function $f : \mathbb{Z}\times X \to X$ is said to be discrete almost automorphic in $k \in \mathbb{Z}$ for each $x \in X$, if for every integer sequence $(k_n')$, there exists a subsequence $(k_n)$ such that $\lim_{n\to\infty} f(k + k_n, x) := g(k, x)$ is well defined for each $k \in \mathbb{Z}$, $x \in X$ and $\lim_{n\to\infty} g(k - k_n, x) = f(k, x)$ for each $k \in \mathbb{Z}$, $x \in X$. We denoted by $AA_d(\mathbb{Z}\times X, X)$ the spaces of all discrete almost automorphic in $k \in \mathbb{Z}$ for each $x \in X$.

**Lemma 6** (see [11]). *Let $f \in AA_d(\mathbb{Z}\times X, X) \cap \mathscr{UC}(\mathbb{Z}\times X, X)$; then $f(\cdot, h(\cdot)) \in AA_d(\mathbb{Z}, X)$ if $h \in AA_d(\mathbb{Z}, X)$.*

## 3. Discrete Weighted Pseudo Almost Automorphy

Let $U$ denote the collection of functions (weights) $\rho : \mathbb{Z} \to (0,+\infty)$. For $\rho \in U$ and $T \in \mathbb{Z}^+ = \{n \in \mathbb{Z}, n \ge 0\}$, set

$$\mu(T,\rho) := \sum_{k=-T}^{T} \rho(k). \tag{6}$$

Denote

$$U_\infty := \left\{\rho \in U : \lim_{T\to+\infty}\mu(T,\rho) = +\infty\right\},$$
$$U_B := \left\{\rho \in U_\infty : 0 < \inf_{k\in\mathbb{Z}}\rho(k) \le \sup_{k\in\mathbb{Z}}\rho(k) < +\infty\right\}. \tag{7}$$

*Definition 7.* Let $\rho_1, \rho_2 \in U_\infty$. $\rho_1$ is said to be equivalent to $\rho_2$ (i.e., $\rho_1 \sim \rho_2$) if $\rho_1/\rho_2 \in U_B$.

It is trivial to show that "$\sim$" is a binary equivalence relation on $U_\infty$. The equivalence class of a given weight $\rho \in U_\infty$ which is denoted by $\mathrm{cl}(\rho) = \{\varrho \in U_\infty : \rho \sim \varrho\}$. It is clear that $U_\infty = \bigcup_{\rho\in U_\infty}\mathrm{cl}(\rho)$.

For $\rho_1 \in U_\infty$, define the weighted ergodic space [17]

$$\overline{WPAA}_0(\mathbb{Z},X) := \left\{f : \mathbb{Z} \to X \text{ is bounded}, \lim_{T\to+\infty}\frac{1}{\mu(T,\rho_1)}\sum_{k=-T}^{T}\|f(k)\|\rho_1(k) = 0\right\}. \tag{8}$$

Particularly, for $\rho_1, \rho_2 \in U_\infty$, define

$$WPAA_0(\mathbb{Z},X) := \left\{f : \mathbb{Z} \to X \text{ is bounded}, \lim_{T\to+\infty}\frac{1}{\mu(T,\rho_1)}\sum_{k=-T}^{T}\|f(k)\|\rho_2(k) = 0\right\}. \tag{9}$$

Clearly, when $\rho_1 \sim \rho_2$, $WPAA_0(\mathbb{Z},X)$ coincide with $\overline{WPAA}_0(\mathbb{Z},X)$; that is, $WPAA_0(\mathbb{Z},X) = \overline{WPAA}_0(\mathbb{Z},X)$, this fact suggests that $WPAA_0(\mathbb{Z},X)$ are more interesting when $\rho_1$ and $\rho_2$ are not necessarily equivalent. So $WPAA_0(\mathbb{Z},X)$ are general and richer than $\overline{WPAA}_0(\mathbb{Z},X)$.

*Definition 8.* Let $\rho_1, \rho_2 \in U_\infty$. A function $f : \mathbb{Z} \to X$ is called discrete weighted pseudo almost automorphic if it can be expressed as $f = g + \varphi$, where $g \in AA_d(\mathbb{Z},X)$ and $\varphi \in WPAA_0(\mathbb{Z},X)$. The set of such functions is denoted by $WPAA_d(\mathbb{Z},X)$.

*Remark 9.* If $\rho_1 \sim \rho_2$, $WPAA_d(\mathbb{Z},X)$ coincide with the discrete weighted pseudo almost automorphic functions defined in [17].

Throughout the rest of the paper, we denote by $V_\infty$ the set of all the functions $\rho_1, \rho_2 \in U_\infty$ satisfying that there exists an unbounded set $\Omega \subset \mathbb{Z}$ such that for all $m \in \mathbb{Z}$,

$$\limsup_{|k|\to+\infty, k\in\Omega}\frac{\rho_2(k+m)}{\rho_1(k)} := \inf_{N\in\mathbb{N}}\left[\sup_{|k|\ge N, k\in\Omega}\frac{\rho_2(k+m)}{\rho_1(k)}\right] < +\infty,$$
$$\lim_{T\to+\infty}\frac{\sum_{k\in([-T,T]\setminus\Omega)+m}\rho_2(k)}{\mu(T,\rho_1)} = 0. \tag{10}$$

Next, we show some properties of the space $WPAA_d(\mathbb{Z},X)$.

Similar to the proof of [22], one has the following.

**Lemma 10.** *Let $\rho_1, \rho_2 \in V_\infty$, then*

(i) *for each $m \in \mathbb{Z}$, one has*

$$\limsup_{T\to+\infty}\frac{\mu(T+m,\rho_2)}{\mu(T,\rho_1)} < +\infty. \tag{11}$$

(ii) *$WPAA_0(\mathbb{Z},X)$ is translation invariant; that is, $f(\cdot + m) \in WPAA_0(\mathbb{Z},X)$ for each $m \in \mathbb{Z}$ if $f \in WPAA_0(\mathbb{Z},X)$.*

(iii) *$WPAA_d(\mathbb{Z},X)$ is translation invariant.*

(iv) *If $f = g + \varphi \in WPAA_d(\mathbb{Z},X)$, where $g \in AA_d(\mathbb{Z},X)$, $\varphi \in WPAA_0(\mathbb{Z},X)$, then*

$$\{g(n) : n \in \mathbb{Z}\} \subset \overline{\{f(n) : n \in \mathbb{Z}\}}. \tag{12}$$

(v) *$WPAA_d(\mathbb{Z},X)$ is a Banach space under the supremum norm; that is,*

$$\|f\|_d := \sup_{k\in\mathbb{Z}}\|f(k)\|. \tag{13}$$

**Lemma 11.** *If $A \in L(X)$ and $u \in WPAA_d(\mathbb{Z},X)$, then $Au \in WPAA_d(\mathbb{Z},X)$.*

The proof is straightforward and is therefore omitted.

Similarly, define

$$WPAA_0(\mathbb{Z}\times X,X) := \left\{f : \mathbb{Z} \longrightarrow X \text{ is bounded}, \lim_{T\to+\infty}\frac{1}{\mu(T,\rho_1)} \times \sum_{k=-T}^{T}\|f(k,x)\|\rho_2(k) = 0 \text{ uniform in } x \in X\right\}. \tag{14}$$

*Definition 12.* Let $\rho_1, \rho_2 \in U_\infty$. A function $f : \mathbb{Z}\times X \to X$ is said to be discrete weighted pseudo almost automorphic in $k \in \mathbb{Z}$ for each $x \in X$, if it can be decomposed as $f = g + \varphi$, where $g \in AA_d(\mathbb{Z}\times X,X)$ and $\varphi \in WPAA_0(\mathbb{Z}\times X,X)$. Denote by $WPAA_d(\mathbb{Z}\times X,X)$ the set of such functions.

We will establish composition theorem for discrete weighted pseudo almost automorphic functions.

**Lemma 13.** *Let $f : \mathbb{Z} \to X$ be bounded and $\rho_1, \rho_2 \in V_\infty$, then $f \in WPAA_0(\mathbb{Z},X)$ if and only if for any $\varepsilon > 0$,*

$$\lim_{T\to+\infty}\frac{1}{\mu(T,\rho_1)}\sum_{k\in E_f(T,\varepsilon)}\rho_2(k) = 0, \tag{15}$$

*where $E_f(T,\varepsilon) = \{k \in [-T,T]\cap\mathbb{Z} : \|f(k)\| \ge \varepsilon\}$.*

*Proof. Sufficiency.* It is clear that $M = \sup_{k\in\mathbb{Z}}\|f(k)\| < +\infty$ and $\mu(T,\rho_2)/\mu(T,\rho_1) < +\infty$ by Lemma 10. It follows from (15) that $\forall\varepsilon > 0$; there exists $N \in \mathbb{N}$ such that

$$\frac{1}{\mu(T,\rho_1)}\sum_{k\in E_f(T,\varepsilon)}\rho_2(k) < \frac{\varepsilon}{2M}, \quad T > N; \tag{16}$$

then

$$\begin{aligned}
&\frac{1}{\mu(T,\rho_1)}\sum_{k=-T}^{T}\|f(k)\|\rho_2(k)\\
&\quad= \frac{1}{\mu(T,\rho_1)}\sum_{k\in E_f(T,\varepsilon)}\|f(k)\|\rho_2(k)\\
&\qquad+ \frac{1}{\mu(T,\rho_1)}\sum_{k\in([-T,T]\cap\mathbb{Z})\setminus E_f(T,\varepsilon)}\|f(k)\|\rho_2(k)\\
&\quad\le M\cdot\frac{1}{\mu(T,\rho_1)}\sum_{k\in E_f(T,\varepsilon)}\rho_2(k)\\
&\qquad+ \varepsilon\cdot\frac{1}{\mu(T,\rho_1)}\sum_{k\in([-T,T]\cap\mathbb{Z})\setminus E_f(T,\varepsilon)}\rho_2(k)\\
&\quad\le \frac{\varepsilon}{2} + \varepsilon\cdot\frac{\mu(T,\rho_2)}{\mu(T,\rho_1)},
\end{aligned} \tag{17}$$

so

$$\lim_{T\to+\infty}\frac{1}{\mu(T,\rho_1)}\sum_{k=-T}^{T}\|f(k)\|\rho_2(k) = 0. \tag{18}$$

That is, $f \in WPAA_0(\mathbb{Z}, X)$.

*Necessity*. Suppose the contrary, that there exists $\varepsilon_0 > 0$, such that $(1/\mu(T,\rho_1))\sum_{k\in E_f(T,\varepsilon)}\rho_2(k)$ does not converge to 0 as $T \to +\infty$. That is, there exists $\delta > 0$, such that for each $m \in \mathbb{Z}$,

$$\frac{1}{\mu(T_m,\rho_1)}\sum_{k\in E_f(T_m,\varepsilon_0)}\rho_2(k) \ge \delta, \quad \text{for some } T_m > m. \tag{19}$$

Then

$$\begin{aligned}
&\frac{1}{\mu(T_m,\rho_1)}\sum_{k=-T_m}^{T_m}\|f(k)\|\rho_2(k)\\
&\quad= \frac{1}{\mu(T_m,\rho_1)}\sum_{k\in E_f(T_m,\varepsilon_0)}\|f(k)\|\rho_2(k)\\
&\qquad+ \frac{1}{\mu(T_m,\rho_1)}\sum_{k\in([-T_m,T_m]\cap\mathbb{Z})\setminus E_f(T_m,\varepsilon_0)}\|f(k)\|\rho_2(k)\\
&\quad\ge \frac{1}{\mu(T_m,\rho_1)}\sum_{k\in E_f(T_m,\varepsilon_0)}\|f(k)\|\rho_2(k)\\
&\quad\ge \varepsilon_0\cdot\frac{1}{\mu(T_m,\rho_1)}\sum_{k\in E_f(T_m,\varepsilon_0)}\rho_2(k)\\
&\quad\ge \delta\varepsilon_0,
\end{aligned} \tag{20}$$

which contradicts the fact that

$$\lim_{T\to+\infty}\frac{1}{\mu(T,\rho_1)}\sum_{k=-T}^{T}\|f(k)\|\rho_2(k) = 0. \tag{21}$$

Thus (15) holds. □

*Remark 14.* If $\rho_1 = \rho_2 = 1$ and $X = \mathbb{R}$, the result of Lemma 13 is obtained by [23, Lemma 2.9].

**Lemma 15.** *Let $f \in WPAA_0(\mathbb{Z}\times X, X)$ and $K \subseteq X$ be compact; if $f \in \mathscr{UC}(\mathbb{Z}\times K, X)$, then $\tilde{f}(\cdot) \in WPAA_0(\mathbb{Z}, X)$, where*

$$\tilde{f}(k) = \sup_{x\in K}\|f(k,x)\|, \quad k \in \mathbb{Z}. \tag{22}$$

*Proof.* Since $f \in \mathscr{UC}(\mathbb{Z}\times K, X)$, $\forall\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|f(k,u) - f(k,v)\| < \varepsilon \tag{23}$$

for all $k \in \mathbb{Z}$ and $u, v \in K$ with $\|u - v\| < \delta$. Note that $K$ is compact; for the above $\delta > 0$, there exists $x_1, x_2, \ldots, x_m \in K$ such that

$$K \subset \bigcup_{i=1}^{m} B(x_i, \delta). \tag{24}$$

Since $f \in WPAA_0(\mathbb{Z}\times X, X)$, for the above $\varepsilon > 0$, there exists $T_0 \in \mathbb{N}$ such that for all $T > T_0$,

$$\frac{1}{\mu(T,\rho_1)}\sum_{k=-T}^{T}\|f(k,x_i)\|\rho_2(k) < \frac{\varepsilon}{m}, \quad i = 1, 2\ldots, m. \tag{25}$$

For each $x \in K$, there exists $i \in \{1, 2, \ldots, m\}$ such that $\| x - x_i \| < \delta$; hence

$$\begin{aligned}
\|f(k,x)\| &\le \|f(k,x) - f(k,x_i)\| + \|f(k,x_i)\|\\
&\le \varepsilon + \|f(k,x_i)\|, \quad k \in \mathbb{Z},
\end{aligned} \tag{26}$$

which implies that

$$\tilde{f}(k) = \sup_{x\in K}\|f(k,x)\| \le \sum_{i=1}^{m}\|f(k,x_i)\| + \varepsilon, \quad k \in \mathbb{Z}. \tag{27}$$

Thus $\tilde{f}$ is bounded and

$$\begin{aligned}
&\frac{1}{\mu(T,\rho_1)}\sum_{k=-T}^{T}\|\tilde{f}(k)\|\rho_2(k)\\
&\quad\le \sum_{i=1}^{m}\left(\frac{1}{\mu(T,\rho_1)}\sum_{k=-T}^{T}\|f(k,x_i)\|\rho_2(k)\right)\\
&\qquad+ \varepsilon \le 2\varepsilon, \quad T > T_0,
\end{aligned} \tag{28}$$

which means that $\tilde{f}(\cdot) \in WPAA_0(\mathbb{Z}, X)$. □

**Theorem 16.** *Assume that* $\rho_1, \rho_2 \in V_\infty$, $f \in WPAA_d(\mathbb{Z} \times X, X) \cap \mathscr{UC}(\mathbb{Z} \times X, X)$; *then* $f(\cdot, h(\cdot)) \in WPAA_d(\mathbb{Z}, X)$ *if* $h \in WPAA_d(\mathbb{Z}, X)$.

*Proof.* Let

$$f = g + \varphi, \qquad h = x + y, \tag{29}$$

where $g \in AA_d(\mathbb{Z} \times X, X)$, $\varphi \in WPAA_0(\mathbb{Z} \times X, X)$, $x \in AA_d(\mathbb{Z}, X)$ and $y \in WPAA_0(\mathbb{Z}, X)$. The function $f$ can be decomposed as

$$\begin{aligned} f(n, h(n)) &= g(n, x(n)) + f(n, h(n)) - g(n, x(n)) \\ &= g(n, x(n)) + f(n, h(n)) - f(n, x(n)) \\ &\quad + \varphi(n, x(n)). \end{aligned} \tag{30}$$

Set

$$\begin{aligned} F(n) &= g(n, x(n)), \\ G(n) &= f(n, h(n)) - f(n, x(n)), \\ H(n) &= \varphi(n, x(n)), \quad n \in \mathbb{Z}. \end{aligned} \tag{31}$$

By Lemma 10, $\{g(n) : n \in \mathbb{Z}\} \subset \overline{\{f(n) : n \in \mathbb{Z}\}}$, it is not difficult to see that $g \in \mathscr{UC}(\mathbb{Z} \times X, X)$; hence $F(\cdot) \in AA_d(\mathbb{Z}, X)$ by Lemma 6.

We claim that $G(\cdot) \in WPAA_0(\mathbb{Z}, X)$. In fact, since $f \in \mathscr{UC}(\mathbb{Z} \times X, X)$, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$\|G(n)\| = \|f(n, h(n)) - f(n, x(n))\| < \varepsilon \tag{32}$$

for all $n \in \mathbb{Z}$ and $\|y(n)\| = \|h(n) - x(n)\| < \delta$; hence

$$E_G(n, \varepsilon) \subset E_y(n, \delta). \tag{33}$$

Since $y \in WPAA_0(\mathbb{Z}, X)$, by Lemma 13, one has

$$\lim_{T \to +\infty} \frac{1}{\mu(T, \rho_1)} \sum_{k \in E_y(T, \delta)} \rho_2(k) = 0, \tag{34}$$

which implies that

$$\lim_{T \to +\infty} \frac{1}{\mu(T, \rho_1)} \sum_{k \in E_G(T, \varepsilon)} \rho_2(k) = 0. \tag{35}$$

By Lemma 13, $G(\cdot) \in WPAA_0(\mathbb{Z}, X)$.

Since $f \in \mathscr{UC}(\mathbb{Z} \times X, X)$, then $\varphi \in \mathscr{UC}(\mathbb{Z} \times X, X)$. Let $K = \overline{\{x(n) : n \in \mathbb{Z}\}}$; by Lemma 15, one has

$$\lim_{T \to +\infty} \frac{1}{\mu(T, \rho_1)} \sum_{k=-T}^{T} \sup_{z \in K} \|\varphi(k, z)\| \rho_2(k) = 0, \tag{36}$$

so

$$\begin{aligned} &\lim_{T \to +\infty} \frac{1}{\mu(T, \rho_1)} \sum_{k=-T}^{T} \|\varphi(k, x(k))\| \rho_2(k) \\ &\quad \le \lim_{T \to +\infty} \frac{1}{\mu(T, \rho_1)} \sum_{k=-T}^{T} \sup_{z \in K} \|\varphi(k, z)\| \rho_2(k) = 0. \end{aligned} \tag{37}$$

That is $H(\cdot) \in WPAA_0(\mathbb{Z}, X)$. Then $F(\cdot) \in AA_d(\mathbb{Z}, X)$, $G(\cdot) + H(\cdot) \in WPAA_0(\mathbb{Z}, X)$. Hence $f(\cdot, h(\cdot)) \in WPAA_d(\mathbb{Z}, X)$. □

**Corollary 17.** *Let* $\rho_1, \rho_2 \in V_\infty$, $f \in WPAA_d(\mathbb{Z} \times X, X)$ *and satisfies the Lipschitz condition*

$$\|f(k, u) - f(k, v)\| \le L_f \|u - v\|, \quad \forall k \in \mathbb{Z}, \ u, v \in X, \tag{38}$$

*then* $f(\cdot, h(\cdot)) \in WPAA_d(\mathbb{Z}, X)$ *if* $h \in WPAA_d(\mathbb{Z}, X)$.

The following Lemma is the essential property to study the existence of $WPAA_d(\mathbb{Z}, X)$ solutions of Volterra difference equations of convolution type.

**Lemma 18.** *Let* $\nu : \mathbb{Z}^+ \to \mathbb{C}$ *be a summable function; if* $u \in WPAA_d(\mathbb{Z}, X)$, $\rho_1, \rho_2 \in V_\infty$, *then* $\Lambda(\cdot) \in WPAA_d(\mathbb{Z}, X)$, *where*

$$\Lambda(k) = \sum_{l=-\infty}^{k} |\nu(k - l)| u(l), \quad k \in \mathbb{Z}. \tag{39}$$

*Proof.* Note that

$$\|\Lambda(k)\| \le \sum_{l=0}^{\infty} |\nu(l)| \|u(k - l)\| \le \|u\|_d \cdot \sum_{l=0}^{\infty} |\nu(l)| < \infty; \tag{40}$$

hence $\Lambda$ is bounded.

Let $u = u_1 + u_2$, where $u_1 \in AA_d(\mathbb{Z}, X)$, $u_2 \in WPAA_0(\mathbb{Z}, X)$; then

$$\Lambda = \Lambda_1 + \Lambda_2, \tag{41}$$

where

$$\begin{aligned} \Lambda_1(k) &= \sum_{l=-\infty}^{k} |\nu(k - l)| u_1(l), \\ \Lambda_2(k) &= \sum_{l=-\infty}^{k} |\nu(k - l)| u_2(l), \quad k \in \mathbb{Z}. \end{aligned} \tag{42}$$

The almost automorphy of $\Lambda_1$ follows from [10].

Next, we show that $\Lambda_2 \in WPAA_0(\mathbb{Z}, X)$. In fact,

$$\begin{aligned} &\frac{1}{\mu(T, \rho_1)} \sum_{k=-T}^{T} \|\Lambda_2(k)\| \rho_2(k) \\ &\quad \le \frac{1}{\mu(T, \rho_1)} \sum_{k=-T}^{T} \sum_{l=-\infty}^{k} |\nu(k - l)| \|u_2(l)\| \rho_2(k) \\ &\quad = \frac{1}{\mu(T, \rho_1)} \sum_{k=-T}^{T} \sum_{l=0}^{\infty} |\nu(l)| \|u_2(k - l)\| \rho_2(k) \\ &\quad = \sum_{l=0}^{\infty} |\nu(l)| \left( \frac{1}{\mu(T, \rho_1)} \sum_{k=-T}^{T} \|u_2(k - l)\| \rho_2(k) \right). \end{aligned} \tag{43}$$

Since $u_2 \in WPAA_0(\mathbb{Z}, X)$, $u_2(\cdot - l) \in WPAA_0(\mathbb{Z}, X)$ for each $l \in \mathbb{Z}$ by Lemma 10, then

$$\lim_{T\to\infty} \frac{1}{\mu(T,\rho_1)} \sum_{k=-T}^{T} \|u_2(k-l)\| \rho_2(k) = 0; \quad (44)$$

by Lebesgue dominated convergence theorem, one has

$$\lim_{T\to\infty} \frac{1}{\mu(T,\rho_1)} \sum_{k=-T}^{T} \|\Lambda_2(k)\| \rho_2(k)$$

$$\leq \lim_{T\to\infty} \sum_{l=0}^{\infty} |v(l)| \left( \frac{1}{\mu(T,\rho_1)} \sum_{k=-T}^{T} \|u_2(k-l)\| \rho_2(k) \right) = 0, \quad (45)$$

which implies that $\Lambda_2 \in WPAA_0(\mathbb{Z}, X)$. The proof is completed. □

*Remark 19.* If $\rho_1 \sim \rho_2$, the results of Corollary 17 and Lemma 18 are obtained by [17, Theorem 2.8] and [17, Theorem 2.9], respectively.

## 4. Applications in Difference Equations

As an application, the main goal of this section is to establish some sufficient criteria for the existence and uniqueness of $WPAA_d$ solution to the Volterra difference equations and nonautonomous semilinear difference equations.

*4.1. Volterra Difference Equation.* This subsection is devoted to establish some sufficient criteria for the existence and uniqueness of $WPAA_d$ solutions of (46).

Consider the Volterra difference equations of convolution type

$$u(n+1) = \lambda \sum_{j=-\infty}^{n} a(n-j)u(j) + f(n, Au(n)), \quad n \in \mathbb{Z}, \quad (46)$$

where $\lambda \in \mathbb{C}$, $a(\cdot)$ is a summable function; $A \in L(X)$ and $f \in WPAA_d(\mathbb{Z} \times X, X)$. Its associated homogeneous linear equation is given by

$$u(n+1) = \lambda \sum_{j=-\infty}^{n} a(n-j)u(j) + f(n), \quad n \in \mathbb{Z}, \quad (47)$$

where $\lambda \in \mathbb{C}$, $a(\cdot)$ is a summable function.

For a given $\lambda \in \mathbb{C}$, let $s(\lambda, k) \in \mathbb{C}$ be the solution of the difference equation

$$s(\lambda, k+1) = \lambda \sum_{j=0}^{k} a(k-j)s(\lambda, j), \quad k = 0, 1, 2, \ldots \quad (48)$$

$$s(\lambda, 0) = 1.$$

In this case, $s(\lambda, k)$ is called the fundamental solution to (47) generated by $a(\cdot)$. We define the set

$$\Omega_s = \left\{ \lambda \in \mathbb{C}, \|s(\lambda,\cdot)\|_1 := \sum_{k=0}^{\infty} |s(\lambda,k)| < \infty \right\}. \quad (49)$$

By [13], if $\lambda \in \Omega_s$, the solution of (47) is given by

$$u(n+1) = \sum_{k=-\infty}^{n} s(\lambda, n-k) f(k). \quad (50)$$

To establish our results, we introduce the following condition.

($H_1$) $\lambda \in \Omega_s$, $A \in L(X)$.

($H_2$) $f \in WPAA_d(\mathbb{Z} \times X, X)$, $\rho_1, \rho_2 \in V_\infty$.

($H_{31}$) There exists a constant $L_f > 0$ such that

$$\|f(k,u) - f(k,v)\| \leq L_f \|u - v\|, \quad \forall k \in \mathbb{Z}, \ u, \ v \in X. \quad (51)$$

($H_{32}$) There exists a linear nondecreasing function $\Phi : [0,\infty) \to [0,\infty)$ and $f$ satisfies

$$\|f(k,u) - f(k,v)\| \leq \Phi(\|u - v\|), \quad \forall k \in \mathbb{Z}, \ u, \ v \in X. \quad (52)$$

**Theorem 20.** *Assume that ($H_1$), ($H_2$), and ($H_{31}$) hold and $L_f \|A\| |s(\lambda,\cdot)|_1 < 1$; then* (46) *has a unique solution $u(n) \in WPAA_d(\mathbb{Z}, X)$ which is given by*

$$u(n+1) = \sum_{k=-\infty}^{n} s(\lambda, n-k) f(k, Au(k)). \quad (53)$$

*Proof.* Similar to the proof of [13], it can be shown that a solution of (53) is the solution of (46).

Define the operator $\mathcal{F} : WPAA_d(\mathbb{Z}, X) \to WPAA_d(\mathbb{Z}, X)$ by

$$(\mathcal{F}u)(n) = \sum_{k=-\infty}^{n-1} s(\lambda, n-1-k) f(k, Au(k)). \quad (54)$$

Since $u \in WPAA_d(\mathbb{Z}, X)$ and ($H_{31}$) holds, $f(\cdot, Au(\cdot)) \in WPAA_d(\mathbb{Z}, X)$ by Lemma 11, Corollary 17. By Lemma 18, $\mathcal{F}u \in WPAA_d(\mathbb{Z}, X)$. Hence $\mathcal{F}$ is well-defined.

For $u, v \in WPAA_d(\mathbb{Z}, X)$,

$$\|\mathcal{F}u - \mathcal{F}v\|_d$$

$$\leq \sup_{n\in\mathbb{Z}} \sum_{k=-\infty}^{n-1} |s(\lambda, n-1-k)| \|f(k, Au(k)) - f(k, Av(k))\|$$

$$\leq L_f \|A\| \sup_{n\in\mathbb{Z}} \sum_{k=-\infty}^{n-1} |s(\lambda, n-1-k)| \|u(k) - v(k)\|$$

$$\leq L_f \|A\| \|u - v\|_d \sup_{n\in\mathbb{Z}} \sum_{k=-\infty}^{n-1} |s(\lambda, n-1-k)|$$

$$\leq L_f \|A\| |s(\lambda,\cdot)|_1 \|u - v\|_d. \quad (55)$$

By the Banach contraction mapping principle, $\mathcal{F}$ has a unique fixed point $u \in WPAA_d(\mathbb{Z}, X)$, which is the unique $WPAA_d$ solution to (46). □

**Theorem 21.** *Assume that* $(H_1)$, $(H_2)$, *and* $(H_{32})$ *hold; then* (46) *has a unique solution* $u(n) \in WPAA_d(\mathbb{Z}, X)$ *if* $(\|A\| |s(\lambda,\cdot)|_1 \Phi)^n(t) \to 0$ *as* $n \to \infty$ *for each* $t > 0$.

*Proof.* Define the operator $\mathcal{F}$ as in (54), so $\mathcal{F}$ is well defined. For $u, v \in WPAA_d(\mathbb{Z}, X)$, one has

$$\begin{aligned}
&\|(\mathcal{F}u)(n) - (\mathcal{F}v)(n)\| \\
&\quad \le \sum_{k=-\infty}^{n-1} |s(\lambda, n-1-k)| \left\| f(k, Au(k)) - f(k, Av(k)) \right\| \\
&\quad \le \sum_{k=-\infty}^{n-1} |s(\lambda, n-1-k)| \,\Phi(\|Au(k) - Av(k)\|) \\
&\quad \le \|A\| \,|s(\lambda,\cdot)|_1 \Phi(\|u(k) - v(k)\|).
\end{aligned} \tag{56}$$

Since $(\|A\| |s(\lambda,\cdot)|_1 \Phi)^n(t) \to 0$ as $n \to \infty$ for each $t > 0$, by Matkowski fixed point theorem (Theorem 1), $\mathcal{F}$ has a unique fixed point $u \in WPAA_d(\mathbb{Z}, X)$, which is the unique $WPAA_d$ solution to (46). □

*4.2. Nonautonomous Semilinear Difference Equations.* In this subsection, consider the following nonautonomous semilinear difference equations

$$u(n+1) = A(n)u(n) + f(n, u(n)), \quad n \in \mathbb{Z}. \tag{57}$$

Its associated homogeneous linear difference equation is given by

$$u(n+1) = A(n)u(n), \quad n \in \mathbb{Z}. \tag{58}$$

To establish our results, we introduce the following condition.

$(A_1)$ $A \in AA_d(\mathbb{Z}, X)$.

$(A_2)$ $f \in WPAA_d(\mathbb{Z} \times X, X)$, $\rho_1, \rho_2 \in V_\infty$.

$(A_3)$ There exists a constant $L_f > 0$ such that

$$\left\| f(k, u) - f(k, v) \right\| \le L_f \|u - v\|, \quad \forall k \in \mathbb{Z}, u, \ v \in X. \tag{59}$$

$(A_4)$ Equation (58) admits an exponential dichotomy on $\mathbb{Z}$ with positive constants $\eta, \nu, \alpha, \beta$.

**Theorem 22.** *Assume that* $(A_1)$–$(A_4)$ *hold and* $(\eta/(1-e^{-\alpha}) + \nu e^{-\beta}/(1-e^{-\beta}))L_f < 1$; *then* (57) *has a unique solution* $u(n) \in WPAA_d(\mathbb{Z}, X)$ *which is given by*

$$\begin{aligned}
u(n) = &\sum_{j=-\infty}^{n-1} \mathcal{A}(n, j+1) P(j+1) f(j, u(j)) \\
&- \sum_{j=n}^{\infty} \mathcal{A}(n, j+1) Q(j+1) f(j, u(j)), \quad n \in \mathbb{Z}.
\end{aligned} \tag{60}$$

*Proof.* Similar to the proof of [14, 15], it can be shown that $u(\cdot)$ given by (60) is the solution of (57).

Define an operator $\Gamma : WPAA_d(\mathbb{Z}, X) \to WPAA_d(\mathbb{Z}, X)$ as follows:

$$\begin{aligned}
(\Gamma u)(n) := &\sum_{j=-\infty}^{n-1} \mathcal{A}(n, j+1) P(j+1) f(j, u(j)) \\
&- \sum_{j=n}^{\infty} \mathcal{A}(n, j+1) Q(j+1) f(j, u(j)), \quad n \in \mathbb{Z}.
\end{aligned} \tag{61}$$

Since $u \in WPAA_d(\mathbb{Z}, X)$ and $(A_3)$ holds, $\Psi(\cdot) = f(\cdot, u(\cdot)) \in WPAA_d(\mathbb{Z}, X)$ by Corollary 17. Let $\Psi = \Psi_1 + \Psi_2$, where $\Psi_1 \in AA_d(\mathbb{Z}, X)$, $\Psi_2 \in WPAA_0(\mathbb{Z}, X)$; then

$$(\Gamma u)(n) := (\Gamma_1 u)(n) + (\Gamma_2 u)(n), \tag{62}$$

where

$$\begin{aligned}
(\Gamma_1 u)(n) = &\sum_{j=-\infty}^{n-1} \mathcal{A}(n, j+1) P(j+1) \Psi_1(j) \\
&- \sum_{j=n}^{\infty} \mathcal{A}(n, j+1) Q(j+1) \Psi_1(j), \\
(\Gamma_2 u)(n) = &\sum_{j=-\infty}^{n-1} \mathcal{A}(n, j+1) P(j+1) \Psi_2(j) \\
&- \sum_{j=n}^{\infty} \mathcal{A}(n, j+1) Q(j+1) \Psi_2(j).
\end{aligned} \tag{63}$$

Similar to the proof of [14, 15], $\Gamma_1 u \in AA_d(\mathbb{Z}, X)$.

Next, we show that $\Gamma_2 u \in WPAA_0(\mathbb{Z}, X)$. In fact, let

$$(\Gamma_2 u)(n) := (\Gamma_{21} u)(n) + (\Gamma_{22} u)(n), \tag{64}$$

where

$$\begin{aligned}
(\Gamma_{21} u)(n) &= \sum_{j=-\infty}^{n-1} \mathcal{A}(n, j+1) P(j+1) \Psi_2(j), \\
(\Gamma_{22} u)(n) &= \sum_{j=n}^{\infty} \mathcal{A}(n, j+1) Q(j+1) \Psi_2(j).
\end{aligned} \tag{65}$$

Then,

$$\begin{aligned}
&\frac{1}{\mu(T, \rho_1)} \sum_{k=-T}^{T} \|(\Gamma_{21} u)(k)\| \rho_2(k) \\
&\quad = \frac{1}{\mu(T, \rho_1)} \sum_{k=-T}^{T} \left\| \sum_{j=-\infty}^{k-1} \mathcal{A}(k, j+1) P(j+1) \Psi_2(j) \right\| \rho_2(k)
\end{aligned}$$

$$\leq \frac{1}{\mu(T,\rho_1)} \sum_{k=-T}^{T} \sum_{j=-\infty}^{k-1} \eta e^{-\alpha(k-j-1)} \left\| \Psi_2(j) \right\| \rho_2(k)$$
$$= \frac{1}{\mu(T,\rho_1)} \sum_{k=-T}^{T} \sum_{j=0}^{\infty} \eta e^{-\alpha j} \left\| \Psi_2(k-1-j) \right\| \rho_2(k)$$
$$= \sum_{j=0}^{\infty} \eta e^{-\alpha j} \left( \frac{1}{\mu(T,\rho_1)} \sum_{k=-T}^{T} \left\| \Psi_2(k-1-j) \right\| \rho_2(k) \right). \tag{66}$$

Since $\Psi_2 \in WPAA_0(\mathbb{Z}, X)$, $\Psi_2(\cdot - 1 - j) \in WPAA_0(\mathbb{Z}, X)$ for each $j \in \mathbb{Z}$ by Lemma 10; then

$$\lim_{T\to\infty} \frac{1}{\mu(T,\rho_1)} \sum_{k=-T}^{T} \left\| \Psi_2(k-1-j) \right\| \rho_2(k) = 0; \tag{67}$$

by Lebesgue dominated convergence theorem, one has

$$\lim_{T\to\infty} \frac{1}{\mu(T,\rho_1)} \sum_{k=-T}^{T} \left\| (\Gamma_{21}u)(k) \right\| \rho_2(k)$$
$$\leq \sum_{j=0}^{\infty} \eta e^{-\alpha j} \left( \lim_{T\to\infty} \frac{1}{\mu(T,\rho_1)} \sum_{k=-T}^{T} \left\| \Psi_2(k-1-j) \right\| \rho_2(k) \right)$$
$$= 0; \tag{68}$$

hence $\Gamma_{21}u \in WPAA_0(\mathbb{Z}, X)$. Similarly, one can prove $\Gamma_{22}u \in WPAA_0(\mathbb{Z}, X)$. So $\Gamma$ is well defined.

For $u, v \in WPAA_d(\mathbb{Z}, X)$, by the exponential dichotomy and Lipschitz condition, one has

$$\left\| (\Gamma u)(n) - (\Gamma v)(n) \right\|$$
$$= \Bigg\| \sum_{j=-\infty}^{n-1} \mathscr{A}(n, j+1) P(j+1) \left[ f(j, u(j)) - f(j, v(j)) \right]$$
$$- \sum_{j=n}^{\infty} \mathscr{A}(n, j+1) Q(j+1) \times \left[ f(j, u(j)) - f(j, v(j)) \right] \Bigg\|$$
$$\leq \sum_{j=-\infty}^{n-1} \eta e^{-\alpha(n-j-1)} L_f \left\| u(j) - v(j) \right\| + \sum_{j=n}^{+\infty} \nu e^{-\beta(j+1-n)} L_f \left\| u(j) - v(j) \right\|$$
$$\leq \eta L_f \|u - v\|_d \sum_{j=0}^{+\infty} e^{-\alpha j} + \nu L_f \|u - v\|_d \sum_{j=1}^{+\infty} e^{-\beta j}$$
$$\leq \frac{\eta L_f}{1 - e^{-\alpha}} \|u - v\|_d + \frac{\nu L_f e^{-\beta}}{1 - e^{-\beta}} \|u - v\|_d$$
$$= \left( \frac{\eta}{1 - e^{-\alpha}} + \frac{\nu e^{-\beta}}{1 - e^{-\beta}} \right) L_f \|u - v\|_d; \tag{69}$$

hence $\Gamma$ is a contraction. By the Banach contraction mapping principle, $\Gamma$ has a unique fixed point $u \in WPAA_d(\mathbb{Z}, X)$, which is the unique $WPAA_d$ solution to (57). The proof is completed. □

## 5. Examples

In this section, we provide some examples to illustrate our main results.

*Example 1.* For $a(k) = p^k$, where $|p| < 1$, after a calculation using in (48) the unilateral-$Z$ transform, that $s(\lambda, k) = \lambda(\lambda + p)^{k-1}$, $k \geq 1$, and define

$$\mathbb{D}(-p, 1) := \left\{ z \in \mathbb{C} : \left| z + p \right| < 1 \right\} \subseteq \Omega_s. \tag{70}$$

Consider the following difference equation:

$$u(n+1) = \lambda \sum_{k=-\infty}^{n} p^{n-k} u(k) + \kappa g(k) u(k), \quad n \in \mathbb{Z}, \tag{71}$$

where $|p| < 1$, $\lambda \in \mathbb{D}(-p, 1)$, $g \in WPAA_d(\mathbb{Z}, X)$. It is easy to see that $(\mathrm{H}_1)$, $(\mathrm{H}_2)$, and $(\mathrm{H}_{31})$ hold with $L_f = |\kappa| \|g\|_d$. By Theorem 20, if $|\lambda||\kappa| \parallel g\|_d \sum_{k=0}^{\infty} |\lambda + p|^{k-1} < 1$, then (71) has a unique solution $u(n) \in WPAA_d(\mathbb{Z}, X)$.

*Example 2.* Consider the system

$$u(k+1) = Au(k) + h(k) g(u), \quad k \in \mathbb{Z}, \tag{72}$$

where $A$ is a nonsingular $n \times n$ matrix such that $\sigma_p(A) \cap S^1 = \emptyset$, $h \in WPAA_d(\mathbb{Z}, \mathbb{R}^n)$, and there exists a constant $L_f > 0$ such that

$$\left\| g(u) - g(v) \right\| \leq L_g \|u - v\|, \quad u, v \in \mathbb{R}^n. \tag{73}$$

Since $\sigma_p(A) \cap S^1 = \emptyset$, the system

$$u(k+1) = Au(k), \quad k \in \mathbb{Z} \tag{74}$$

admits an exponential dichotomy with positive constants $\eta$, $\nu$, $\alpha$, $\beta$ [19] and $(\mathrm{A}_3)$ holds with $L_f = L_g \|h\|_d$. By Theorem 22, If we suppose that $(\eta/(1 - e^{-\alpha}) + \nu e^{-\beta}/(1 - e^{-\beta})) L_g \|h\|_d < 1$, then (72) has a unique discrete weighted pseudo almost automorphic solution.

## Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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