Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 982378, 6 pages
http://dx.doi.org/10.1155/2014/982378

Research Article

On the Periodicity of Some Classes of Systems of Nonlinear Difference Equations

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Received 25 September 2013; Revised 3 December 2013; Accepted 4 December 2013; Published 23 January 2014

Academic Editor: Josef Diblik

Some classes of systems of difference equations whose all well-defined solutions are periodic are presented in this note.

1. Introduction

There has been a great recent interest in studying difference equations and systems of difference equations which do not stem from differential ones (see, e.g., [1–19] and the references therein). For some results on concrete systems of nonlinear difference equations, see, for example, [1, 3–5, 9–12, 18, 19]. Some classical results in the topic can be found, for example, in book [20].

Solution $(x_n^{(1)},\dots,x_n^{(l)})_{n\ge -k}$, of the system of difference equations

$$\begin{aligned} x_n^{(1)} &= f_1\left(x_{n-1}^{(1)},\dots,x_{n-k_1}^{(1)},\dots,x_{n-1}^{(l)},\dots,x_{n-k_l}^{(l)}\right),\\ x_n^{(2)} &= f_2\left(x_{n-1}^{(1)},\dots,x_{n-k_1}^{(1)},\dots,x_{n-1}^{(l)},\dots,x_{n-k_l}^{(l)}\right),\\ &\vdots\\ x_n^{(l)} &= f_l\left(x_{n-1}^{(1)},\dots,x_{n-k_1}^{(1)},\dots,x_{n-1}^{(l)},\dots,x_{n-k_l}^{(l)}\right), \end{aligned} \tag{1}$$

where $n\in\mathbb{N}_0$ and $k=\max\{k_1,\dots,k_l\}$, is called eventually periodic with period p, if there is an $n_1\ge -k$ such that

$$x_{n+p}^{(j)} = x_n^{(j)}, \tag{2}$$

for every $j=\overline{1,l}$, and $n\ge n_1$. It is periodic with period p if $n_1=-k$. Period p is prime if there is no $\widehat{p}\in\mathbb{N}$, $\widehat{p}<p$, which is a period. If all well-defined solutions of an equation or a system of difference equations are eventually periodic with the same period, then such an equation or system is called periodic. For some results on the periodicity, asymptotic periodicity and periodic equations or systems of difference equations see, for example, [1–10, 12–14, 16–19] and the related references therein.

In recent paper [19], the authors formulated four results which claim that the following systems of difference equations are periodic with period ten:

$$x_{n+1}=\frac{y_n}{x_{n-1}(1+y_n)},\quad y_{n+1}=\frac{x_n}{y_{n-1}(1+x_n)},\quad n\in\mathbb{N}_0; \tag{3}$$

$$x_{n+1}=\frac{y_n}{x_{n-1}(-1+y_n)},\quad y_{n+1}=\frac{x_n}{y_{n-1}(-1+x_n)},\quad n\in\mathbb{N}_0; \tag{4}$$

$$x_{n+1}=\frac{y_n}{x_{n-1}(1+y_n)},\quad y_{n+1}=\frac{x_n}{y_{n-1}(-1+x_n)},\quad n\in\mathbb{N}_0; \tag{5}$$

$$x_{n+1}=\frac{y_n}{x_{n-1}(-1+y_n)},\quad y_{n+1}=\frac{x_n}{y_{n-1}(1+x_n)},\quad n\in\mathbb{N}_0. \tag{6}$$

First, we show that all the results in [19] follow from known ones in the literature and also present some extensions of these results in the spirit of systems (3)–(6). To do this, we will use a system of difference equations related to the following, so called, Lyness difference equation:

$$x_{n+1} = \frac{1+x_n}{x_{n-1}}, \quad n \in \mathbb{N}_0. \tag{7}$$

It is easy to see that every well-defined solution of (7) is periodic with period five. The equation arises in frieze patterns (for the original sources, see [21–23]).

Studying max-type equations and systems of difference equations is another topic of a recent interest (see, e.g, [2, 3, 5–7, 10, 11, 15–19]).

Some special cases of the following max-type difference equation:

$$x_n = \max\left\{\frac{A_n}{x_{n-s}}, x_{n-k}\right\}, \quad n \in \mathbb{N}_0, \tag{8}$$

where $s, k \in \mathbb{N}$, and $(A_n)_{n\in\mathbb{N}_0} \subset \mathbb{R}$, have been studied, for example, in [2, 16]. Positive solutions of (8) are periodic in many cases. However, if $(A_n)_{n\in\mathbb{N}_0}$ is not a positive sequence, it was shown in [2] that (8) can have unbounded solutions.

In [5], it was shown that all solutions of the following max-type system of difference equations:

$$x_{n+1} = \max\left\{\frac{A_n}{y_n}, x_{n-1}\right\}, \qquad cy_{n+1} = \max\left\{\frac{B_n}{x_n}, y_{n-1}\right\}, \quad n \in \mathbb{N}_0, \tag{9}$$

where $x_0, x_{-1}, y_0, y_{-1} \in (0, +\infty)$ and $(A_n)_{n\in\mathbb{N}_0}$, $(B_n)_{n\in\mathbb{N}_0}$ are positive two-periodic sequences, are eventually periodic with, not necessarily prime, period two. This was done by direct calculation.

By using direct calculation, it can be easily shown that positive solutions of the following max-type system of difference equations:

$$x_{n+1} = \max\left\{\frac{A_n}{x_n}, y_{n-1}\right\}, \qquad y_{n+1} = \max\left\{\frac{B_n}{y_n}, x_{n-1}\right\}, \quad n \in \mathbb{N}_0, \tag{10}$$

where $(A_n)_{n\in\mathbb{N}_0}$ and $(B_n)_{n\in\mathbb{N}_0}$ are positive two-periodic sequences, are also periodic.

Here, we give a noncalculatory explanation of the fact by proving that positive solutions of the following max-type system of difference equations:

$$x_n = \max\left\{\frac{A_n}{x_{n-s}}, y_{n-k}\right\}, \qquad y_n = \max\left\{\frac{B_n}{y_{n-s}}, x_{n-k}\right\}, \quad n \in \mathbb{N}_0, \tag{11}$$

where $s, k \in \mathbb{N}$, and $(A_n)_{n\in\mathbb{N}_0}$, $(B_n)_{n\in\mathbb{N}_0}$ are positive periodic sequences of a certain period, are also periodic. We also present another extension of the result.

2. Some Extensions of Systems (3)–(6)

In this section, we present some periodic systems of difference equations in the spirit of systems (3)–(6).

Theorem 1. *Consider the following system of difference equations*

$$\begin{aligned} x_{n+1}^{(1)} &= f_1^{-1}\left(\frac{1+f_2\left(x_n^{(2)}\right)}{f_3\left(x_{n-1}^{(3)}\right)}\right), \\ &\vdots \\ x_{n+1}^{(k-1)} &= f_{k-1}^{-1}\left(\frac{1+f_k\left(x_n^{(k)}\right)}{f_1\left(x_{n-1}^{(1)}\right)}\right), \\ x_{n+1}^{(k)} &= f_k^{-1}\left(\frac{1+f_1\left(x_n^{(1)}\right)}{f_2\left(x_{n-1}^{(2)}\right)}\right), \quad n \in \mathbb{N}_0, \end{aligned} \tag{12}$$

where $k \in \mathbb{N}\setminus\{1\}$, and functions f_j, $j = \overline{1,k}$, are continuous on their domains; map the set $\mathbb{R}\setminus\{0\}$ onto itself and, for each fixed $j \in \{1,\dots,k\}$, f_j is simultaneously increasing or decreasing on the intervals $(-\infty, 0)$ and $(0, +\infty)$.

Then the following statements hold.

(a) *If $k \not\equiv 0 \pmod 5$, then every well-defined solution of system* (12) *is periodic with period* $5k$.

(b) *If $k \equiv 0 \pmod 5$, then every well-defined solution of system* (12) *is periodic with period* k.

Proof. From the conditions of the theorem, it follows that for each $j \in \{1,\dots,k\}$, there is f_j^{-1} which continuously map the set $\mathbb{R}\setminus\{0\}$ onto itself. Using the change of variables

$$y_n^{(j)} = f_j\left(x_n^{(j)}\right), \quad j = \overline{1,k}, \tag{13}$$

system (12) is easily transformed into the next one

$$y_{n+1}^{(1)} = \frac{1+y_n^{(2)}}{y_{n-1}^{(1)}}, \qquad y_{n+1}^{(2)} = \frac{1+y_n^{(1)}}{y_{n-1}^{(2)}}, \tag{14}$$

for $k = 2$,

$$y_{n+1}^{(1)} = \frac{1+y_n^{(2)}}{y_{n-1}^{(3)}}, \qquad y_{n+1}^{(2)} = \frac{1+y_n^{(3)}}{y_{n-1}^{(1)}}, \qquad y_{n+1}^{(3)} = \frac{1+y_n^{(1)}}{y_{n-1}^{(2)}}, \tag{15}$$

for $k = 3$, and

$$y_{n+1}^{(1)} = \frac{1+y_n^{(2)}}{y_{n-1}^{(3)}}, \qquad y_{n+1}^{(2)} = \frac{1+y_n^{(3)}}{y_{n-1}^{(4)}}, \dots, y_{n+1}^{(k)} = \frac{1+y_n^{(1)}}{y_{n-1}^{(2)}}, \tag{16}$$

for $k \geq 4$. In [4], it was proved that, if $k \not\equiv 0 \pmod 5$, then every well-defined solution of systems (14)–(16) is periodic with period $5k$, and, if $k \equiv 0 \pmod 5$, then every well-defined solution of systems (14)–(16) is periodic with period k. Using this along with the fact

$$x_n^{(j)} = f_j^{-1}\left(y_n^{(j)}\right), \quad j = \overline{1,k}, \tag{17}$$

following from (13), the results in (a) and (b) follow. □

The following theorem is proved in a similar way. Therefore, the proof will be omitted.

Theorem 2. *Consider the following system of difference equations*

$$\begin{aligned} x_{n+1}^{(1)} &= f_1^{-1}\left(\frac{1+f_k\left(x_n^{(k)}\right)}{f_{k-1}\left(x_{n-1}^{(k-1)}\right)}\right), \\ x_{n+1}^{(2)} &= f_2^{-1}\left(\frac{1+f_1\left(x_n^{(1)}\right)}{f_k\left(x_{n-1}^{(k)}\right)}\right), \\ &\vdots \\ x_{n+1}^{(k)} &= f_k^{-1}\left(\frac{1+f_{k-1}\left(x_n^{(k-1)}\right)}{f_{k-2}\left(x_{n-1}^{(k-2)}\right)}\right), \quad n \in \mathbb{N}_0, \end{aligned} \tag{18}$$

where $k \in \mathbb{N}\setminus\{1\}$, and functions f_j, $j = \overline{1,k}$, are continuous on their domains; map the set $\mathbb{R}\setminus\{0\}$ onto itself and, for each fixed $j \in \{1,\ldots,k\}$, f_j is simultaneously increasing or decreasing on the intervals $(-\infty, 0)$ and $(0, +\infty)$.

Then the following statements hold.

(a) *If $k \not\equiv 0 \pmod 5$, then every well-defined solution of system (18) is periodic with period $5k$.*

(b) *If $k \equiv 0 \pmod 5$, then every well-defined solution of system (18) is periodic with period k.*

Now, we show that all the results on the periodicity of the solutions of systems (3)–(6) in [19] follow from Theorems 1 and 2.

Corollary 3. *Systems of difference equations (3)–(6) are all periodic with period ten.*

Proof. For the systems of difference equations (3)–(6), we use the following changes of variables, respectively:

$$\begin{aligned} x_n &= \frac{1}{u_n}, \quad y_n = \frac{1}{v_n}, \quad n \in \mathbb{N}_0; \\ x_n &= -\frac{1}{u_n}, \quad y_n = -\frac{1}{v_n}, \quad n \in \mathbb{N}_0; \\ x_n &= -\frac{1}{u_n}, \quad y_n = \frac{1}{v_n}, \quad n \in \mathbb{N}_0; \\ x_n &= \frac{1}{u_n}, \quad y_n = -\frac{1}{v_n}, \quad n \in \mathbb{N}_0. \end{aligned} \tag{19}$$

By using them, systems (3)–(6) are transformed into system (14). By applying Theorem 1(a), ten-periodicity of all well-defined solutions of system (14) follows, from which ten-periodicity of all well-defined solutions of systems (3)–(6) follows. □

The following four examples are natural extensions of systems (3)–(6).

Example 4. If we use the following functions:

$$f_1(x) = \frac{1}{x^l}, \qquad f_2(x) = \frac{1}{x^m}, \tag{20}$$

where l and m are odd integers, we see that all the conditions in Theorem 1 are applied with $k = 2$, so by using the theorem we obtain that the system of difference equations

$$\begin{aligned} x_{n+1} &= \frac{1}{x_{n-1}}\left(\frac{y_n^m}{1+y_n^m}\right)^{1/l}, \\ y_{n+1} &= \frac{1}{y_{n-1}}\left(\frac{x_n^l}{1+x_n^l}\right)^{1/m}, \end{aligned} \tag{21}$$

$n \in \mathbb{N}_0$, is also periodic with period ten.

Example 5. For

$$f_1(x) = -\frac{1}{x^l}, \qquad f_2(x) = -\frac{1}{x^m}, \tag{22}$$

where l and m are also odd integers, all the conditions of Theorem 1 are again satisfied with $k = 2$. Using the theorem, it follows that the system

$$\begin{aligned} x_{n+1} &= \frac{1}{x_{n-1}}\left(\frac{y_n^m}{-1+y_n^m}\right)^{1/l}, \\ y_{n+1} &= \frac{1}{y_{n-1}}\left(\frac{x_n^l}{-1+x_n^l}\right)^{1/m}, \end{aligned} \tag{23}$$

$n \in \mathbb{N}_0$, is ten-periodic.

Example 6. For

$$f_1(x) = \frac{1}{x^l}, \qquad f_2(x) = -\frac{1}{x^m}, \tag{24}$$

where l and m are odd integers, all the conditions of Theorem 1 are also satisfied with $k = 2$. Applying the theorem, it follows that the system

$$\begin{aligned} x_{n+1} &= \frac{1}{x_{n-1}}\left(\frac{y_n^m}{-1+y_n^m}\right)^{1/l}, \\ y_{n+1} &= \frac{1}{y_{n-1}}\left(\frac{x_n^l}{1+x_n^l}\right)^{1/m}, \end{aligned} \tag{25}$$

$n \in \mathbb{N}_0$, is ten-periodic.

Example 7. Finally, for

$$f_1(x) = -\frac{1}{x^l}, \qquad f_2(x) = \frac{1}{x^m}, \tag{26}$$

where l and m are odd integers and applying Theorem 1 with $k = 2$, we get that the system

$$x_{n+1} = \frac{1}{x_{n-1}}\left(\frac{y_n^m}{1+y_n^m}\right)^{1/l},$$
$$y_{n+1} = \frac{1}{y_{n-1}}\left(\frac{x_n^l}{-1+x_n^l}\right)^{1/m}, \qquad (27)$$

$n \in \mathbb{N}_0$, is ten-periodic.

The main results in [4] can be relatively easily extended to a very general situation, which have been noticed by Iričanin and Stević soon after publishing [4], and later also proved by several other authors. Namely, the following result holds (see, e.g., [1]).

Theorem 8. *Assume that the following difference equation*

$$x_n = f\left(x_{n-1}, \dots, x_{n-k}\right), \quad n \in \mathbb{N}_0, \qquad (28)$$

is periodic with period p.

Then the following system of difference equations

$$x_n^{(i)} = f\left(x_{n-1}^{(\sigma(i))}, x_{n-2}^{(\sigma^{[2]}(i))}, \dots, x_{n-k}^{(\sigma^{[k]}(i))}\right), \quad i = \overline{1,l}, \quad n \in \mathbb{N}_0, \qquad (29)$$

where $\sigma(i) = i+1$, for $1 \le i \le l-1$, $\sigma(l) = 1$ and $\sigma^{[j]}(i) = \sigma(\sigma^{[j-1]}(i))$, $j = \overline{1,k}$, and $\sigma^{[0]}(i) = i$, $i = \overline{1,l}$, is periodic with period $\operatorname{lcm}(p,l)$ *(the least common multiple of numbers p and l).*

Theorem 8 can be used in constructing numerous periodic cyclic systems of difference equations based on scalar periodic difference equations, which, with some changes of variables, give some other periodic cyclic systems of difference equations.

3. Periodicity of Positive Solutions of System (11)

In this section, we study positive solutions of system (11). By $\gcd(s,k)$, we denote the greatest common divisor of natural numbers s and k.

Theorem 9. *Consider system* (11). *Assume that s, $k \in \mathbb{N}$, and $(A_n)_{n\in\mathbb{N}_0}$ and $(B_n)_{n\in\mathbb{N}_0}$ are positive $k\gcd(s,k)$-periodic sequences. Then every positive solution of system* (11) *is periodic with, not necessarily prime, period*

$$p = 2k\gcd(s,k). \qquad (30)$$

Proof. Let $r = \gcd(s,k)$. Then we have that $s = rs_1$ and $k = rk_1$ for some $s_1, k_1 \in \mathbb{N}$ such that

$$\gcd\left(s_1, k_1\right) = 1. \qquad (31)$$

Since every $n \in \mathbb{N}_0$ can be written as $n = mr + i$, for some $m \in \mathbb{N}_0$ and $i = \overline{0, r-1}$, system (11) becomes

$$x_{mr+i} = \max\left\{\frac{A_{mr+i}}{x_{r(m-s_1)+i}}, y_{r(m-k_1)+i}\right\},$$
$$y_{mr+i} = \max\left\{\frac{B_{mr+i}}{y_{r(m-s_1)+i}}, x_{r(m-k_1)+i}\right\}, \qquad (32)$$

for every $m \in \mathbb{N}_0$ and $i = \overline{0, r-1}$.

Using the next change of variables

$$x_t^{(i)} = x_{tr+i}, \qquad y_t^{(i)} = y_{tr+i}, \qquad (33)$$

where $t \ge -\max\{s_1, k_1\}$, $i = \overline{0, r-1}$, in (32), we have that $(x_t^{(i)})_{t\ge -\max\{s_1,k_1\}}$, $(y_t^{(i)})_{t\ge -\max\{s_1,k_1\}}$, $i = \overline{0, r-1}$, are r independent solutions of the next systems

$$x_t = \max\left\{\frac{A_{tr+i}}{x_{t-s_1}}, y_{t-k_1}\right\}, \qquad y_t = \max\left\{\frac{B_{tr+i}}{y_{t-s_1}}, x_{t-k_1}\right\}, \qquad (34)$$

which are systems of the form in (11) with s_1 and k_1 instead of s and k, and where the sequences $(A_{tr+i})_{t\in\mathbb{N}_0}$ and $(B_{tr+i})_{t\in\mathbb{N}_0}$, $i = \overline{1, r}$, are k-periodic.

Hence, it is enough to prove the theorem when $\gcd(s,k) = 1$ and the sequences $(A_n)_{n\in\mathbb{N}_0}$, and $(B_n)_{n\in\mathbb{N}_0}$ are positive k-periodic.

Now note that from the equations in (11), we have that

$$x_n \ge y_{n-k}, \quad y_n \ge x_{n-k}, \quad \text{for } n \in \mathbb{N}_0. \qquad (35)$$

Further, by using the equations in (11), we also get

$$x_n = \max\left\{\frac{A_n}{x_{n-s}}, y_{n-k}\right\} = \max\left\{\frac{A_n}{x_{n-s}}, \frac{B_{n-k}}{y_{n-k-s}}, x_{n-2k}\right\},$$
$$y_n = \max\left\{\frac{B_n}{y_{n-s}}, x_{n-k}\right\} = \max\left\{\frac{B_n}{y_{n-s}}, \frac{A_{n-k}}{x_{n-k-s}}, y_{n-2k}\right\}, \qquad (36)$$

for $n \ge k$.

Using relations (36), we get

$$\begin{aligned} x_n &= \max\left\{\frac{A_n}{x_{n-s}}, \frac{B_{n-k}}{y_{n-k-s}}, x_{n-2k}\right\} \\ &= \max\left\{\frac{A_n}{x_{n-s}}, \frac{B_{n-k}}{y_{n-k-s}}, \frac{A_{n-2k}}{x_{n-2k-s}}, \frac{B_{n-3k}}{y_{n-3k-s}}, x_{n-4k}\right\}, \\ y_n &= \max\left\{\frac{B_n}{y_{n-s}}, \frac{A_{n-k}}{x_{n-k-s}}, y_{n-2k}\right\} \\ &= \max\left\{\frac{B_n}{y_{n-s}}, \frac{A_{n-k}}{x_{n-k-s}}, \frac{B_{n-2k}}{y_{n-2k-s}}, \frac{A_{n-3k}}{x_{n-3k-s}}, y_{n-4k}\right\}, \end{aligned} \qquad (37)$$

for $n \ge 3k$.

Now, note that, from the inequalities in (35), we have that

$$x_n \ge x_{n-2k}, \quad y_n \ge y_{n-2k}, \quad \text{for } n \ge k. \qquad (38)$$

Using (38) and k-periodicity of the sequences A_n and B_n, we obtain

$$\begin{aligned} \frac{A_n}{x_{n-s}} &= \frac{A_{n-2k}}{x_{n-s}} \leq \frac{A_{n-2k}}{x_{n-2k-s}}, \\ \frac{B_{n-k}}{y_{n-k-s}} &= \frac{B_{n-3k}}{y_{n-k-s}} \leq \frac{B_{n-3k}}{y_{n-3k-s}}, \\ \frac{B_n}{y_{n-s}} &= \frac{B_{n-2k}}{y_{n-s}} \leq \frac{B_{n-2k}}{y_{n-2k-s}}, \\ \frac{A_{n-k}}{x_{n-k-s}} &= \frac{A_{n-3k}}{x_{n-k-s}} \leq \frac{A_{n-3k}}{x_{n-3k-s}}. \end{aligned} \tag{39}$$

Employing (39) into (37), we get

$$\begin{aligned} x_n &= \max\left\{\frac{A_{n-2k}}{x_{n-2k-s}}, \frac{B_{n-3k}}{y_{n-3k-s}}, x_{n-4k}\right\} = x_{n-2k}, \\ y_n &= \max\left\{\frac{B_{n-2k}}{y_{n-2k-s}}, \frac{A_{n-3k}}{x_{n-3k-s}}, y_{n-4k}\right\} = y_{n-2k}, \end{aligned} \tag{40}$$

from which it follows that in this case the solutions of system (11) are $2k$-periodic. From all the above, the theorem follows. □

By a slight modification of the proof of Theorem 9, the next result can be proved. We omit the proof.

Theorem 10. *Consider the following system of difference equations*

$$\begin{gathered} x_n = \max\left\{\frac{A_n}{x_{n-s}}, y_{n-k}\right\}, \qquad y_n = \max\left\{\frac{B_n}{y_{n-s}}, z_{n-k}\right\}, \\ z_n = \max\left\{\frac{C_n}{z_{n-s}}, x_{n-k}\right\}, \quad n \in \mathbb{N}_0, \end{gathered} \tag{41}$$

where $s, k \in \mathbb{N}$, and $(A_n)_{n\in\mathbb{N}_0}$, $(B_n)_{n\in\mathbb{N}_0}$, and $(C_n)_{n\in\mathbb{N}_0}$ are positive $k \gcd(s,k)$-periodic sequences. Then, every positive solution of system (41) *is periodic with, not necessarily prime, period*

$$p = 3k \gcd(s,k). \tag{42}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, under Grant no. (11-130/1433 HiCi). The authors, therefore, acknowledge technical and financial support of KAU.

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