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Research Article

On Some Vector-Valued Inequalities of Gronwall Type

Dah-Chin Luor

Department of Applied Mathematics, I-Shou University, No. 1, Sec. 1, Syuecheng Road, Dashu District, Kaohsiung City 84001, Taiwan

Correspondence should be addressed to Dah-Chin Luor; dclour@isu.edu.tw

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In this paper we established some vector-valued inequalities of Gronwall type in ordered Banach spaces. Our results can be applied to investigate systems of real-valued Gronwall-type inequalities. We also show that the classical Gronwall-Bellman-Bihari integral inequality can be generalized from composition operators to a variety of operators, which include integral operators, maximal operators, geometric mean operators, and geometric maximal operators.

1. Introduction

It is well known that the Gronwall-type inequalities play an important role in the study of qualitative properties of solutions to differential equations and integral equations. The Gronwall inequality was established in 1919 by Gronwall [1] and then it was generalized by Bellman [2]. In fact, if

$$u(t) \leq \eta + \int_0^t g(s)\, u(s)\, ds, \quad 0 \leq t \leq b, \tag{1}$$

where $\eta \geq 0$ and u, and g are nonnegative continuous functions on $[0, b]$, then

$$u(t) \leq \eta \exp\left(\int_0^t g(s)\, ds\right), \quad 0 \leq t \leq b. \tag{2}$$

This result plays a key role in studying stability and asymptotic behavior of solutions to differential equations and integral equations. One of the important nonlinear generalizations of (1) and (2) was established by Bihari [3]. Assume that $0 \leq k \leq \eta < C$, u and g are nonnegative continuous functions on $[a, b]$, and r is a positive increasing continuous function on $[0, \infty)$. Bihari showed that if

$$u(t) \leq \eta + \int_a^t g(s)\, r(u(s))\, ds, \quad a \leq t \leq b, \tag{3}$$

and $\int_a^b g(s)ds < \int_\eta^C r(s)^{-1} ds$, then

$$u(t) \leq G^{-1}\left(G(\eta) + \int_a^t g(s)\, ds\right) < C, \quad a \leq t \leq b, \tag{4}$$

where $G(x) = \int_k^x r(s)^{-1} ds$, $x \geq k$. By choosing $r(s) = s$, inequality (4) can be reduced to the form (2). Many results on the various generalizations of real-valued Gronwall-Bellman-Bihari type inequalities are established. See [4–12], [13, CH.XII], [14–16], and the references given in this literature.

Another direction of generalizations is the development of the abstract Gronwall lemma. These results are closely related to the fixed points of operators. See [17, 18], [13, CH.XIV], [19], [20, Proposition 7.15], and the references given in this literature.

Inequality (3) can be written in a general form

$$u(t) \leq \eta + \int_a^t g(s)\, A[u](s)\, ds, \quad a \leq t \leq b, \tag{5}$$

where A is a positive operator on continuous functions. If A is a composition operator defined by $A[u](s) = r(u(s))$, then (5) is reduced to (3). We show that if A belongs to the class $\mathscr{F}$ of operators which is defined in Section 5, then we have an upper estimate of u which is similar to the form (4). It is worth pointing out that the class $\mathscr{F}$ includes integral operators

$$\mathbb{T}[f](t) = \int_a^t k(t, s)\, r(f(s))\, ds, \tag{6}$$

maximal operators

$$\mathbb{M}[f](t) = \sup_{a<x<t} \int_x^t k(t,s)\, r(f(s))\, ds, \quad (7)$$

geometric mean operators

$$\mathbb{G}[f](t) = \exp\left(\frac{1}{\int_a^t k(t,s)\, ds} \int_a^t k(t,s) \log r(f(s))\, ds \right), \quad (8)$$

and geometric maximal operators

$$\mathscr{G}[f](t) = \sup_{a<x<t} \exp\left(\frac{1}{\int_x^t k(t,s)\, ds} \int_x^t k(t,s) \log r(f(s))\, ds \right). \quad (9)$$

We discuss these operators and the class $\mathscr{F}$ in Section 5. We extend the Gronwall-Bellman-Bihari inequality (3) and (4) to the form (5) and the operator A in (5) is generalized from a composition operator to the class $\mathscr{F}$ of operators.

The aim of this paper is to investigate vector-valued inequalities in ordered Banach spaces. Under suitable conditions we have estimates for u which are similar to (4). By restricting our results to Euclidean spaces we study systems of real-valued Gronwall-type inequalities. For the one-dimensional case, real-valued inequalities of the form (5) with $A \in \mathscr{F}$ are discussed.

Throughout this paper, let $\mathbb{R}^+ = [0, \infty)$, $I = [a, b]$, where $-\infty < a < b < \infty$, let $Y_1, \ldots, Y_n$ be Banach spaces, $\mathbb{Y} = Y_1 \times \cdots \times Y_n$; and Y be an ordered Banach space with an order cone K (see [20, Definition 7.1]). We denote by $C(X_1, X_2)$ and $C^1(X_1, X_2)$ the space of continuous operators and the space of continuously *Fréchet* differentiable operators, respectively, from X_1 into X_2.

2. Preliminaries

For $f \in C(I, Y)$, the integral of f on I and the derivative of f at $t_0 \in I$, which are denoted by $\int_a^b f(t)dt$ and $f'(t_0)$, respectively, can be defined as generalizations of the usual definitions of integral and derivative for real-valued functions. One may see [20, Sections 3.1 and 3.2] for the definitions and properties of the integral and derivative. In particular, if $f'(t)$ exists for all $t \in I$, where $f'(a)$ and $f'(b)$ are defined by one-sided limits, and if f' is continuous on I, then $f(b) - f(a) = \int_a^b f'(t)dt$. If we define $F(t) = \int_a^t f(s)ds$ for all $t \in [a, b]$, then $F'(t)$ exists for all $t \in (a, b)$ and $F'(t) = f(t)$. See [20, Propositions 3.5 and 3.7]. By defining

$$\mathbb{K} = \{ f : f \in C(I, Y),\ f(t) \in K\ \forall t \in I \} \subseteq C(I, Y), \quad (10)$$

we see that $C(I, Y)$ is an ordered Banach space with sup-norm $\|\cdot\|_\infty$ and the order cone $\mathbb{K}$. If $f \le g$ in $C(I, Y)$, then $\int_a^b f(t)dt \le \int_a^b g(t)dt$.

By [21, Theorem 7.1.9] we see that, for $t \in (a, b)$, the *Fréchet* derivative $D_{\mathscr{F}} f(t)$ exists if and only if $f'(t)$ exists and

$$(D_{\mathscr{F}} f(t))(h) = h f'(t) \quad \forall h \in \mathbb{R}. \quad (11)$$

Here we denote by $(D_{\mathscr{F}} f(t))(h)$ the value of $D_{\mathscr{F}} f(t)$ at h.

For $y_i \in Y_i$, $i = 1, \ldots, n$, let $y = (y_1, \ldots, y_n) \in \mathbb{Y}$ and we define $\|y\|_{\mathbb{Y}} = \|y_1\|_{Y_1} + \cdots + \|y_n\|_{Y_n}$. Then $\|\cdot\|_{\mathbb{Y}}$ is a norm and $\mathbb{Y}$, with coordinatewise linear operations, is a Banach space. If $Y_1, \ldots, Y_n$ are ordered Banach spaces with ordered cones $K_1, \ldots, K_n$, respectively, then $\mathbb{Y}$ is an ordered Banach space with an order cone $K_1 \times \cdots \times K_n$. It is easy to see that a sequence of points $y^{(m)} = (y_1^{(m)}, \ldots, y_n^{(m)})$ in $\mathbb{Y}$ converges to a point $(c_1, \ldots, c_n) \in \mathbb{Y}$ if and only if $y_i^{(m)}$ converges to c_i in Y_i for each $i = 1, \ldots, n$. For $f_i \in C(I, Y_i)$, $i = 1, \ldots, n$, we define $f(t) = (f_1(t), \ldots, f_n(t))$. Then $f \in C(I, \mathbb{Y})$. If $f_1'(t), \ldots, f_n'(t)$ all exist for $t \in I$, then the derivative of f at t exists and $f'(t) = (f_1'(t), \ldots, f_n'(t))$.

Let X, Y, and Z be Banach spaces, and U, V are nonempty open subsets of X, Y, respectively, and let $g : U \to V$ and $f : V \to Z$. Suppose that g is continuous and *Fréchet* differentiable at a point $z \in U$ and that f is continuous and *Fréchet* differentiable at the point $g(z) \in V$. Then $f \circ g$ is continuous and *Fréchet* differentiable at z, and

$$D_{\mathscr{F}}(f \circ g)(z) = D_{\mathscr{F}} f(g(z)) \circ D_{\mathscr{F}} g(z). \quad (12)$$

Let X_1, X_2, and Y be Banach spaces, and A is a nonempty open subset of $X_1 \times X_2$, and let $f : A \to Y$ be given by $(x_1, x_2) \mapsto f(x_1, x_2)$. For $(z_1, z_2) \in A$, let $A_1 = \{x_1 \in X_1 : (x_1, z_2) \in A\}$ and let $g(x_1) = f(x_1, z_2)$ for all $x_1 \in A_1$. It is clear that $A_1 \subseteq X_1$ is open and $g : A_1 \to Y$. If g has a *Fréchet* derivative at z_1, then we define the partial *Fréchet* derivative of f at (z_1, z_2) with respect to the variable x_1 to be $D_{\mathscr{F},1} f(z_1, z_2) = D_{\mathscr{F}} g(z_1)$; it is a linear operator of X_1 into Y. The derivative $D_{\mathscr{F},2} f(z_1, z_2)$ is defined similarly. If f is *Fréchet* differentiable at (z_1, z_2), then f is *Fréchet* differentiable with respect to both variables at (z_1, z_2) and

$$(D_{\mathscr{F}} f(z_1, z_2))((x_1, x_2)) = (D_{\mathscr{F},1} f(z_1, z_2))(x_1) + (D_{\mathscr{F},2} f(z_1, z_2))(x_2) \quad (13)$$

for all $(x_1, x_2) \in X_1 \times X_2$. Moreover, f is continuously *Fréchet* differentiable in a neighborhood of (z_1, z_2) if and only if all partial *Fréchet* derivatives are continuous in a neighborhood of (z_1, z_2). Similar results hold for maps of the form $(x_1, \ldots, x_n) \mapsto f(x_1, \ldots, x_n)$.

Notation. Here we give notations used in this paper for reader's convenience. $u \in C(I, Y)$, $\Phi : C(I, Y) \to C(I, Y)$, $F \in C^1(I \times \mathbb{Y}, Y)$, $S_i : C(I, Y) \to C^1(I, Y_i)$, $i = 1, \ldots, n$, and $\mathbb{S} : C(I, Y) \to C^1(I, I \times \mathbb{Y})$ is defined by

$$\mathbb{S}[u](t) = (t, S_1[u](t), \ldots, S_n[u](t)),$$
$$T_i : C(I, Y) \longrightarrow C(I, Y),$$
$$i = 0, \ldots, n, \quad \ell \in C^1(Y, Y), \quad U_i \in C(I, Y), \quad i = 0, \ldots, n. \quad (14)$$

3. Some Vector-Valued Gronwall-Type Inequalities

In this section we consider vector-valued inequalities of the form

$$\Phi[u] \leq F \circ \mathbb{S}[u] \quad \text{in } C(I, Y). \tag{15}$$

Theorem 1 gives an estimate of u which is similar to the form (4).

Theorem 1. *Let $L_0 \in Y$. Suppose that $F(\mathbb{S}[u](a)) \leq L_0$ and Φ is bijective and is of monotone type, and suppose that there exist monotone increasing operators $T_i : C(I, Y) \to C(I, Y)$, $i = 0, \ldots, n$, such that for all $t \in I$,*

$$\left(D_{\mathcal{F},1} F(\mathbb{S}[u](t))\right)(1) \leq T_0[u](t), \tag{16}$$

$$\left(D_{\mathcal{F},i+1} F(\mathbb{S}[u](t))\right)\left(S_i[u]'(t)\right) \leq T_i[u](t), \quad i = 1, \ldots, n. \tag{17}$$

We also suppose that there exist $\ell \in C^1(Y, Y)$ and $U_i \in C(I, Y)$, $i = 0, \ldots, n$, such that ℓ is bijective and monotone increasing and is of monotone type, and for $i = 0, 1, \ldots, n$,

$$\left(D_{\mathcal{F}} \ell((F \circ \mathbb{S}[u])(t))\right)\left(T_i\left[\Phi^{-1}[F \circ \mathbb{S}[u]]\right](t)\right) \leq U_i(t) \quad \forall t \in I. \tag{18}$$

If u satisfies (15), then

$$u \leq \Phi^{-1}[V] \quad \text{in } C(I, Y), \tag{19}$$

where $V \in C(I, Y)$ is defined by

$$V(t) = \ell^{-1}\left(\ell(L_0) + \sum_{i=0}^{n} \int_a^t U_i(y)\, dy\right), \quad t \in I. \tag{20}$$

Moreover, if there exists $M \in Y$ such that $\ell(V(t)) \leq M$ for all $t \in I$, then

$$u \leq \Phi^{-1}\left[\ell^{-1}(M)\right] \quad \text{in } C(I, Y), \tag{21}$$

where $\ell^{-1}(M)$ in (21) is the constant function in $C(I, Y)$ with value $\ell^{-1}(M)$.

Proof of Theorem 1. Let $\alpha_u = F \circ \mathbb{S}[u] \in C^1(I, Y)$. By (15) we have $\Phi[u](t) \leq \alpha_u(t)$ for all $t \in I$. By the chain rule we see that, for $t \in (a, b)$ and $h \in \mathbb{R}$,

$$\begin{aligned} h\alpha_u'(t) &= \left(D_{\mathcal{F}} \alpha_u(t)\right)(h) = \left(D_{\mathcal{F}}(F \circ \mathbb{S}[u])(t)\right)(h) \\ &= \left(D_{\mathcal{F}} F(\mathbb{S}[u](t)) \circ D_{\mathcal{F}} \mathbb{S}[u](t)\right)(h). \end{aligned} \tag{22}$$

Since $\mathbb{S}[u] \in C^1(I, I \times \mathbb{Y})$, $(D_{\mathcal{F}} \mathbb{S}[u](t))(h) = h\mathbb{S}[u]'(t)$ and we see that

$$\begin{aligned} h\alpha_u'(t) &= \left(D_{\mathcal{F}} F(\mathbb{S}[u](t))\right)\left(h\mathbb{S}[u]'(t)\right) \\ &= \left(D_{\mathcal{F},1} F(\mathbb{S}[u](t))\right)(h) \\ &\quad + \left(D_{\mathcal{F},2} F(\mathbb{S}[u](t))\right)\left(hS_1[u]'(t)\right) \\ &\quad + \cdots + \left(D_{\mathcal{F},n+1} F(\mathbb{S}[u](t))\right)\left(hS_n[u]'(t)\right) \\ &= h\left(D_{\mathcal{F},1} F(\mathbb{S}[u](t))\right)(1) \\ &\quad + h\left(D_{\mathcal{F},2} F(\mathbb{S}[u](t))\right)\left(S_1[u]'(t)\right) \\ &\quad + \cdots + h\left(D_{\mathcal{F},n+1} F(\mathbb{S}[u](t))\right)\left(S_n[u]'(t)\right). \end{aligned} \tag{23}$$

This implies

$$\begin{aligned} \alpha_u'(t) &= \left(D_{\mathcal{F},1} F(\mathbb{S}[u](t))\right)(1) \\ &\quad + \left(D_{\mathcal{F},2} F(\mathbb{S}[u](t))\right)\left(S_1[u]'(t)\right) \\ &\quad + \cdots + \left(D_{\mathcal{F},n+1} F(\mathbb{S}[u](t))\right)\left(S_n[u]'(t)\right). \end{aligned} \tag{24}$$

Since Φ is bijective, we write $\Phi^{-1}[\alpha_u]$ to be the solution of the equation $\Phi[x] = \alpha_u$. By $\Phi[u] \leq \alpha_u$ in $C(I, Y)$ and [20, Proposition 7.37] we see that $u \leq \Phi^{-1}[\alpha_u]$. This shows that $T_i[u] \leq T_i[\Phi^{-1}[\alpha_u]]$ for $i = 0, 1, \ldots, n$ and hence

$$\alpha_u'(t) \leq \sum_{i=0}^{n} T_i[u](t) \leq \sum_{i=0}^{n} T_i\left[\Phi^{-1}[\alpha_u]\right](t). \tag{25}$$

Since ℓ is monotone increasing, we see that $D_{\mathcal{F}} \ell(y)$ is positive and hence monotone increasing for each $y \in Y$. Therefore for $h > 0$,

$$\begin{aligned} h(\ell \circ \alpha_u)'(t) &= \left(D_{\mathcal{F}}(\ell \circ \alpha_u)(t)\right)(h) \\ &= \left(D_{\mathcal{F}} \ell(\alpha_u(t)) \circ D_{\mathcal{F}} \alpha_u(t)\right)(h) \\ &= \left(D_{\mathcal{F}} \ell(\alpha_u(t))\right)\left(h\alpha_u'(t)\right) \\ &\leq h\left(D_{\mathcal{F}} \ell(\alpha_u(t))\right)\left(\sum_{i=0}^{n} T_i\left[\Phi^{-1}[\alpha_u]\right](t)\right) \\ &= h\sum_{i=0}^{n}\left(D_{\mathcal{F}} \ell(\alpha_u(t))\right)\left(T_i\left[\Phi^{-1}[\alpha_u]\right](t)\right) \\ &\leq h\sum_{i=0}^{n} U_i(t). \end{aligned} \tag{26}$$

For $a < d < t < b$ we obtain

$$(\ell \circ \alpha_u)(t) - (\ell \circ \alpha_u)(d) \leq \sum_{i=0}^{n} \int_d^t U_i(y)\, dy. \tag{27}$$

By letting $d \to a$ and the condition $\alpha_u(a) \le L_0$,

$$\ell(\alpha_u(t)) \le \ell(L_0) + \sum_{i=0}^{n} \int_a^t U_i(y)\,dy = \ell(V(t)) \tag{28}$$

for all $t \in (a,b)$. Since $\ell \circ \alpha_u$ and $\ell \circ V$ are continuous on I, inequality (28) holds for all $t \in I$. This implies $\alpha_u(t) \le V(t)$ for all $t \in I$. Therefore $\Phi[u] \le \alpha_u \le V$ and $u \le \Phi^{-1}[V]$ in $C(I,Y)$.

Since ℓ^{-1} and Φ^{-1} are monotone increasing, it follows that if $\ell(V(t)) \le M$ for all $t \in I$ then $V \le \ell^{-1}(M)$ in $C(I,Y)$ and we obtain (21). This completes the proof. □

In the following we consider two particular cases as examples of Theorem 1. We see that under these cases, conditions in Theorem 1 can be reduced to more simpler forms.

In the case $Y_1 = Y_2 = \cdots = Y_n = Y$ and $F(t, y_1, \ldots, y) = y_1 + \cdots + y_n$, we see that $F \circ \mathbb{S}[u] \in C^1(I,Y)$ is given by

$$F \circ \mathbb{S}[u](t) = F(\mathbb{S}[u](t)) = \sum_{i=1}^{n} S_i[u](t), \quad t \in I. \tag{29}$$

Moreover, $D_{\mathcal{F},1}F(\cdot)$ is the zero operator of I into Y and $D_{\mathcal{F},i+1}F(\cdot)$ is the identity operator of Y into Y. We have the following corollary.

Corollary 2. *Let $L_0 \in Y$ and let u, Φ, ℓ, S_i, T_i, U_i, $i = 1, \ldots, n$, and $\mathbb{S}$ be given as in Theorem 1 with $Y_1 = Y_2 = \cdots = Y_n = Y$. Suppose that $\sum_{i=1}^{n} S_i[u](a) \le L_0$, and*

$$S_i[u]'(t) \le T_i[u](t), \tag{30}$$

$$\left(D_{\mathcal{F}}\ell\left(\sum_{j=1}^{n} S_j[u](t)\right)\right)\left(T_i\left[\Phi^{-1}\left[\sum_{j=1}^{n} S_j[u]\right]\right](t)\right) \le U_i(t) \tag{31}$$

for each $i = 1, \ldots, n$ and for all $t \in I$. If u satisfies

$$\Phi[u] \le \sum_{i=1}^{n} S_i[u] \quad \text{in } C(I,Y), \tag{32}$$

then we have (19)–(21).

Remark 3. If $K_i : C(I,Y) \to C(I,Y)$ and $S_i[u](t) = \int_a^t K_i[u](z)dz$ for each $i = 1, \ldots, n$, then (32) is reduced to the following integral inequality:

$$\Phi[u](t) \le \sum_{i=1}^{n} \int_a^t K_i[u](z)\,dz, \quad t \in I. \tag{33}$$

By [20, Proposition 3.7] the item $S_i[u]'(t)$ in (17) and (30) can be replaced by $K_i[u](t)$. Moreover, if K_i is monotone increasing for each i, then we can choose $T_i = K_i$, $i = 1, \ldots, n$, in Corollary 2 and condition (30) is redundant.

Consider the case that $Y_1, \ldots, Y_n$ are ordered Banach spaces, $Y = \mathbb{Y}$, and $F(t, y_1, \ldots, y_n) = (y_1, \ldots, y_n)$. We see that $F \circ \mathbb{S}[u] \in C^1(I, \mathbb{Y})$ is given by

$$(F \circ \mathbb{S}[u])(t) = F(\mathbb{S}[u](t)) = (S_1[u](t), \ldots, S_n[u](t)), \quad t \in I. \tag{34}$$

Moreover, $D_{\mathcal{F},1}F(\cdot)$ is the zero operator of I into $\mathbb{Y}$ and $D_{\mathcal{F},i+1}F(\cdot)$ is the operator of Y_i into $\mathbb{Y}$ such that for $h \in Y_i$, the ith element of $(D_{\mathcal{F},i+1}F(\cdot))(h)$ is h and the other elements are zero.

Remark 4. Let $\Phi_i : C(I,\mathbb{Y}) \to C(I,Y_i)$, $i = 1, \ldots, n$, and $\Phi : C(I,\mathbb{Y}) \to C(I,\mathbb{Y})$ is defined by $\Phi[u](t) = (\Phi_1[u](t), \ldots, \Phi_n[u](t))$ for $t \in I$. Suppose that each Φ_i is injective and is of monotone type. Suppose that for any $w = (w_1, \ldots, w_n) \in C(I,\mathbb{Y})$, where $w_i \in C(I,Y_i)$, there exist $v \in C(I,\mathbb{Y})$ such that $\Phi_i[v] = w_i$ for each $i = 1, \ldots, n$. Then Φ is bijective and is of monotone type.

The following corollary can be obtained by Theorem 1.

Corollary 5. *Let $L_0 = (L_1, \ldots, L_n) \in \mathbb{Y}$ and let u, ℓ, S_i, $i = 1, \ldots, n$, and $\mathbb{S}$ be given as in Theorem 1 with $Y = \mathbb{Y}$. Let $\Phi_1, \ldots, \Phi_n$, Φ be given as in Remark 4. Suppose that $S_i[u](a) \le L_i$, $i = 1, \ldots, n$, and there exist $U_i \in C(I,\mathbb{Y})$ and monotone increasing operators $G_i : C(I,\mathbb{Y}) \to C(I,Y_i)$, $i = 1, \ldots, n$, such that*

$$S_i[u]'(t) \le G_i[u](t), \tag{35}$$

$$(D_{\mathcal{F}}\ell((F \circ \mathbb{S}[u])(t)))\left(T_i\left[\Phi^{-1}[F \circ \mathbb{S}[u]]\right](t)\right) \le U_i(t), \tag{36}$$

for $i = 1, \ldots, n$ and for all $t \in I$, where $T_i[\cdot](t) \in \mathbb{Y}$ and the ith element of $T_i[\cdot](t)$ is $G_i[\cdot](t)$ and the other elements are zero. If u satisfies the systems

$$\Phi_i[u](t) \le S_i[u](t), \quad i = 1, \ldots, n, \tag{37}$$

for all $t \in I$, then we have (19)–(21).

Remark 6. If $K_i : C(I,\mathbb{Y}) \to C(I,Y_i)$ and $S_i[u](t) = \int_a^t K_i[u](z)dz$ for each $i = 1, \ldots, n$, then (37) is reduced to the system of integral inequalities

$$\Phi_i[u](t) \le \int_a^t K_i[u](z)\,dz, \quad t \in I,\ i = 1, \ldots, n. \tag{38}$$

The item $S_i[u]'(t)$ in (35) can be replaced by $K_i[u](t)$. Moreover, if K_i is monotone increasing for each i, then we can choose $G_i = K_i$, $i = 1, \ldots, n$, in Corollary 5 and condition (35) is redundant.

4. Systems of Real-Valued Gronwall-Type Inequalities

In this section we apply results in Section 3 to obtain systems of real-valued Gronwall-type inequalities. Consider the case

$Y = \mathbb{R}^m$, $Y_i = \mathbb{R}$, $i = 1,\ldots,n$, $\mathbb{Y} = \mathbb{R}^n$, and $K = \{(x_1,\ldots,x_m) \in \mathbb{R}^m : x_i \geq 0,\ i = 1,\ldots,m\}$. Let $u_i \in C(I,\mathbb{R})$, $\Phi_i : C(I,\mathbb{R}^m) \to C(I,\mathbb{R})$, $F_i \in C^1(I \times \mathbb{R}^n,\mathbb{R})$, $\ell_i \in C^1(\mathbb{R}^m,\mathbb{R})$, $i = 1,\ldots,m$, and $u \in C(I,\mathbb{R}^m)$, $u(t) = (u_1(t),\ldots,u_m(t))$, $F \in C^1(I \times \mathbb{R}^n,\mathbb{R}^m)$, $F = (F_1,\ldots,F_m)$. Let $S_i : C(I,\mathbb{R}^m) \to C^1(I,\mathbb{R})$, $i = 1,\ldots,n$.

Define $\Phi : C(I,\mathbb{R}^m) \to C(I,\mathbb{R}^m)$ by

$$\Phi[u](t) = (\Phi_1[u](t),\ldots,\Phi_m[u](t)), \quad t \in I. \tag{39}$$

Then (15) can be reduced to the system

$$\Phi_i[u](t) \leq (F_i \circ \mathbb{S}[u])(t), \quad t \in I,\ i = 1,\ldots,m. \tag{40}$$

Remark 7. Suppose that Φ_i is injective and is of monotone type for each $i = 1,\ldots,m$. Suppose that for any $w = (w_1,\ldots,w_m) \in C(I,\mathbb{R}^m)$, where $w_i \in C(I,\mathbb{R})$, there exist $v \in C(I,\mathbb{R}^m)$ such that $\Phi_i[v] = w_i$ for each $i = 1,\ldots,m$. Then Φ is bijective and is of monotone type.

Since

$$(D_{\mathcal{F},1}F(t,y_1,\ldots,y_n))(1) = \frac{\partial F}{\partial t}(t,y_1,\ldots,y_n), \tag{41}$$

and, for $i = 1,\ldots,n$, the linear transform $D_{\mathcal{F},i+1}F(t,y_1,\ldots,y_n) : \mathbb{R} \to \mathbb{R}^m$ can be represented by the $m \times 1$ matrix $[(\partial F_k/\partial y_i)(t,y_1,\ldots,y_n)]$, conditions (16)-(17) are reduced to

$$\frac{\partial F}{\partial t}(\mathbb{S}[u](t)) \leq T_0[u](t), \tag{42}$$

$$\left[\frac{\partial F_k}{\partial y_i}(\mathbb{S}[u](t))\right] S_i[u]'(t) \leq T_i[u](t), \quad i = 1,\ldots,n, \tag{43}$$

where $T_i : C(I,\mathbb{R}^m) \to C(I,\mathbb{R}^m)$, $i = 0,\ldots,n$, are monotone increasing operators. Here the vector $T_i[u](t)$ in (43) is written as a column matrix. In particular, if $m = 1$, then $F \in C^1(I \times \mathbb{R}^n,\mathbb{R})$ and (43) is reduced to

$$\frac{\partial F}{\partial y_i}(\mathbb{S}[u](t)) \cdot S_i[u]'(t) \leq T_i[u](t), \quad i = 1,\ldots,n. \tag{44}$$

Remark 8. Define $\ell \in C^1(\mathbb{R}^m,\mathbb{R}^m)$ by $\ell = (\ell_1,\ldots,\ell_m)$. Suppose that ℓ_k is injective and monotone increasing and is of monotone type for each k, and, for any $z = (z_1,\ldots,z_m) \in \mathbb{R}^m$, there exist $y \in \mathbb{R}^m$ such that $\ell_k(y) = z_k$ for each $k = 1,\ldots,m$. Then ℓ is bijective and monotone increasing and is of monotone type. Moreover, the linear transform $D_{\mathcal{F}}\ell(\xi_1,\ldots,\xi_m)$ can be represented by the $m \times m$ Jacobian matrix $[D_j\ell_k(\xi_1,\ldots,\xi_m)]$, $1 \leq j,k \leq m$. Hence (18) is reduced to

$$\left[D_j\ell_k((F \circ \mathbb{S}[u])(t))\right] T_i\left[\Phi^{-1}[F \circ \mathbb{S}[u]]\right](t) \leq U_i(t), \quad i = 0,\ldots,n, \tag{45}$$

where $U_i \in C(I,\mathbb{R}^m)$. Here the vectors $T_i[\Phi^{-1}[F \circ \mathbb{S}[u]]](t)$ and $U_i(t)$ in (45) are written as column matrices.

Theorem 9. *Let $L_0 \in \mathbb{R}^m$ and the conditions of $\Phi_1,\ldots,\Phi_m$, Φ be given in Remark 7. Suppose that $F(\mathbb{S}[u](a)) \leq L_0$, and there exist monotone increasing operators $T_i : C(I,\mathbb{R}^m) \to C(I,\mathbb{R}^m)$, $i = 0,\ldots,n$, such that (42)-(43) are satisfied. We also suppose that conditions for $\ell_1,\ldots,\ell_m$, ℓ in Remark 8 hold and (45) is satisfied. If u satisfies the system (40), then one has (19)–(21) with $Y = \mathbb{R}^m$.*

Let $\Psi_i : C(I,\mathbb{R}) \to C(I,\mathbb{R})$, $i = 1,\ldots,m$. If we define $\Phi : C(I,\mathbb{R}^m) \to C(I,\mathbb{R}^m)$ by

$$\Phi[u](t) = (\Psi_1[u_1](t),\ldots,\Psi_m[u_m](t)), \quad t \in I, \tag{46}$$

then (15) can be reduced to the system

$$\Psi_i[u_i](t) \leq (F_i \circ \mathbb{S}[u])(t), \quad t \in I,\ i = 1,\ldots,m. \tag{47}$$

Remark 10. Suppose that Ψ_i is bijective and is of monotone type for each $i = 1,\ldots,m$. Then Φ defined by (46) is bijective and is of monotone type, and

$$\Phi^{-1}[f](t) = \left(\Psi_1^{-1}[f_1](t),\ldots,\Psi_m^{-1}[f_m](t)\right) \quad \forall t \in I, \tag{48}$$

where $f = (f_1,\ldots,f_m) \in C(I,\mathbb{R}^m)$. In particular, the function $\Phi^{-1}[F \circ \mathbb{S}[u]] \in C(I,\mathbb{R}^m)$ is given by

$$\begin{aligned}\Phi^{-1}&[F \circ \mathbb{S}[u]](t) \\ &= \left(\Psi_1^{-1}[F_1 \circ \mathbb{S}[u]](t),\ldots,\Psi_m^{-1}[F_m \circ \mathbb{S}[u]](t)\right) \\ &\qquad \forall t \in I.\end{aligned} \tag{49}$$

Example 11. Define $\Psi_j[u_j](t) = \psi_j(u_j(t))$, where $\psi_j \in C(\mathbb{R},\mathbb{R})$. If ψ_j is strictly increasing from $\mathbb{R}$ onto $\mathbb{R}$, then Ψ_j is bijective and is of monotone type, and (48) can be reduced to

$$\Phi^{-1}[f](t) = \left(\psi_1^{-1}(f_1(t)),\ldots,\psi_m^{-1}(f_m(t))\right) \quad \forall t \in I, \tag{50}$$

where $f = (f_1,\ldots,f_m) \in C(I,\mathbb{R}^m)$.

Remark 12. Let $l_i \in C^1(\mathbb{R},\mathbb{R})$, $i = 1,\ldots,m$. Define $\ell \in C^1(\mathbb{R}^m,\mathbb{R}^m)$ by

$$\ell(\xi_1,\ldots,\xi_m) = (l_1(\xi_1),\ldots,l_m(\xi_m)). \tag{51}$$

Suppose that l_i is strictly increasing from $\mathbb{R}$ onto $\mathbb{R}$ for $i = 1,\ldots,m$; then it is easy to see that ℓ is bijective and monotone increasing and is of monotone type, and

$$\ell^{-1}(\xi_1,\ldots,\xi_m) = \left(l_1^{-1}(\xi_1),\ldots,l_m^{-1}(\xi_m)\right). \tag{52}$$

Moreover, the Jacobian matrix $[D_jl_k(\xi_1,\ldots,\xi_m)]$ is a diagonal matrix with diagonal entries $l_k'(\xi_k)$, $k = 1,\ldots,m$. Hence $D_{\mathcal{F}}\ell((F \circ \mathbb{S}[u])(t))$ in (18) can be represented by the $m \times m$ diagonal matrix with diagonal entries $l_k'((F_k \circ \mathbb{S}[u])(t))$, $k = 1,\ldots,m$.

The following theorem can be obtained by Theorem 1.

Theorem 13. *Let $L_1, \ldots, L_m \in \mathbb{R}$ and the conditions of $\Psi_1, \ldots, \Psi_m$, Φ be given as in Remark 10. Suppose that $F_j(\mathbb{S}[u](a)) \leq L_j$, $j = 1, \ldots, m$, and there exist monotone increasing operators $T_i : C(I, \mathbb{R}^m) \to C(I, \mathbb{R}^m)$, $i = 0, \ldots, n$, such that (42)-(43) are satisfied. We also suppose that there exist $l_j \in C^1(\mathbb{R}, \mathbb{R})$, $j = 1, \ldots, m$, and $U_{ij} \in C(I, \mathbb{R})$, $i = 0, \ldots, n$, $j = 1, \ldots, m$, such that conditions for l_j and ℓ in Remark 12 hold, and (18) with $U_i = (U_{i1}, \ldots, U_{im})$ are satisfied. If u satisfies (47), then we have*

$$u_j(t) \leq \Psi_j^{-1}\left[V_j\right](t), \quad t \in I, \ j = 1, \ldots, m, \tag{53}$$

where

$$V_j(t) = l_j^{-1}\left(l_j\left(L_j\right) + \sum_{i=0}^{n} \int_a^t U_{ij}(y)\, dy\right), \quad t \in I. \tag{54}$$

Moreover, if there exists $M_1, \ldots, M_m \in \mathbb{R}$ such that $l_j(V_j(t)) \leq M_j$ for all $t \in I$, $j = 1, \ldots, m$, then

$$u_j(t) \leq \Psi_j^{-1}\left[l_j^{-1}\left(M_j\right)\right] \quad \forall t \in I, \ j = 1, \ldots, m, \tag{55}$$

where $l_j^{-1}(M_j)$ in (55) is the constant function in $C(I, \mathbb{R})$ with value $l_j^{-1}(M_j)$.

Consider the particular case $m = n$ and $F_j(t, y_1, \ldots, y_m) = y_j$, $j = 1, \ldots, m$ of Theorem 13. In this case $(F_i \circ \mathbb{S}[u])(t) = S_i[u](t)$ and (47) is reduced to

$$\Psi_i\left[u_i\right](t) \leq S_i[u](t), \quad t \in I, \ i = 1, \ldots, m. \tag{56}$$

Assume that

$$S_i[u](a) \leq L_i, \quad i = 1, \ldots, m. \tag{57}$$

We also assume that there exist monotone increasing operators $G_i : C(I, \mathbb{R}^m) \to C(I, \mathbb{R})$, $i = 1, \ldots, m$, such that

$$S_i[u]'(t) \leq G_i[u](t), \quad i = 1, \ldots, m. \tag{58}$$

We also suppose that there exist $l_i \in C^1(\mathbb{R}, \mathbb{R})$ and $W_i \in C(I, \mathbb{R})$, $i = 1, \ldots, m$, such that conditions for l_i in Remark 12 hold and satisfy the following condition:

$$l_i'\left(S_i[u](t)\right) G_i\left[\Phi^{-1}\left[F \circ \mathbb{S}[u]\right]\right](t) \leq W_i(t), \quad i = 1, \ldots, m, \tag{59}$$

where

$$\Phi^{-1}\left[F \circ \mathbb{S}[u]\right](t) = \left(\Psi_1^{-1}\left[S_1[u]\right](t), \ldots, \Psi_m^{-1}\left[S_m[u]\right](t)\right) \quad \forall t \in I. \tag{60}$$

Then Theorem 13 implies that if u satisfies (56) then we have (53) with

$$V_j(t) = l_j^{-1}\left(l_j\left(L_j\right) + \int_a^t W_j(y)\, dy\right), \quad t \in I. \tag{61}$$

Remark 14. Condition (59) can be satisfied if there exist $r_i \in C(\mathbb{R}, (0, \infty))$, $i = 1, \ldots, m$, such that

$$G_i\left[\Phi^{-1}\left[F \circ \mathbb{S}[u]\right]\right](t) \leq W_i(t)\, r_i\left(S_i[u](t)\right), \quad t \in I, \tag{62}$$

and we choose

$$l_i(t) = \int_{c_i}^{t} \frac{1}{r_i(s)}\, ds, \quad i = 1, \ldots, m, \tag{63}$$

where c_i is some constant. Moreover, V_j defined by (61) satisfies $l_j(V_j(t)) \leq M_j$, where

$$M_j = \max_{a \leq t \leq b}\left\{l_j\left(L_j\right) + \int_a^t W_j(y)\, dy\right\}, \quad j = 1, \ldots, m. \tag{64}$$

By a little modification of Theorem 13 we have Corollary 15.

Corollary 15. *Let $L_1, \ldots, L_m \in \mathbb{R}$ and the conditions of $\Psi_1, \ldots, \Psi_m$ be given as in Remark 10. Suppose that (57) holds, and there exist monotone increasing operators $G_i : C(I, \mathbb{R}^m) \to C(I, \mathbb{R})$, $i = 1, \ldots, m$, such that (58) is satisfied. For $i = 1, \ldots, m$, we suppose that there exist $W_i \in C(I, \mathbb{R})$ and $r_i \in C(\mathbb{R}, (0, \infty))$ such that (62) holds. Define l_i by (63). Assume that $l_i(-\infty) < l_i(V_i(t)) < l_i(\infty)$ for all $t \in I$, where V_i is defined by (61). If u satisfies (56), then we have (53) and (55) with (64).*

As an example we consider the system of integral inequalities

$$\psi_j\left(u_j(t)\right) \leq L_j + \int_a^t g_j(s)\, A_j[u](s)\, ds, \quad t \in I, \ j = 1, \ldots, m. \tag{65}$$

Here $\psi_j \in C(\mathbb{R}, \mathbb{R})$ is strictly increasing from $\mathbb{R}$ onto $\mathbb{R}$, $L_j \in \mathbb{R}$, $g_j \in C(I, \mathbb{R}^+)$, and $A_j : C(I, \mathbb{R}^m) \to C(I, \mathbb{R})$. Assume that for each $j = 1, \ldots, m$, A_j is monotone increasing and there exist $w_j \in C(I, \mathbb{R})$ and $r_j \in C(\mathbb{R}, (0, \infty))$ such that

$$A_j\left[\left(\psi_1^{-1} \circ f_1, \ldots, \psi_m^{-1} \circ f_m\right)\right](t) \leq w_j(t)\, r_j\left(f_j(t)\right), \quad t \in I, \tag{66}$$

for all $f_1, \ldots, f_m \in C(I, \mathbb{R})$. Moreover, we suppose that $l_j(-\infty) < l_j(L_j) + \int_a^t g_j(y) w_j(y) dy < l_j(\infty)$ for all $t \in I$, where l_j is given in (63). By Example 11 and Corollary 15 with $\Psi_j[u_j](t) = \psi_j(u_j(t))$, $S_j[u](t) = L_j + \int_a^t g_j(s) A_j[u](s) ds$, $G_j[u](t) = g_j(t) A_j[u](t)$, and $W_j(t) = g_j(t) w_j(t)$, we have

$$u_j(t) \leq \psi_j^{-1}\left(l_j^{-1}\left(l_j\left(L_j\right) + \int_a^t g_j(y)\, w_j(y)\, dy\right)\right), \quad t \in I, \ j = 1, \ldots, m. \tag{67}$$

5. A Real-Valued Gronwall-Type Inequality

In this section we establish a real-valued Gronwall-type inequality which extends (3) from composition operators $A[u](s) = r(u(s))$ to $A \in \mathscr{F}$. Consider the case $m = 1$ of (65):

$$\psi(u(t)) \leq L + \int_a^t g(s) A[u](s)\, ds, \quad t \in I. \tag{68}$$

Here $u \in C(I, \mathbb{R}^+)$, $\psi \in C(\mathbb{R}^+, \mathbb{R}^+)$ is strictly increasing and $\psi(\infty) = \infty$, $L \geq 0$, $g \in C(I, \mathbb{R}^+)$, and $A : C(I, \mathbb{R}) \to C(I, \mathbb{R})$. Let $\mathscr{F}$ be the class of operators $A : C(I, \mathbb{R}) \to C(I, \mathbb{R})$ such that A is positive and monotone increasing, and there exist $w \in C(I, \mathbb{R}^+)$ and $r \in C(\mathbb{R}^+, (0, \infty))$ such that for all increasing $f \in C(I, \mathbb{R}^+)$, we have

$$A[f](t) \leq w(t)\, r(f(t)), \quad t \in I. \tag{69}$$

By some modification of results in Section 4 we have Theorem 16.

Theorem 16. *Let $L \geq 0$ and u, ψ, g, and A be given as above. Suppose $A \in \mathscr{F}$. Let $l(t) = \int_c^t (1/r(\psi^{-1}(s)))ds$, where $0 \leq c \leq L$ and $t \geq c$. Assume that $l(\infty) > l(L) + \int_a^b g(y)w(y)dy$. If u satisfies* (68)*, then for all $t \in I$ we have*

$$\begin{aligned} u(t) &\leq \psi^{-1}\left(l^{-1}\left(l(L) + \int_a^t g(y)\, w(y)\, dy\right)\right) \\ &\leq \psi^{-1}\left(l^{-1}\left(l(L) + \int_a^b g(y)\, w(y)\, dy\right)\right). \end{aligned} \tag{70}$$

It is easy to see that if $A \in \mathscr{F}$, then $A^m \in \mathscr{F}$ for $m > 0$. If $A_1, \ldots, A_k \in \mathscr{F}$, then $A = \prod_{i=1}^k A_i \in \mathscr{F}$. Moreover, we also have $A = \sum_{i=1}^k b_i A_i \in \mathscr{F}$, where $b_i \geq 0$. Hence the class of operators $\mathscr{F}$ is closed under multiplication and linear combination with nonnegative coefficients. This and the following examples show that Theorem 16 can be applied to a variety of operators A in (68).

Example 17. Let

$$A[f](t) = r(f(t)), \tag{71}$$

where $r \in C(\mathbb{R}^+, (0, \infty))$ is increasing. Then $A \in \mathscr{F}$ with $w(t) = 1$ in (69). In this case and $\psi \equiv 1$, Theorem 16 is reduced to (3)-(4).

Example 18. Let

$$A[f](t) = \sup_{a \leq s \leq t} f(s). \tag{72}$$

Then $A \in \mathscr{F}$ with $w(t) = 1$ and $r(t) = 1$ in (69).

Example 19. Let

$$\mathbb{T}[f](t) = \int_a^t k(t,s)\, r(f(s))\, ds, \tag{73}$$

where $k \in C(I^2, \mathbb{R}^+)$ and $r \in C(\mathbb{R}^+, (0, \infty))$ is increasing. Then $\mathbb{T} \in \mathscr{F}$ with $w(t) = \int_a^t k(t,s)ds$ in (69).

Example 20. Let

$$\mathbb{G}[f](t) = \exp\left(\frac{1}{\int_a^t k(t,s)\, ds} \int_a^t k(t,s) \log r(f(s))\, ds\right), \tag{74}$$

where $k \in C(I^2, (0, \infty))$ and $r \in C(\mathbb{R}^+, (0, \infty))$ is increasing. Then $\mathbb{G} \in \mathscr{F}$ with $w(t) = 1$ in (69).

Example 21. Let

$$\mathbb{M}[f](t) = \sup_{a<x<t} \int_x^t k(t,s)\, r(f(s))\, ds, \tag{75}$$

where k and r are given as in Example 19. Then $\mathbb{M} \in \mathscr{F}$ with $w(t) = \int_a^t k(t,s)ds$ in (69).

Example 22. Let

$$\begin{aligned} \mathscr{G}[f](t) \\ = \sup_{a<x<t} \exp\left(\frac{1}{\int_x^t k(t,s)\, ds} \int_x^t k(t,s) \log r(f(s))\, ds\right), \end{aligned} \tag{76}$$

where k and r are given as in Example 20. Then $\mathscr{G} \in \mathscr{F}$ with $w(t) = 1$ in (69).

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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