Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 960276, 12 pages
http://dx.doi.org/10.1155/2014/960276

Research Article

Existence and Multiplicity of Fast Homoclinic Solutions for a Class of Damped Vibration Problems with Impulsive Effects

Qiongfen Zhang

College of Science, Guilin University of Technology, Guilin, Guangxi 541004, China

Correspondence should be addressed to Qiongfen Zhang; qfzhangcsu@163.com

Received 11 April 2014; Accepted 26 May 2014; Published 15 June 2014

Academic Editor: Jianqing Chen

This paper is concerned with the existence and multiplicity of fast homoclinic solutions for a class of damped vibration problems with impulsive effects. Some new results are obtained under more relaxed conditions by using Mountain Pass Theorem and Symmetric Mountain Pass Theorem in critical point theory. The results obtained in this paper generalize and improve some existing works in the literature.

1. Introduction

Consider fast homoclinic solutions of the following problem:

$$\begin{aligned} &u''(t) + q(t)u'(t) - a(t)u(t) \\ &\quad + \nabla V(t,u(t)) = 0, \quad \text{a.e. } t \in (t_j, t_{j+1}), \ j \in \mathbb{Z}, \\ &\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-) = I(u(t_j)), \quad j \in \mathbb{Z}, \end{aligned} \tag{1}$$

where $V : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is of class C^1, $\nabla V(t,u(t)) = \partial V(t,u(t))/\partial u$, $I : \mathbb{R} \to \mathbb{R}$, $a \in C(\mathbb{R},(0,+\infty))$, $a(t) \to +\infty$ as $|t| \to +\infty$, $\mathbb{Z}$ denotes the sets of integers, and t_j $(j \in \mathbb{Z})$ are impulsive points. Moreover, there exist a positive integer p and a positive constant T such that $0 < t_0 < t_1 < \cdots < t_{p-1} < T$, $t_{l+kp} = t_l + kT$, $\forall k \in \mathbb{Z}$, $l = 0,1,\ldots,p-1$, $u'(t_j^+) = \lim_{h\to 0^+} u'(t_j + h)$ and $u'(t_j^-) = \lim_{h\to 0^-} u'(t_j - h)$ represent the right and left limits of $u'(t)$ at $t = t_j$, respectively, $q : \mathbb{R} \to \mathbb{R}$ is a continuous function, and $Q(t) = \int_0^s q(s)ds$ with

$$\lim_{|t|\to+\infty} Q(t) = +\infty. \tag{2}$$

When $I \equiv 0$, problem (1) becomes the following damped vibration problem:

$$\begin{aligned} &u''(t) + q(t)u'(t) - a(t)u(t) \\ &\quad + \nabla V(t,u(t)) = 0, \quad \text{a.e. } t \in \mathbb{R}. \end{aligned} \tag{3}$$

Chen et al. [1] investigated problem (3) and obtained some results of fast homoclinic solutions by critical point theory.

When $q(t) \equiv 0$ and $a(t) \equiv 0$, problem (1) becomes the following problem:

$$\begin{aligned} &u''(t) + \nabla V(t,u(t)) = 0, \quad \text{a.e. } t \in (t_j, t_{j+1}), \ j \in \mathbb{Z}, \\ &\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-) = I(u(t_j)), \quad j \in \mathbb{Z}. \end{aligned} \tag{4}$$

Fang and Duan [2] obtained the following result of homoclinic solutions for (4) by employing Mountain Pass Theorem, a weak convergence argument, and a weak version of Lieb's methods.

Theorem A (see [2]). *Assume that the following conditions hold:*

(V1) *there exists a positive number T such that*

$$\begin{aligned} &\nabla V(t+T,x) = \nabla V(t,x), \\ &V(t+T,x) = V(t,x), \\ &\forall (t,x) \in \mathbb{R}^2; \end{aligned} \tag{5}$$

(V2) $\lim_{x\to 0}(\nabla V(t,x)/x) = 0$ *uniformly for $t \in \mathbb{R}$;*

(V3) *there exists a constant $\mu > 0$ such that*

$$\mu V(t,x) \le (\nabla V(t,x), x), \quad \forall (t,x) \in \mathbb{R}\times\mathbb{R}\setminus\{0\}; \tag{6}$$

(V4) *there exist constants $b_0 > 0$ and $b > 0$ such that*

$$V(t,x) \geq b_0 |x|^{\mu}, \quad \forall t \in \mathbb{R},\ |x| \geq 1; \qquad V(t,x) \leq b|x|^{\mu}, \quad \forall t \in \mathbb{R},\ |x| \leq 1; \tag{7}$$

(I) *there exists a constant b_1 with $0 < b_1 < (\mu-2)/(\mu+2)pT$ such that*

$$|I(x)| \leq b_1 |x|, \quad 2\int_0^x I(t)\,dt - I(x)x \leq 0. \tag{8}$$

Then, problem (4) *possesses a nontrivial weak homoclinic orbit.*

For $I \neq 0$, problem (1) involves impulsive effects. Impulsive differential equations are suitable for the mathematical simulation of evolutionary processes in which the parameters undergo relatively long periods of smooth variation followed by a short-term rapid change (that is jumps) in their values. Impulsive differential equations are often investigated in various fields of science and technology, for example, many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics, pharmacokinetics, and frequency modulated systems, and so on. For more details of impulsive differential equations, we refer the readers to the books [3, 4].

In recent years, some researchers have paid attention to the existence and multiplicity of solutions for impulsive differential equations via variational methods. See, for example, [1, 5–17] and references therein. However, there are few papers [2, 18, 19] concerning homoclinic solutions of impulsive differential equations by variational methods. So, it is a novel method to employ variational methods to investigate the existence of homoclinic solutions for impulsive differential equations.

Motivated by the above papers, we will establish some new results for (1). In order to introduce the concept of fast homoclinic solutions for (1), we first state some properties of the weighted Sobolev space E on which certain variational functional associated with (1) is defined and the fast homoclinic solutions are the critical points of the certain functional.

Let

$$E = \Big\{ u \in H^{1,2}(\mathbb{R},\mathbb{R}) \mid \int_{\mathbb{R}} e^{Q(t)} \left[\left|u'(t)\right|^2 + a(t)|u(t)|^2 \right] dt < +\infty, \left\{u(t_j)\right\}_{j=-\infty}^{+\infty} \in l^2, u(\pm\infty) = 0, u(kT) = 0, k \in \mathbb{Z} \Big\}, \tag{9}$$

where $Q(t)$ is defined in (2) and l^2 denotes the space of sequences whose second powers are summable on $\mathbb{Z}$; that is,

$$\sum_{j\in\mathbb{Z}} \left|a_j\right|^2 < +\infty, \quad \forall a = \left\{a_j\right\}_{j=-\infty}^{+\infty} \in l^2. \tag{10}$$

The space l^2 is equipped with the following norm:

$$\|a\|_{l^2} = \left(\sum_{j\in\mathbb{Z}} \left|a_j\right|^2 \right)^{1/2}. \tag{11}$$

For $u, v \in E$, let

$$\langle u, v\rangle = \int_{\mathbb{R}} e^{Q(t)} \left[\left(u'(t), v'(t)\right) + (a(t)u(t), v(t)) \right] dt. \tag{12}$$

Similar to [2], it is easy to check that E is a Hilbert space with the norm given by

$$\|u\| = \left(\int_{\mathbb{R}} e^{Q(t)} \left[\left|u'(t)\right|^2 + a(t)|u(t)|^2 \right] dt \right)^{1/2}. \tag{13}$$

It is obvious that

$$E \subset L^2\left(e^{Q(t)}\right) \tag{14}$$

with the embedding being continuous. Here, $L^p(e^{Q(t)})(2 \leq p < +\infty)$ denotes the Banach spaces of functions on $\mathbb{R}$ with values in $\mathbb{R}$ under the norm

$$\|u\|_p = \left\{ \int_{\mathbb{R}} e^{Q(t)} |u(t)|^p dt \right\}^{1/p}. \tag{15}$$

Similar to [1], we have the following definition of fast homoclinic solutions.

Definition 1. If (2) holds, a solution of (1) is called a fast homoclinic solution if $u \in E$.

Here and in subsequence, $(,)$ and $|\cdot|$ denote the inner product and norm in $\mathbb{R}$, respectively. C_i $(i = 0, 1, \ldots)$ denote different positive constants. Now, we state our main results.

Theorem 2. *Suppose that q, a, I, and V satisfy* (2), *(V1), and the following conditions:*

(A) *$a \in C(\mathbb{R}, (0,+\infty))$ and $a(t) \to +\infty$ as $|t| \to +\infty$,*

(V2)′ *$V(t,x) = V_1(t,x) - V_2(t,x)$, $V_1, V_2 \in C^1(\mathbb{R}\times\mathbb{R},\mathbb{R})$, and there exists a constant $R > 0$ such that*

$$|\nabla V(t,x)| = o(|x|) \quad \text{as } x \longrightarrow 0 \text{ uniformly in } t \in (-\infty,-R] \cup [R,+\infty), \tag{16}$$

(V3)′ *there is a constant $\mu > 2$ such that*

$$0 < \mu V_1(t,x) \leq \left(\nabla V_1(t,x), x\right), \quad \forall (t,x) \in \mathbb{R}\times\mathbb{R}\setminus\{0\}, \tag{17}$$

(V5) *$V_2(t,0) = 0$ and there exists a constant $\varrho \in (2,\mu)$ such that*

$$V_2(t,x) \geq 0, \quad \left(\nabla V_2(t,x), x\right) \leq \varrho V_2(t,x), \quad \forall (t,x) \in \mathbb{R}\times\mathbb{R}, \tag{18}$$

(I)′ *$I \in C(\mathbb{R},\mathbb{R})$ and there exists a constant c with $0 < c < (\varrho-2)e_0/(\varrho+2)pT$ such that*

$$|I(x)| \leq c|x|, \quad \left(I(x) - I(y), x - y\right) \geq 0, \quad \forall x, y \in \mathbb{R}, \tag{19}$$

where $e_0 = e^{\min\{Q(t): t\in\mathbb{R}\}}$.

Then, problem (1) *has at least one nontrivial fast homoclinic solution.*

Theorem 3. *Suppose that q, a, V, and I satisfy* (2), *(V1), (A), (V3)′, (I)′, and the following conditions:*

(V2)″ $V(t,x) = V_1(t,x) - V_2(t,x)$, $V_1, V_2 \in C^1(\mathbb{R}\times\mathbb{R},\mathbb{R})$, *and*

$$|\nabla V(t,x)| = o(|x|) \quad \text{as } x \longrightarrow 0 \text{ uniformly in } t \in \mathbb{R}, \tag{20}$$

(V5)′ $V_2(t,0) = 0$ *and there exists a constant* $\varrho \in (2,\mu)$ *such that*

$$(\nabla V_2(t,x), x) \le \varrho V_2(t,x), \quad \forall (t,x) \in \mathbb{R}\times\mathbb{R}. \tag{21}$$

Then, problem (1) *has at least one nontrivial fast homoclinic solution.*

Theorem 4. *Suppose that q, a, V, and I satisfy* (2), *(V1), (A), (V2)′, (V3)′, (V5), (I)′, and the following condition:*

(V6) $I(-t) = -I(t), V(t,-x) = V(t,x), \forall (t,x) \in \mathbb{R}\times\mathbb{R}$.

Then, problem (1) *has an unbounded sequence of fast homoclinic solutions.*

Theorem 5. *Suppose that q, a, V, and I satisfy* (2), *(V1), (A), (V2)″, (V3)′, (V5)′, (I)′, and (V6). Then, problem* (1) *has an unbounded sequence of fast homoclinic solutions.*

The rest of this paper is organized as follows. In Section 2, some preliminaries are presented. In Section 3, we give the proofs of our results. In Section 4, some examples are given to illustrate our results.

2. Preliminaries

Let E and $\|\cdot\|$ be given in Section 1. By a similar argument in [2, 20], we have the following important lemma.

Lemma 6. *For any* $u \in E$,

$$\begin{aligned}\|u\|_\infty &\le \frac{1}{\sqrt{2e_0\sqrt{a_0}}}\|u\| \\ &= \frac{1}{\sqrt{2e_0\sqrt{a_0}}}\left\{\int_{\mathbb{R}} e^{Q(s)}\left[\left|u'(s)\right|^2 + a(s)|u(s)|^2\right]ds\right\}^{1/2},\end{aligned} \tag{22}$$

$$\begin{aligned}|u(t)| \le \Bigg\{&\int_t^{+\infty} e^{-Q(s)}[a(s)]^{-1/2} \\ &\times\left[e^{Q(s)}\left(\left|u'(s)\right|^2 + a(s)|u(s)|^2\right)\right]ds\Bigg\}^{1/2},\end{aligned} \tag{23}$$

$$\begin{aligned}|u(t)| \le \Bigg\{&\int_{-\infty}^{t} e^{-Q(s)}[a(s)]^{-1/2} \\ &\times\left[e^{Q(s)}\left(\left|u'(s)\right|^2 + a(s)|u(s)|^2\right)\right]ds\Bigg\}^{1/2},\end{aligned} \tag{24}$$

$$\sum_{j=-\infty}^{\infty}\left|u(t_j)\right|^2 \le \frac{pT}{e_0}\left\|u'\right\|_2^2, \tag{25}$$

where $\|u\|_\infty = \operatorname{ess\,sup}_{t\in\mathbb{R}}|u(t)|$, $a_0 = \min_{t\in\mathbb{R}}\{a(t)\}$, *and* e_0 *is the same as that in assumption (I)′.*

The following two lemmas are Mountain Pass Theorem and Symmetric Mountain Pass Theorem, which are useful in the proofs of our theorems.

Lemma 7 (see [21]). *Let E be a real Banach space and* $\varphi \in C^1(E,\mathbb{R})$ *satisfying (PS)-condition. Suppose that* $\varphi(0) = 0$ *and*

(i) *there exist constants* $\rho, \alpha > 0$ *such that* $\varphi_{\partial B_\rho(0)} \ge \alpha$;

(ii) *there exists an* $e \in E \setminus \overline{B}_\rho(0)$ *such that* $\varphi(e) \le 0$.

Then, φ *possesses a critical value* $c \ge \alpha$ *which can be characterized as* $c = \inf_{h\in\Phi}\max_{s\in[0,1]}\varphi(h(s))$, *where* $\Phi = \{h \in C([0,1],E) \mid h(0) = 0, h(1) = e\}$ *and* $B_\rho(0)$ *is an open ball in E of radius* ρ *centered at* 0.

Lemma 8 (see [21]). *Let E be a real Banach space and* $\varphi \in C^1(E,\mathbb{R})$ *with I even. Assume that* $\varphi(0) = 0$ *and* φ *satisfies (PS)-condition, assumption (i) of Lemma 7, and the following condition:*

(iii) *for each finite dimensional subspace* $E' \subset E$, *there is* $r = r(E') > 0$ *such that* $\varphi(u) \le 0$, *for* $u \in E' \setminus B_r(0)$, *and* $B_r(0)$ *is an open ball in E of radius r centered at* 0.

Then, φ *possesses an unbounded sequence of critical values.*

Remark 9. Since it is very difficult to check condition (iii) of Lemma 8, few results about infinitely many homoclinic solutions can be seen in the literature by using Lemma 8, let alone infinitely many fast homoclinic solutions obtained by this lemma. Motivated by the idea of [22], we will use Lemma 8 to prove that problem (1) has infinitely many homoclinic fast solutions.

Lemma 10. *Assume that (V3)′ and (V5) or (V5)′ hold. Then, for every* $(t,x) \in \mathbb{R}\times\mathbb{R}$,

(i) $s^{-\mu}V_1(t,sx)$ *is nondecreasing on* $(0,+\infty)$;

(ii) $s^{-\varrho}V_2(t,sx)$ *is nonincreasing on* $(0,+\infty)$.

The proof of Lemma 10 is routine and we omit it.

The functional φ corresponding to (1) on E is given by

$$\varphi(u) = \int_{\mathbb{R}} \frac{1}{2} e^{Q(t)} \left[\left| u'(t) \right|^2 + a(t) |u(t)|^2 \right] dt - \int_{\mathbb{R}} V(t, u(t))\, dt + \sum_{j=-\infty}^{\infty} \int_0^{u(t_j)} I(s)\, ds, \quad u \in E. \tag{26}$$

We now show that $\varphi \in C^1(E, \mathbb{R})$ and, for $u, v \in E$,

$$\begin{aligned} \left\langle \varphi'(u), v \right\rangle &= \int_{\mathbb{R}} e^{Q(t)} \left[\left(u'(t), v'(t) \right) + (a(t) u(t), v(t)) - (\nabla V(t, u(t)), v(t)) \right] dt \\ &\quad + \sum_{j=-\infty}^{\infty} \left(I\left(u\left(t_j\right)\right), v\left(t_j\right) \right). \end{aligned} \tag{27}$$

Firstly, we show that $\varphi : E \to \mathbb{R}$. By $(V2)'$, for any given $\varepsilon_0 > 0$, there exists $\gamma_0 > 0$ such that

$$|\nabla V(t, x)| \leq 2\varepsilon e_0 a_0 |x|, \quad t \in (-\infty, -R] \cup [R, +\infty), \quad |x| \leq \gamma_0, \tag{28}$$

where $a_0 = \min_{t \in \mathbb{R}} \{a(t)\}$, e_0 is the same as that in assumption $(I)'$. Then, by $V(t, 0) = 0$ and (28), we have

$$|V(t, x)| = \left| \int_0^1 (\nabla V(t, sx), x)\, ds \right| \leq \varepsilon e_0 a_0 |x|^2, \quad \forall t \in (-\infty, -R] \cup [R, +\infty), \quad x \in \mathbb{R}. \tag{29}$$

Therefore, from (29), we have

$$\begin{aligned} \left| \int_{\mathbb{R} \setminus (-R, R)} V(t, u(t))\, dt \right| &\leq \int_{\mathbb{R} \setminus (-R, R)} |V(t, u(t))|\, dt \\ &\leq \int_{\mathbb{R} \setminus (-R, R)} \varepsilon e_0 a_0 |u(t)|^2 dt \\ &\leq \int_{\mathbb{R} \setminus (-R, R)} \varepsilon e^{Q(t)} a(t) |u(t)|^2 dt \\ &\leq \int_{\mathbb{R}} \varepsilon e^{Q(t)} a(t) |u(t)|^2 dt \leq \varepsilon \|u\|^2, \quad u \in E. \end{aligned} \tag{30}$$

From $(I)'$ and Lemma 6, we have

$$\begin{aligned} \sum_{j=-\infty}^{\infty} \left| \int_0^{u(t_j)} I(s)\, ds \right| &\leq \sum_{j=-\infty}^{\infty} \int_{\min\{0, u(t_j)\}}^{\max\{0, u(t_j)\}} |I(s)|\, ds \\ &\leq \frac{1}{2} \sum_{j=-\infty}^{\infty} c \left| u\left(t_j\right) \right|^2 \leq \frac{cpT}{2e_0} \left\| u' \right\|_2^2 \leq \frac{cpT}{2e_0} \|u\|^2. \end{aligned} \tag{31}$$

It follows from (26), (30), and (31) that $\varphi : E \to \mathbb{R}$. Next, we prove that $\varphi \in C^1(E, \mathbb{R})$. Rewrite φ as the following:

$$\varphi(u) = \varphi_1(u) - \varphi_2(u) + \varphi_3(u), \tag{32}$$

where

$$\varphi_1(u) := \int_{\mathbb{R}} \frac{1}{2} e^{Q(t)} \left[\left| u'(t) \right|^2 + a(t) |u(t)|^2 \right] dt, \qquad \varphi_2(u) := \int_{\mathbb{R}} V(t, u(t))\, dt, \qquad \varphi_3(u) := \sum_{j=-\infty}^{\infty} \int_0^{u(t_j)} I(s)\, ds. \tag{33}$$

It is easy to check that $\varphi_1, \varphi_3 \in C^1(E, \mathbb{R})$ and

$$\begin{aligned} \left\langle \varphi_1'(u), v \right\rangle &= \int_{\mathbb{R}} e^{Q(t)} \left[\left(u'(t), v'(t) \right) + (a(t) u(t), v(t)) \right] dt, \\ \left\langle \varphi_3'(u), v \right\rangle &= \sum_{j=-\infty}^{\infty} \left(I\left(u\left(t_j\right)\right), v\left(t_j\right) \right), \\ &\qquad \forall u, v \in E. \end{aligned} \tag{34}$$

Next, we prove that $\varphi_2 \in C^1(E, \mathbb{R})$ and

$$\left\langle \varphi_2'(u), v \right\rangle = \int_{\mathbb{R}} (\nabla V(t, u(t)), v(t))\, dt, \quad \forall u, v \in E. \tag{35}$$

Let $u_n \to u$ in E, without loss of generality, and we can assume that $\|u_n\| \leq \gamma_0$. Since $V \in C^1(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, we have

$$\begin{aligned} \left| \nabla V\left(t, u_n(t)\right), v(t) \right| &\leq \max_{|x| \leq \gamma_0 / \sqrt{2e_0 \sqrt{a_0}}} |\nabla V(t, x)|\, |v(t)| \\ &:= C_0 |v(t)|, \quad \forall t \in [-R, R]. \end{aligned} \tag{36}$$

By (29) and (36), we have

$$\begin{aligned} &\left| \nabla V\left(t, u_n(t)\right), v(t) \right| \\ &\leq \varepsilon e_0 a_0 \left| u_n(t) \right| |v(t)| + C_0 |v(t)| \\ &\leq \varepsilon e^{Q(t)} a(t) \left(\left| u_n(t) - u(t) \right| + |u(t)| \right) |v(t)| + C_0 |v(t)| \\ &:= g_n(t), \quad t \in \mathbb{R}. \end{aligned} \tag{37}$$

Since $u_n(t) \to u(t)$, for almost every $t \in \mathbb{R}$, we have

$$\lim_{n\to+\infty} g_n(t) = \varepsilon e^{Q(t)} a(t)|u(t)||v(t)| + C_0|v(t)| := g(t),$$

$$\begin{aligned}
&\lim_{n\to+\infty}\int_{\mathbb{R}} g_n(t)\,dt \\
&= \lim_{n\to+\infty}\int_{\mathbb{R}} \left[\varepsilon e^{Q(t)} a(t)\left(|u_n(t)-u(t)| + |u(t)|\right)|v(t)| + C_0|v(t)|\right]dt \\
&= \lim_{n\to+\infty}\int_{\mathbb{R}} \varepsilon e^{Q(t)} a(t)\left(|u_n(t)-u(t)|\right)dt + \int_{\mathbb{R}}\left[\varepsilon e^{Q(t)} a(t)|u(t)||v(t)| + C_0|v(t)|\right]dt \\
&= \int_{\mathbb{R}}\left[\varepsilon e^{Q(t)} a(t)|u(t)||v(t)| + C_0|v(t)|\right]dt \\
&:= \int_{\mathbb{R}} g(t)\,dt < +\infty.
\end{aligned} \tag{38}$$

Then, by (37), (38), and Lebesgue's dominated convergence theorem, we have

$$\lim_{n\to+\infty}\int_{\mathbb{R}}(\nabla V(t,u_n(t)), v(t))\,dt = \int_{\mathbb{R}}(\nabla V(t,u(t)), v(t))\,dt. \tag{39}$$

Therefore, for any $u, v \in E$ and for any function $\theta : \mathbb{R} \to (0,1)$, from (39), we have

$$\begin{aligned}
&\langle \varphi_2'(u), v\rangle \\
&= \lim_{h\to 0^+}\frac{\varphi_2(u+hv)-\varphi_2(u)}{h} \\
&= \lim_{h\to 0^+}\frac{1}{h}\int_{\mathbb{R}}[V(t,u(t)+hv(t)) - V(t,u(t))]\,dt \\
&= \lim_{h\to 0^+}\int_{\mathbb{R}}(\nabla V(t,u(t)+\theta(t)hv(t)), v(t))\,dt \\
&= \int_{\mathbb{R}}(\nabla V(t,u(t)), v(t))\,dt, \quad \forall u,v\in E.
\end{aligned} \tag{40}$$

Finally, we prove that $\varphi_2 \in C^1(E,\mathbb{R})$. From (40), $u_n \to u$ in E, and $V \in C^1(\mathbb{R}\times\mathbb{R},\mathbb{R})$, we have

$$\begin{aligned}
&\lim_{n\to+\infty}\left|\langle \varphi_2'(u_n) - \varphi_2'(u), v\rangle\right| \\
&= \lim_{n\to+\infty}\left|\int_{\mathbb{R}}(\nabla V(t,u_n(t)) - \nabla V(t,u(t)), v(t))\,dt\right| \\
&\le \lim_{n\to+\infty}\int_{\mathbb{R}}|\nabla V(t,u_n(t)) - \nabla V(t,u(t))||v(t)|\,dt \\
&= \int_{\mathbb{R}}\lim_{n\to+\infty}|\nabla V(t,u_n(t)) - \nabla V(t,u(t))| \times |v(t)|\,dt = 0, \quad \forall v\in E.
\end{aligned} \tag{41}$$

This shows that $\varphi_2 \in C^1(E,\mathbb{R})$. Therefore, $\varphi \in C^1(E,\mathbb{R})$ and (27) holds. Similarly, we can prove $\varphi \in C^1(E,\mathbb{R})$ and (27) holds by (V1), (V2)$''$, and (I)$'$. Furthermore, the critical points of φ in E are classical solutions of (1) with $u(\pm\infty) = 0$.

3. Proofs of Theorems

Proof of Theorem 2. It is clear that $\varphi(0) = 0$. We first show that the functional φ satisfies the (PS)-condition. Let $\{u_n\} \subset E$ satisfying $\varphi(u_n)$ which is bounded and let $\varphi'(u_n) \to 0$ as $n \to \infty$. Then, there exists a constant $C_1 > 0$ such that

$$|\varphi(u_n)| \le C_1, \qquad \|\varphi'(u_n)\|_{E^*} \le \mu C_1. \tag{42}$$

From (I)$'$ and (25), we have

$$\begin{aligned}
\sum_{j=-\infty}^{\infty}\left(I(u(t_j)), u(t_j)\right) &\le \sum_{j=-\infty}^{\infty}\left|I(u(t_j))\right|\left|u(t_j)\right| \\
&\le \sum_{j=-\infty}^{\infty} c\left|u(t_j)\right|^2 \le \frac{cpT}{e_0}\|u'\|_2^2.
\end{aligned} \tag{43}$$

From (26), (27), (31), (42), (43), (V3)$'$, and (V5), we have

$$\begin{aligned}
&2C_1 + 2C_1\|u_n\| \\
&\ge 2\varphi(u_n) - \frac{2}{\mu}\langle \varphi'(u_n), u_n\rangle \\
&= \frac{\mu-2}{\mu}\|u_n\|^2 + 2\sum_{j=-\infty}^{\infty}\int_0^{u(t_j)} I(s)\,ds - \frac{2}{\mu}\sum_{j=-\infty}^{\infty}\left(I(u_n(t_j)), u_n(t_j)\right) \\
&\quad - 2\int_{\mathbb{R}} e^{Q(t)}\left[V_1(t,u_n(t)) - \frac{1}{\mu}(\nabla V_1(t,u_n(t)), u_n(t))\right]dt \\
&\quad + 2\int_{\mathbb{R}} e^{Q(t)}\left[V_2(t,u_n(t)) - \frac{1}{\mu}(\nabla V_2(t,u_n(t)), u_n(t))\right]dt \\
&\ge \frac{\mu-2}{\mu}\|u_n\|^2 - \frac{cpT}{e_0}\|u_n'\|_2^2 - \frac{2cpT}{\mu e_0}\|u_n'\|_2^2 \\
&\ge \left(\frac{\mu-2}{\mu} - \frac{(\mu+2)cpT}{\mu e_0}\right)\|u_n\|^2.
\end{aligned} \tag{44}$$

Since $\mu > \varrho > 2$ and $0 < c < (\varrho-2)e_0/(\varrho+2)pT$, the above inequalities imply that there exists a constant $C_2 > 0$ such that

$$\|u_n\| \le C_2, \quad n\in\mathbb{N}. \tag{45}$$

Now, we prove that $u_n \to u_0$ in E. Passing to a subsequence if necessary, it can be assumed that $u_n \rightharpoonup u_0$ in E. For any given number $\varepsilon > 0$, by (V2)$'$, we can choose $\xi > 0$ such that

$$|\nabla V(t,x)| \le \varepsilon a_0|x| \quad \text{for } |t|\ge R,\ |x|\le\xi. \tag{46}$$

Since $Q(t) \to \infty$ as $|t| \to \infty$, we can choose $R_0 > R$ such that

$$Q(t) \geq \ln\left(\frac{C_2}{\xi}\right) \quad \text{for } |t| \geq R_0. \tag{47}$$

From (A), we can choose $R_0' > R$ such that

$$a(t) \geq \frac{C_2}{\xi} \quad \text{for } |t| \geq R_0'. \tag{48}$$

It follows from (23), (45), (47), and (48) that

$$\begin{aligned} &|u_n(t)|^2 \\ &\leq \left\{ \int_t^{+\infty} e^{-Q(s)} [a(s)]^{-1/2} \right. \\ &\qquad \left. \times \left[e^{Q(s)} \left(|u'(s)|^2 + a(s) |u(s)|^2 \right) \right] ds \right\}^{1/2} \\ &\leq \frac{\xi^2}{C_2^2} \|u_n\|^2 \leq \xi^2 \quad \text{for } t \geq R_1,\ n \in \mathbb{N}, \end{aligned} \tag{49}$$

where $R_1 = \max\{R_0, R_0'\}$. Similarly, by (24), (45), (47), and (48), we have

$$|u_n(t)|^2 \leq \xi^2 \quad \text{for } t \leq -R_1,\ n \in \mathbb{N}. \tag{50}$$

Since $u_n \rightharpoonup u_0$ in E, it is easy to verify that $u_n(t)$ converges to $u_0(t)$ pointwise for all $t \in \mathbb{R}$. Hence, it follows from (49) and (50) that

$$|u_0(t)| \leq \xi \quad \text{for } t \in (-\infty, -R_1] \cup [R_1, +\infty). \tag{51}$$

Since $e^{Q(t)} \geq e_0 > 0$ on $[-R_1, R_1] = J$, the operator defined by $S : E \to H^1(J) : u \to u|_J$ is a linear continuous map. So, $u_n \to u_0$ in $H^1(J)$. Sobolev theorem implies that $u_n \to u_0$ uniformly on J, so there is $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} &\int_{-R_1}^{R_1} e^{Q(t)} |\nabla V(t, u_n(t)) - \nabla V(t, u_0(t))| \\ &\qquad \times |u_n(t) - u_0(t)| \, dt < \varepsilon \quad \text{for } n \geq n_0. \end{aligned} \tag{52}$$

From (45), (46), (49), (50), and (51), we have

$$\begin{aligned} &\int_{\mathbb{R}\setminus[-R_1,R_1]} e^{Q(t)} |\nabla V(t, u_n(t)) - \nabla V(t, u_0(t))| \\ &\qquad \times |u_n(t) - u_0(t)| \, dt \\ &\leq \int_{\mathbb{R}\setminus[-R_1,R_1]} e^{Q(t)} \left(|\nabla V(t, u_n(t))| + |\nabla V(t, u_0(t))| \right) \\ &\qquad \times \left(|u_n(t)| + |u_0(t)| \right) dt \\ &\leq \varepsilon \int_{\mathbb{R}\setminus[-R_1,R_1]} e^{Q(t)} a_0 \left(|u_n(t)| + |u_0(t)| \right) \\ &\qquad \times \left(|u_n(t)| + |u_0(t)| \right) dt \\ &\leq 2\varepsilon \int_{\mathbb{R}\setminus[-R_1,R_1]} e^{Q(t)} a_0 \left(|u_n(t)|^2 + |u_0(t)|^2 \right) dt \\ &\leq 2\varepsilon \int_{\mathbb{R}\setminus[-R_1,R_1]} e^{Q(t)} \left[a(t) |u_n(t)|^2 + a(t) |u_0(t)|^2 \right] dt \\ &\leq 2\varepsilon \left(\|u_n\|^2 + \|u_0\|^2 \right) \leq 2\varepsilon \left(C_2^2 + \|u_0\|^2 \right), \quad n \in \mathbb{N}. \end{aligned} \tag{53}$$

It follows from (52) and (53) that

$$\begin{aligned} &\int_{\mathbb{R}} e^{Q(t)} |\nabla V(t, u_n(t)) - \nabla V(t, u_0(t))| \\ &\qquad |u_n(t) - u_0(t)| \, dt \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{54}$$

From (27) and $(\mathrm{I})'$, as $n \to \infty$, we have

$$\begin{aligned} 0 &\longleftarrow \left\langle \varphi'(u_n) - \varphi'(u_0), u_n - u_0 \right\rangle \\ &= \|u_n - u_0\|^2 \\ &\quad + \sum_{j=-\infty}^{\infty} \left(I\left(u_n(t_j)\right) - I\left(u_0(t_j)\right), u_n(t_j) - u_0(t_j) \right) \\ &\quad - \int_{\mathbb{R}} e^{Q(t)} \left(\nabla V(t, u_n(t)) - \nabla V(t, u_0(t)), u_n(t) - u_0(t) \right) dt \\ &\geq \|u_n - u_0\|^2 \\ &\quad - \int_{\mathbb{R}} e^{Q(t)} \left(\nabla V(t, u_n(t)) - \nabla V(t, u_0(t)), u_n(t) - u_0(t) \right) dt. \end{aligned} \tag{55}$$

It follows from (54) and (55) that

$$\|u_n - u_0\|^2 \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{56}$$

Hence, $u_n \to u_0$ in E by (56). This shows that φ satisfies (PS)-condition.

We now show that there exist constants $\rho, \alpha > 0$ such that assumption (i) of Lemma 7 holds. From $(\mathrm{V2})'$, there exists $\delta \in (0, 1)$ such that

$$|\nabla V(t, x)| \leq \frac{a_0}{2} |x| \quad \text{for } |t| \geq R,\ |x| \leq \delta. \tag{57}$$

By $V(t, 0) = 0$ and (57), we have

$$|V(t, x)| \leq \frac{a_0}{4} |x|^2 \quad \text{for } |t| \geq R,\ |x| \leq \delta. \tag{58}$$

Let

$$C_3 = \sup\left\{ \frac{V_1(t, x)}{a_0} \mid t \in [-R, R], x \in \mathbb{R}, |x| = 1 \right\}. \tag{59}$$

Setting $\sigma = \min\{1/(4C_3+1)^{1/(\mu-2)}, \delta\}$ and $\|u\| = \sqrt{2e_0\sqrt{a_0}}\sigma := \rho$, it follows from Lemma 6 that $|u(t)| \le \sigma \le \delta < 1$ for $t \in \mathbb{R}$. From Lemma 10 (i) and (59), we have

$$\begin{aligned}
&\int_{-R}^{R} e^{Q(t)} V_1(t, u(t))\, dt \\
&\le \int_{\{t\in[-R,R]:u(t)\neq 0\}} e^{Q(t)} V_1\left(t, \frac{u(t)}{|u(t)|}\right) |u(t)|^{\mu} dt \\
&\le C_3 \int_{-R}^{R} e^{Q(t)} a_0 |u(t)|^{\mu} dt \\
&\le C_3 \sigma^{\mu-2} \int_{-R}^{R} e^{Q(t)} a_0 |u(t)|^2 dt \\
&\le C_3 \sigma^{\mu-2} \int_{-R}^{R} e^{Q(t)} a(t) |u(t)|^2 dt \\
&\le \frac{1}{4} \int_{-R}^{R} e^{Q(t)} a(t) |u(t)|^2 dt.
\end{aligned} \tag{60}$$

By (V5), (31), (58), and (60), we have

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\|u\|^2 + \sum_{j=-\infty}^{\infty} \int_0^{u(t_j)} I(s)\, ds - \int_{\mathbb{R}} e^{Q(t)} V(t, u(t))\, dt \\
&\ge \frac{1}{2}\|u\|^2 + \sum_{j=-\infty}^{\infty} \int_0^{u(t_j)} I(s)\, ds - \int_{\mathbb{R}\setminus[-R,R]} e^{Q(t)} V(t, u(t))\, dt - \int_{-R}^{R} e^{Q(t)} V_1(t, u(t))\, dt \\
&\ge \frac{1}{2}\|u\|^2 - \frac{cpT}{2e_0}\left\|u'\right\|_2^2 - \frac{1}{4}\int_{\mathbb{R}\setminus[-R,R]} a_0 e^{Q(t)} |u(t)|^2 dt - \frac{1}{4}\int_{-R}^{R} e^{Q(t)} a(t) |u(t)|^2 dt \\
&\ge \frac{1}{2}\|u\|^2 - \frac{cpT}{2e_0}\left\|u'\right\|_2^2 - \frac{1}{4}\int_{\mathbb{R}\setminus[-R,R]} e^{Q(t)} a(t) |u(t)|^2 dt - \frac{1}{4}\int_{-R}^{R} e^{Q(t)} a(t) |u(t)|^2 dt \\
&= \frac{1}{2}\|u\|^2 - \frac{cpT}{2e_0}\left\|u'\right\|_2^2 - \frac{1}{4}\int_{\mathbb{R}} e^{Q(t)} a(t) |u(t)|^2 dt \\
&\ge \min\left\{\frac{1}{2} - \frac{cpT}{2e_0}, \frac{1}{4}\right\} \|u\|^2.
\end{aligned} \tag{61}$$

Therefore, we can choose a constant $\alpha > 0$ depending on ρ such that $\varphi(u) \ge \alpha$ for any $u \in E$ with $\|u\| = \rho$, which shows that φ satisfies assumption (i) of Lemma 7.

Finally, it remains to show that φ satisfies assumption (ii) of Lemma 7. From Lemma 10 (ii) and (22), we have for any $u \in E$

$$\begin{aligned}
&\int_{-3}^{3} e^{Q(t)} V_2(t, u(t))\, dt \\
&= \int_{\{t\in[-3,3]:|u(t)|>1\}} e^{Q(t)} V_2(t, u(t))\, dt + \int_{\{t\in[-3,3]:|u(t)|\le 1\}} e^{Q(t)} V_2(t, u(t))\, dt \\
&\le \int_{\{t\in[-3,3]:|u(t)|>1\}} e^{Q(t)} V_2\left(t, \frac{u(t)}{|u(t)|}\right) |u(t)|^{\varrho} dt + \int_{-3}^{3} e^{Q(t)} \max_{|x|\le 1} V_2(t, x)\, dt \\
&\le \|u\|_{\infty}^{\varrho} \int_{-3}^{3} e^{Q(t)} \max_{|x|=1} V_2(t, x)\, dt + \int_{-3}^{3} e^{Q(t)} \max_{|x|\le 1} V_2(t, x)\, dt \\
&\le \left(\frac{1}{\sqrt{2e_0\sqrt{a_0}}}\right)^{\varrho} \|u\|^{\varrho} \int_{-3}^{3} e^{Q(t)} \max_{|x|=1} V_2(t, x)\, dt + \int_{-3}^{3} e^{Q(t)} \max_{|x|\le 1} V_2(t, x)\, dt \\
&= C_4 \|u\|^{\varrho} + C_5,
\end{aligned} \tag{62}$$

where $C_4 = (1/\sqrt{2e_0\sqrt{a_0}})^{\varrho} \int_{-3}^{3} e^{Q(t)} \max_{|x|=1} V_2(t,x) dt$, $C_5 = \int_{-3}^{3} e^{Q(t)} \max_{|x|\le 1} V_2(t,x) dt$. Take $\omega \in E$ such that

$$|\omega(t)| = \begin{cases} 1, & \text{for } |t| \le 1, \\ 0, & \text{for } |t| \ge 3, \end{cases} \tag{63}$$

and $|\omega(t)| \le 1$ for $|t| \in (1,3]$. For $s > 1$, from Lemma 10 (i) and (63), we get

$$\begin{aligned}
\int_{-1}^{1} e^{Q(t)} V_1(t, s\omega(t))\, dt &\ge s^{\mu} \int_{-1}^{1} e^{Q(t)} V_1(t, \omega(t))\, dt \\
&= C_6 s^{\mu},
\end{aligned} \tag{64}$$

where $C_6 = \int_{-1}^{1} e^{Q(t)} V_1(t, \omega(t))dt > 0$. From (26), (31), (62), (63), and (64), we get for $s > 1$

$$\begin{aligned}
\varphi(s\omega) &= \frac{s^2}{2}\|\omega\|^2 + \sum_{j=-\infty}^{\infty} \int_0^{s\omega(t)} I(t)\,dt \\
&\quad + \int_{\mathbb{R}} e^{Q(t)} \left[V_2(t, s\omega(t)) - V_1(t, s\omega(t))\right] dt \\
&\le \frac{s^2}{2}\|\omega\|^2 + \frac{cpTs^2}{2e_0}\left\|\omega'\right\|_2^2 \\
&\quad + \int_{-3}^{3} e^{Q(t)} V_2(t, s\omega(t))\,dt \\
&\quad - \int_{-1}^{1} e^{Q(t)} V_1(t, s\omega(t))\,dt \\
&\le \left(\frac{e_0 + cpT}{2e_0}\right) s^2\|\omega\|^2 + C_4 s^{\varrho}\|\omega\|^{\varrho} + C_5 - C_6 s^{\mu}.
\end{aligned} \tag{65}$$

Since $\mu > \varrho > 2$ and $C_6 > 0$, it follows from (65) that there exists $s_1 > 1$ such that $\|s_1\omega\| > \rho$ and $\varphi(s_1\omega) < 0$. Set $e = s_1\omega(t)$, and then $e \in E$, $\|e\| = \|s_1\omega\| > \rho$, and $\varphi(e) = \varphi(s_1\omega) < 0$. By Lemma 7, φ has a critical value $c > \alpha$ given by

$$c = \inf_{g\in\Phi} \max_{s\in[0,1]} \varphi(g(s)), \tag{66}$$

where

$$\Phi = \left\{g \in C([0,1], E) : g(0) = 0, g(1) = e\right\}. \tag{67}$$

Hence, there exists $u^* \in E$ such that

$$\varphi(u^*) = c, \qquad \varphi'(u^*) = 0. \tag{68}$$

The function u^* is a desired solution of problem (1). Since $c > 0$, u^* is a nontrivial fast homoclinic solution. The proof is complete. □

Proof of Theorem 3. In the proof of Theorem 2, the condition $V_2(t, x) \ge 0$ in (V5) is only used in the proofs of (45) and assumption (i) of Lemma 7. Therefore, we only need to prove that (45) and assumption (i) of Lemma 7 still hold if we use (V2)$''$ and (V5)$'$ instead of (V2)$'$ and (V5), respectively. We first prove that (45) holds. From (V3)$'$, (V5)$'$, (26), (27), (31), (42), and (43), we have

$$\begin{aligned}
&2C_1 + \frac{2C_1\mu}{\varrho}\|u_n\| \\
&\ge 2\varphi(u_n) - \frac{2}{\varrho}\left\langle \varphi'(u_n), u_n\right\rangle \\
&= \frac{(\varrho - 2)}{\varrho}\|u_n\|^2 \\
&\quad + 2\int_{\mathbb{R}} e^{Q(t)}\left[V_2(t, u_n(t)) - \frac{1}{\varrho}(\nabla V_2(t, u_n(t)), u_n(t))\right] dt \\
&\quad - 2\int_{\mathbb{R}} e^{Q(t)}\left[V_1(t, u_n(t)) - \frac{1}{\varrho}(\nabla V_1(t, u_n(t)), u_n(t))\right] dt \\
&\quad + 2\sum_{j=-\infty}^{\infty}\int_0^{u(t_j)} I(t)\,dt - \frac{2}{\varrho}\sum_{j=-\infty}^{\infty}\left(I\left(u\left(t_j\right)\right), u\left(t_j\right)\right) \\
&\ge \frac{(\varrho - 2)}{\varrho}\|u_n\|^2 - \frac{cpT}{e_0}\left\|u_n'\right\|_2^2 - \frac{2cpT}{\varrho e_0}\left\|u_n'\right\|_2^2 \\
&\ge \frac{(\varrho - 2)}{\varrho}\|u_n\|^2 - \frac{cpT}{e_0}\|u_n\|^2 - \frac{2cpT}{\varrho e_0}\|u_n\|^2 \\
&= \left(\frac{\varrho - 2}{\varrho} - \frac{(\varrho + 2)cpT}{\varrho e_0}\right)\|u_n\|^2,
\end{aligned} \tag{69}$$

which implies that there exists a constant $C_2 > 0$ such that (45) holds. Next, we prove that assumption (i) of Lemma 7 still holds. From (V2)$''$, there exists $\delta \in (0, 1)$ such that

$$|\nabla V(t, x)| \le \frac{a_0}{2}|x| \quad \text{for } t \in \mathbb{R},\ |x| \le \delta. \tag{70}$$

By $V(t, 0) = 0$ and (70), we have

$$|V(t, x)| \le \frac{a_0}{4}|x|^2 \quad \text{for } t \in \mathbb{R},\ |x| \le \delta. \tag{71}$$

Let $\|u\| = \sqrt{2e_0\sqrt{a_0}}\delta := \rho$, and it follows from Lemma 6 that $|u(t)| \le \delta$. It follows from (31) and (71) that

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\|u\|^2 + \sum_{j=-\infty}^{\infty}\int_0^{u(t_j)} I(t)\,dt \\
&\quad - \int_{\mathbb{R}} e^{Q(t)} V(t, u(t))\,dt \\
&\ge \frac{1}{2}\|u\|^2 - \frac{cpT}{2e_0}\left\|u'\right\|_2^2 \\
&\quad - \frac{1}{4}\int_{\mathbb{R}} a_0 e^{Q(t)}|u(t)|^2 dt \\
&\ge \frac{1}{2}\|u\|^2 - \frac{cpT}{2e_0}\left\|u'\right\|_2^2 \\
&\quad - \frac{1}{4}\int_{\mathbb{R}} a(t)\, e^{Q(t)}|u(t)|^2 dt \\
&\ge \min\left\{\frac{1}{2} - \frac{cpT}{2e_0}, \frac{1}{4}\right\}\|u\|^2.
\end{aligned} \tag{72}$$

Therefore, we can choose a constant $\alpha > 0$ depending on ρ such that $\varphi(u) \ge \alpha$ for any $u \in E$ with $\|u\| = \rho$. The proof of Theorem 3 is complete. □

Proof of Theorem 4. Condition (V6) shows that φ is even. In view of the proof of Theorem 2, we know that $\varphi \in C^1(E, \mathbb{R})$ and satisfies (PS)-condition and assumption (i) of Lemma 7. Now, we prove that (iii) of Lemma 8 holds. Let E' be a finite dimensional subspace of E. Since all norms of a finite dimensional space are equivalent, there exists $d > 0$ such that

$$\|u\| \le d\|u\|_\infty, \quad u \in E'. \tag{73}$$

Assume that $\dim E' = m$ and $\{u_1, u_2, \ldots, u_m\}$ is a base of E' such that

$$\|u_i\| = d, \quad i = 1, 2, \ldots, m. \tag{74}$$

For any $u \in E'$, there exist $\lambda_i \in \mathbb{R}$, $i = 1, 2, \ldots, m$ such that

$$u(t) = \sum_{i=1}^{m} \lambda_i u_i(t) \quad \text{for } t \in \mathbb{R}. \tag{75}$$

Let

$$\|u\|_* = \sum_{i=1}^{m} |\lambda_i| \, \|u_i\|. \tag{76}$$

It is easy to see that $\|\cdot\|_*$ is a norm of E'. Hence, there exists a constant $d' > 0$ such that $d'\|u\|_* \le \|u\|$. Since $u_i \in E$, by Lemma 6, we can choose $R_2 > R$ such that

$$|u_i(t)| < \frac{d'\delta}{m + d'}, \qquad |t| > R_2, \quad i = 1, 2, \ldots, m, \tag{77}$$

where δ is given in (71). Let

$$\begin{aligned} \Theta &= \left\{ \sum_{i=1}^{m} \lambda_i u_i(t) : \lambda_i \in \mathbb{R}, i = 1, 2, \ldots, m; \sum_{i=1}^{m} |\lambda_i| = 1 \right\} \\ &= \left\{ u \in E' : \|u\|_* = d \right\}. \end{aligned} \tag{78}$$

Hence, for $u \in \Theta$, let $t_0 = t_0(u) \in \mathbb{R}$ such that

$$|u(t_0)| = \|u\|_\infty. \tag{79}$$

Then, by (73)–(76), (78), and (79), we have

$$\begin{aligned} dd' &= dd' \sum_{i=1}^{m} |\lambda_i| = d' \sum_{i=1}^{m} |\lambda_i| \, \|u_i\| = d'\|u\|_* \\ &\le \|u\| \le d\|u\|_\infty = d\,|u(t_0)| = d\left|\sum_{i=1}^{m} \lambda_i u_i(t_0)\right| \\ &\le d\sum_{i=1}^{m} |\lambda_i| \, |u_i(t_0)|, \quad u \in \Theta. \end{aligned} \tag{80}$$

This shows that $|u(t_0)| \ge d'$ and there exists $i_0 \in \{1, 2, \ldots, m\}$ such that $|u_{i_0}(t_0)| \ge d'/m$, which together with (77) implies that $|t_0| \le R_2$. Let $R_3 = R_2 + 1$ and

$$\begin{aligned} \gamma = \min \Bigg\{ & e^{Q(t)} V_1(t, x) : -R_3 \le t \le R_3, \\ & \frac{d'}{\sqrt{2}} \le |x| \le \frac{d}{\sqrt{2e_0\sqrt{a_0}}} \Bigg\}. \end{aligned} \tag{81}$$

Since $V_1(t, x) > 0$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R} \setminus \{0\}$ and $V_1 \in C^1(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, it follows that $\gamma > 0$. For any $u \in E$, from Lemmas 6 and 10 (i), we have

$$\begin{aligned} &\int_{-R_3}^{R_3} e^{Q(t)} V_2(t, u(t))\, dt \\ &= \int_{\{t\in[-R_3,R_3]:|u(t)|>1\}} e^{Q(t)} V_2(t, u(t))\, dt \\ &\quad + \int_{\{t\in[-R_3,R_3]:|u(t)|\le 1\}} e^{Q(t)} V_2(t, u(t))\, dt \\ &\le \int_{\{t\in[-R_3,R_3]:|u(t)|>1\}} e^{Q(t)} V_2\left(t, \frac{u(t)}{|u(t)|}\right) |u(t)|^{\varrho} dt \\ &\quad + \int_{-R_3}^{R_3} e^{Q(t)} \max_{|x|\le 1} V_2(t, x)\, dt \\ &\le \|u\|_\infty^{\varrho} \int_{-R_3}^{R_3} e^{Q(t)} \max_{|x|=1} V_2(t, x)\, dt \\ &\quad + \int_{-R_3}^{R_3} e^{Q(t)} \max_{|x|\le 1} V_2(t, x)\, dt \\ &\le \left(\frac{1}{\sqrt{2e_0\sqrt{a_0}}}\right)^{\varrho} \|u\|^{\varrho} \int_{-R_3}^{R_3} e^{Q(t)} \max_{|x|=1} V_2(t, x)\, dt \\ &\quad + \int_{-R_3}^{R_3} e^{Q(t)} \max_{|x|\le 1} V_2(t, x)\, dt \\ &= C_7\|u\|^{\varrho} + C_8, \end{aligned} \tag{82}$$

where $C_7 = (1/\sqrt{2e_0\sqrt{a_0}})^{\varrho} \int_{-R_3}^{R_3} e^{Q(t)} \max_{|x|=1} V_2(t, x) dt$ and $C_8 = \int_{-R_3}^{R_3} e^{Q(t)} \max_{|x|\le 1} V_2(t, x) dt$. Since $u_i' \in L^2(e^{Q(t)})$, $i = 1, 2, \ldots, m$, it follows that there exists $\varepsilon_1 \in (0, (d')^2 e_0/32m^2d^2)$ such that

$$\begin{aligned} &\int_{t-\varepsilon_1}^{t+\varepsilon_1} |u_i'(s)|\, ds \\ &\quad = \int_{t-\varepsilon_1}^{t+\varepsilon_1} e^{-Q(s)/2} e^{Q(s)/2} |u_i'(s)|\, ds \\ &\quad \le \frac{1}{\sqrt{e_0}} \int_{t-\varepsilon_1}^{t+\varepsilon_1} e^{Q(s)/2} |u_i'(s)|\, ds \\ &\quad \le \frac{1}{\sqrt{e_0}} (2\varepsilon_1)^{1/2} \left(\int_{t-\varepsilon}^{t+\varepsilon_1} e^{Q(s)} |u_i'(s)|^2 ds\right)^{1/2} \\ &\quad \le \left(\frac{2\varepsilon_1}{e_0}\right)^{1/2} \|u_i'\|_2 \\ &\quad \le \frac{d'}{4m} \quad \text{for } t \in \mathbb{R},\ i = 1, 2, \ldots, m. \end{aligned} \tag{83}$$

Then, for $u \in \Theta$ with $|u(t_0)| = \|u\|_\infty$ and $t \in [t_0 - \varepsilon_1, t_0 + \varepsilon_1]$, it follows from (75), (78), (79), (80), and (83) that

$$\begin{aligned} |u(t)|^2 &= |u(t_0)|^2 + 2\int_{t_0}^{t} (u'(s), u(s))\, ds \\ &\geq |u(t_0)|^2 - 2\int_{t_0-\varepsilon_1}^{t_0+\varepsilon_1} |u(s)| |u'(s)|\, ds \\ &\geq |u(t_0)|^2 - 2|u(t_0)| \int_{t_0-\varepsilon_1}^{t_0+\varepsilon_1} |u'(s)|\, ds \\ &\geq |u(t_0)|^2 - 2|u(t_0)| \sum_{i=1}^{m} |\lambda_i| \int_{t_0-\varepsilon_1}^{t_0+\varepsilon_1} |u_i'(s)|\, ds \\ &\geq \frac{(d')^2}{2}. \end{aligned} \tag{84}$$

On the other hand, since $\|u\| \leq d$ for $u \in \Theta$, then

$$|u(t)| \leq \|u\|_\infty \leq \frac{d}{\sqrt{2e_0\sqrt{a_0}}}, \quad t \in \mathbb{R},\ u \in \Theta. \tag{85}$$

Therefore, from (81), (84), and (85), we have

$$\begin{aligned} &\int_{-R_3}^{R_3} e^{Q(t)} V_1(t, u(t))\, dt \\ &\quad \geq \int_{t_0-\varepsilon_1}^{t_0+\varepsilon_1} e^{Q(t)} V_1(t, u(t))\, dt \geq 2\varepsilon_1 \gamma \quad \text{for } u \in \Theta. \end{aligned} \tag{86}$$

By (77) and (78), we have

$$|u(t)| \leq \sum_{i=1}^{m} |\lambda_i| |u_i(t)| \leq \delta \quad \text{for } |t| \geq R_2,\ u \in \Theta. \tag{87}$$

By (26), (31), (58), (82), (86), (87), and Lemma 10, we have for $u \in \Theta$ and $r > 1$

$$\begin{aligned} &\varphi(ru) \\ &= \frac{r^2}{2}\|u\|^2 + \sum_{j=-\infty}^{\infty} \int_0^{ru(t_j)} I(t)\, dt \\ &\quad + \int_{\mathbb{R}} e^{Q(t)} [V_2(t, ru(t)) - V_1(t, ru(t))]\, dt \\ &\leq \frac{r^2}{2}\|u\|^2 + \frac{cpTr^2}{2e_0}\|u'\|_2^2 + r^\varrho \int_{\mathbb{R}} e^{Q(t)} V_2(t, u(t))\, dt \\ &\quad - r^\mu \int_{\mathbb{R}} e^{Q(t)} V_1(t, u(t))\, dt \\ &= \frac{r^2}{2}\|u\|^2 + \frac{cpTr^2}{2e_0}\|u'\|_2^2 \\ &\quad + r^\varrho \int_{\mathbb{R}\setminus(-R_3,R_3)} e^{Q(t)} V_2(t, u(t))\, dt \\ &\quad - r^\mu \int_{-R_3}^{R_3} e^{Q(t)} V_1(t, u(t))\, dt \\ &\quad - r^\mu \int_{\mathbb{R}\setminus(-R_3,R_3)} e^{Q(t)} V_1(t, u(t))\, dt \\ &\quad + r^\varrho \int_{-R_3}^{R_3} e^{Q(t)} V_2(t, u(t))\, dt \\ &\leq \frac{r^2}{2}\|u\|^2 + \frac{cpTr^2}{2e_0}\|u'\|_2^2 \\ &\quad - r^\varrho \int_{\mathbb{R}\setminus(-R_3,R_3)} e^{Q(t)} V(t, u(t))\, dt \\ &\quad - r^\mu \int_{-R_3}^{R_3} e^{Q(t)} V_1(t, u(t))\, dt + r^\varrho \int_{-R_3}^{R_3} e^{Q(t)} V_2(t, u(t))\, dt \\ &\leq \frac{r^2}{2}\|u\|^2 + \frac{cpTr^2}{2e_0}\|u'\|_2^2 + \frac{r^\varrho}{4} \int_{\mathbb{R}\setminus(-R_3,R_3)} a_0 e^{Q(t)} |u(t)|^2 dt \\ &\quad + r^\varrho (C_7\|u\|^\varrho + C_8) - 2\varepsilon_1 \gamma r^\mu \\ &\leq \frac{r^2}{2}\|u\|^2 + \frac{cpTr^2}{2e_0}\|u'\|_2^2 \\ &\quad + \frac{r^\varrho}{4} \int_{\mathbb{R}\setminus(-R_3,R_3)} a(t) e^{Q(t)} |u(t)|^2 dt \\ &\quad + r^\varrho (C_7\|u\|^\varrho + C_8) - 2\varepsilon_1 \gamma r^\mu \\ &\leq \left(\frac{1}{2} + \frac{cpT}{2e_0}\right) r^2\|u\|^2 + \frac{r^\varrho}{4}\|u\|^2 + r^\varrho (C_7\|u\|^\varrho + C_8) \\ &\quad - 2\varepsilon_1 \gamma r^\mu \\ &\leq \left(\frac{1}{2} + \frac{cpT}{2e_0}\right) r^2 d^2 + \frac{r^\varrho}{4} d^2 + C_7 (rd)^\varrho + C_8 r^\varrho - 2\varepsilon_1 \gamma r^\mu. \end{aligned} \tag{88}$$

Since $\mu > \varrho > 2$, there exists $r_0 = r_0(c, p, T, e_0, d, d', C_7, C_8, R_2, R_3, \varepsilon_1, \gamma) = r_0(E') > 1$ such that

$$\varphi(ru) < 0 \quad \text{for } u \in \Theta,\ r \geq r_0. \tag{89}$$

It follows that

$$\varphi(u) < 0 \quad \text{for } u \in E',\ \|u\| \geq dr_0, \tag{90}$$

which shows that (iii) of Lemma 8 holds. By Lemma 8, φ possesses an unbounded sequence $\{c_n\}_{n=1}^{\infty}$ of critical values with $c_n = \varphi(u_n)$, where u_n is such that $\varphi'(u_n) = 0$ for $n = 1, 2, \ldots$. If $\{\|u_n\|\}$ is bounded, then there exists $C_9 > 0$ such that

$$\|u_n\| \leq C_9 \quad \text{for } n \in \mathbb{N}. \tag{91}$$

By a similar fashion for the proof of (49) and (50), for the given δ in (58), there exists $R_4 > R$ such that

$$|u_n(t)| \le \delta \quad \text{for } |t| \ge R_4,\ n \in \mathbb{N}. \tag{92}$$

Hence, by (22), (26), (31), (58), (91), and (92), we have

$$\begin{aligned}
&\frac{1}{2}\|u_n\|^2 \\
&= c_n + \int_{\mathbb{R}} e^{Q(t)} V(t, u_n(t))\, dt - \sum_{j=-\infty}^{\infty} \int_0^{u_n(t_j)} I(t)\, dt \\
&= c_n + \int_{\mathbb{R}\setminus(-R_4,R_4)} e^{Q(t)} V(t, u_n(t))\, dt \\
&\quad + \int_{-R_4}^{R_4} e^{Q(t)} V(t, u_n(t))\, dt - \sum_{j=-\infty}^{\infty} \int_0^{u_n(t_j)} I(t)\, dt \\
&\ge c_n - \frac{1}{4}\int_{\mathbb{R}\setminus(-R_4,R_4)} a_0 e^{Q(t)} |u_n(t)|^2 dt \\
&\quad - \int_{-R_4}^{R_4} e^{Q(t)} |V(t, u_n(t))|\, dt - \frac{cpT}{2e_0}\|u_n'\|_2^2 \\
&\ge c_n - \frac{1}{4}\int_{\mathbb{R}\setminus(-R_4,R_4)} e^{Q(t)} a(t) |u_n(t)|^2 dt \\
&\quad - \int_{-R_4}^{R_4} e^{Q(t)} |V(t, u_n(t))|\, dt - \frac{cpT}{2e_0}\|u_n'\|_2^2 \\
&\ge c_n - \frac{1}{4}\|u_n\|^2 - \int_{-R_4}^{R_4} e^{Q(t)} \max_{|x|\le\sqrt{2e_0\sqrt{a_0}}C_9} |V(t,x)|\, dt \\
&\quad - \frac{cpT}{2e_0}\|u_n\|^2.
\end{aligned} \tag{93}$$

It follows from (93) that

$$\begin{aligned}
c_n &\le \left(\frac{3}{4} + \frac{cpT}{2e_0}\right)\|u_n\|^2 \\
&\quad + \int_{-R_3}^{R_3} e^{Q(t)} \max_{|x|\le\sqrt{2e_0\sqrt{a_0}}C_9} |V(t,x)|\, dt < +\infty.
\end{aligned} \tag{94}$$

This contradicts the fact that $\{c_n\}_{n=1}^{\infty}$ is unbounded, and so $\{\|u_n\|\}$ is unbounded. The proof is complete. □

Proof of Theorem 5. In view of the proofs of Theorems 3 and 4, the conclusion of Theorem 5 holds. The proof is complete. □

4. Examples

Example 1. Consider the following system:

$$\begin{aligned}
&u''(t) + t^3 u'(t) - a(t) u(t) \\
&\quad + \nabla V(t, u(t)) = 0, \quad \text{a.e. } t \in (t_j, t_{j+1}),\ j \in \mathbb{Z}, \\
&\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-) = I(u(t_j)), \quad j \in \mathbb{Z},
\end{aligned} \tag{95}$$

where $q(t) = t^3$, $t \in \mathbb{R}$, $u \in \mathbb{R}$, $a \in C(\mathbb{R}, (0, +\infty))$, and $a(t) \to +\infty$ as $|t| \to +\infty$. Let

$$\begin{aligned}
V(t,x) &= \sum_{i=1}^{m} (a_i + 1 + \sin t) |x|^{\mu_i} - \sum_{j=1}^{n} (b_j + 1 + \cos t) |x|^{\varrho_j}, \\
I(x) &= \frac{(\varrho_1 - 2)x}{4(\varrho_1 + 2)p\pi},
\end{aligned} \tag{96}$$

where $\mu_1 > \mu_2 > \cdots > \mu_m > \varrho_1 > \varrho_2 > \cdots > \varrho_n > 2$, $a_i, b_j > 0$, $i = 1, \ldots, m$, and $j = 1, \ldots, n$. Let

$$\begin{aligned}
V_1(t,x) &= \sum_{i=1}^{m} (a_i + 1 + \sin t) |x|^{\mu_i}, \\
V_2(t,x) &= \sum_{j=1}^{n} (b_j + 1 + \cos t) |x|^{\varrho_j}.
\end{aligned} \tag{97}$$

Then, it is easy to check that all the conditions of Theorem 4 are satisfied with $\mu = \mu_m$ and $\varrho = \varrho_1$. Hence, problem (95) has an unbounded sequence of fast homoclinic solutions.

Example 2. Consider the following system:

$$\begin{aligned}
&u''(t) + (t + t^3) u'(t) - a(t) u(t) \\
&\quad + \nabla V(t, u(t)) = 0, \quad \text{a.e. } t \in (t_j, t_{j+1}),\ j \in \mathbb{Z}, \\
&\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-) = I(u(t_j)), \quad j \in \mathbb{Z},
\end{aligned} \tag{98}$$

where $q(t) = t + t^3$, $t \in \mathbb{R}$, $u \in \mathbb{R}$, $a \in C(\mathbb{R}, (0, +\infty))$, and $a(t) \to +\infty$ as $|t| \to +\infty$. Let

$$\begin{aligned}
V(t,x) = &\ (a_1 + 1 + \sin t)|x|^{\mu_1} + (a_2 + 2 + \sin t)|x|^{\mu_2} \\
&- b_1(\cos t)|x|^{\varrho_1} - (b_2 + 1 + \cos t)|x|^{\varrho_2},
\end{aligned} \tag{99}$$

where $\mu_1 > \mu_2 > \varrho_1 > \varrho_2 > 2$, $a_1, a_2 > 0$, b_1, and $b_2 > 0$. Let

$$\begin{aligned}
V_1(t,x) &= (a_1 + 1 + \sin t)|x|^{\mu_1} + (a_2 + 2 + \sin t)|x|^{\mu_2}, \\
V_2(t,x) &= b_1(\cos t)|x|^{\varrho_1} + (b_2 + 1 + \cos t)|x|^{\varrho_2}, \\
I(x) &= \frac{(\varrho_1 - 2)x}{6(\varrho_1 + 2)p\pi}.
\end{aligned} \tag{100}$$

Then, it is easy to check that all the conditions of Theorem 5 are satisfied with $\mu = \mu_2$ and $\varrho = \varrho_1$. Hence, by Theorem 5, problem (98) has an unbounded sequence of fast homoclinic solutions.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work is supported by the NNSF of China (no. 11301108), Guangxi Natural Science Foundation (no. 2013GXNSFBA019004), and the Scientific Research Foundation of Guangxi Education Office (no. 201203YB093).

References

[1] P. Chen, X. Tang, and R. P. Agarwal, "Fast homoclinic solutions for a class of damped vibration problems," *Applied Mathematics and Computation*, vol. 219, no. 11, pp. 6053–6065, 2013.

[2] H. Fang and H. Duan, "Existence of nontrivial weak homoclinic orbits for second-order impulsive differential equations," *Boundary Value Problems*, vol. 2012, article 138, 2012.

[3] D. D. Bainov and P. S. Simeonov, *Impulsive Differential Equations: Periodic Solutions and Applications*, Longman Scientific & Tecgnical, New York, NY, USA, 1993.

[4] V. Lakshmikantham, D. D. Bainov, and P. S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific Press, Singapore, 1989.

[5] P. Chen and X. H. Tang, "New existence and multiplicity of solutions for some Dirichlet problems with impulsive effects," *Mathematical and Computer Modelling*, vol. 55, no. 3-4, pp. 723–739, 2012.

[6] H. Chen and J. Sun, "An application of variational method to second-order impulsive differential equation on the half-line," *Applied Mathematics and Computation*, vol. 217, no. 5, pp. 1863–1869, 2010.

[7] Q. Zhang, W.-Z. Gong, and X. H. Tang, "Existence of subharmonic solutions for a class of second-order p-Laplacian systems with impulsive effects," *Journal of Applied Mathematics*, vol. 2012, Article ID 434938, 18 pages, 2012.

[8] Z. Luo, J. Xiao, and Y. Xu, "Subharmonic solutions with prescribed minimal period for some second-order impulsive differential equations," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 75, no. 4, pp. 2249–2255, 2012.

[9] J. J. Nieto and D. O'Regan, "Variational approach to impulsive differential equations," *Nonlinear Analysis: Real World Applications*, vol. 10, no. 2, pp. 680–690, 2009.

[10] J. Sun and H. Chen, "Variational method to the impulsive equation with neumann boundary conditions," *Boundary Value Problems*, vol. 2009, Article ID 316812, 2009.

[11] J. Sun, H. Chen, and L. Yang, "The existence and multiplicity of solutions for an impulsive differential equation with two parameters via a variational method," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 73, no. 2, pp. 440–449, 2010.

[12] J. Sun, H. Chen, J. J. Nieto, and M. Otero-Novoa, "The multiplicity of solutions for perturbed second-order Hamiltonian systems with impulsive effects," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 72, no. 12, pp. 4575–4586, 2010.

[13] J. Sun, H. Chen, and J. J. Nieto, "Infinitely many solutions for second-order Hamiltonian system with impulsive effects," *Mathematical and Computer Modelling*, vol. 54, no. 1-2, pp. 544–555, 2011.

[14] D. Zhang and B. Dai, "Existence of solutions for nonlinear impulsive differential equations with Dirichlet boundary conditions," *Mathematical and Computer Modelling*, vol. 53, no. 5-6, pp. 1154–1161, 2011.

[15] Z. Zhang and R. Yuan, "An application of variational methods to Dirichlet boundary value problem with impulses," *Nonlinear Analysis: Real World Applications*, vol. 11, no. 1, pp. 155–162, 2010.

[16] J. Zhou and Y. Li, "Existence and multiplicity of solutions for some Dirichlet problems with impulsive effects," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 71, no. 7-8, pp. 2856–2865, 2009.

[17] J. Zhou and Y. Li, "Existence of solutions for a class of second-order Hamiltonian systems with impulsive effects," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 72, no. 3-4, pp. 1594–1603, 2010.

[18] X. Han and H. Zhang, "Periodic and homoclinic solutions generated by impulses for asymptotically linear and sublinear Hamiltonian system," *Journal of Computational and Applied Mathematics*, vol. 235, no. 5, pp. 1531–1541, 2011.

[19] H. Zhang and Z. Li, "Periodic and homoclinic solutions generated by impulses," *Nonlinear Analysis: Real World Applications*, vol. 12, no. 1, pp. 39–51, 2011.

[20] X. H. Tang and X. Lin, "Homoclinic solutions for a class of second-order Hamiltonian systems," *Journal of Mathematical Analysis and Applications*, vol. 354, no. 2, pp. 539–549, 2009.

[21] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, vol. 65 of *CBMS Regional Conference Series in Mathematics*, The American Mathematical Society, Providence, RI, USA, 1986.

[22] X. H. Tang and X. Lin, "Existence of infinitely many homoclinic orbits in Hamiltonian systems," *Proceedings of the Royal Society of Edinburgh A: Mathematics*, vol. 141, no. 5, pp. 1103–1119, 2011.