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Research Article

Multiplicity of Nontrivial Solutions for a Class of Nonlocal Elliptic Operators Systems of Kirchhoff Type

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We investigate the existence and multiplicity of nontrivial solutions for a Kirchhoff type problem involving the nonlocal integrodifferential operators with homogeneous Dirichlet boundary conditions. The main tool used for obtaining our result is Morse theory.

1. Introduction

This paper is concerned with the multiplicity of solutions to the following elliptic systems of Kirchhoff type involving the nonlocal integrodifferential operators:

$$\begin{aligned} &-M_1\left(\int_{\mathbb{R}^{2n}}\left|u(x)-u(y)\right|^2K_1(x-y)\,dx\,dy\right)\mathscr{L}_{K_1}u \\ &\quad = f(x,v) \quad \text{in } \Omega, \\ &-M_2\left(\int_{\mathbb{R}^{2n}}\left|v(x)-v(y)\right|^2K_2(x-y)\,dx\,dy\right)\mathscr{L}_{K_2}v \\ &\quad = g(x,u) \quad \text{in } \Omega, \\ &\qquad u=v=0 \quad \text{in } \mathbb{R}^n\setminus\Omega, \end{aligned} \tag{1}$$

where $\Omega\subset\mathbb{R}^n$ ($n\geq 2$) is a bounded domain with smooth boundary $\partial\Omega$ and $f,g:\Omega\times\mathbb{R}\to\mathbb{R}$ are two continuous functions. $M_i:\mathbb{R}^+\to\mathbb{R}^+$ ($i=1,2$) are two continuous functions whose properties will be introduced later. $\mathscr{L}_{K_i}$ ($i=1,2$) are the nonlocal operators defined by

$$\mathscr{L}_{K_i}u(x)=\frac{1}{2}\int_{\mathbb{R}^n}\left(u(x+y)+u(x-y)-2u(x)\right)K_i(y)\,dy, \quad x\in\mathbb{R}^n, \tag{2}$$

$i=1,2$; here $K_i:\mathbb{R}^n\setminus\{0\}\to(0,+\infty)$ is a function such that

$$mK_i\in L^1(\mathbb{R}^n), \quad i=1,2, \text{ where } m(x)=\min\left\{|x|^2,1\right\}; \tag{3}$$

there exist θ_i and $s_i\in(0,1)$ ($i=1,2$) such that

$$K_i(x)\geq\theta_i|x|^{-(n+2s_i)}, \quad \text{for any } x\in\mathbb{R}^n\setminus\{0\}; \tag{4}$$

$$K_i(x)=K_i(-x), \quad \forall x\in\mathbb{R}^n\setminus\{0\}. \tag{5}$$

A typical example for K_i is given by $K_i(x) = |x|^{-(n+2s_i)}$ $(i = 1, 2)$. In this case $\mathscr{L}_{K_i}$ is the fractional Laplace operator $-(-\Delta)^{s_i}$, where $-(-\Delta)^{s_i}$ is defined by

$$-(-\Delta)^{s_i} u(x) = \frac{1}{2}\int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s_i}} dy, \quad x \in \mathbb{R}^n; \tag{6}$$

here $s_i \in (0, 1)$ and $n > 2s_i$ $(i = 1, 2)$. The fractional Laplacian $-(-\Delta)^{s_i}$ is a classical linear integrodifferential operator of order $2s_i$ which gives the standard Laplacian when $s_i = 1$ (see [1]).

Denote by X_i the linear space of Lebesgue measurable functions $u : \mathbb{R}^n \to \mathbb{R}$ such that

$$\begin{aligned} &\text{the map } (x, y) \rightarrowtail (u(x) - u(y))^2 K_i(x - y) \\ &\text{is in } L^1(Q, dxdy), \end{aligned} \tag{7}$$

where $Q = (\mathbb{R}^n \times \mathbb{R}^n) \setminus \mathscr{O}$ and $\mathscr{O} = (\mathscr{C}\Omega) \times (\mathscr{C}\Omega) \subset \mathbb{R}^n \times \mathbb{R}^n$. The space X_i is endowed with the norm

$$\|u\|_{X_i} = \|u\|_{L^2(\mathbb{R}^n)} + \left(\int_{\mathbb{R}^n\times\mathbb{R}^n} |u(x) - u(y)|^2 K_i(x - y)\, dx\, dy\right)^{1/2}, \quad i = 1, 2. \tag{8}$$

The space Z_i denotes the closure of $C_0^\infty(\Omega)$ in X_i. By Lemmas 6 and 7 in [2], the space Z_i is a Hilbert space which can be endowed with the norm defined as

$$\|u\|_{Z_i} = \left(\int_Q |u(x) - u(y)|^2 K_i(x - y)\, dx\, dy\right)^{1/2}, \quad i = 1, 2. \tag{9}$$

Since $u = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$, we have that the integral in (8) and (9) can be extended to all $\mathbb{R}^{2n}$.

Let $E = Z_1 \times Z_2$ be the Cartesian product of two Hilbert spaces, which is a reflexive Banach space endowed with the norm

$$\|(u, v)\| = \|u\|_{Z_1} + \|v\|_{Z_2}. \tag{10}$$

Denote by $0 < \lambda_1 < \lambda_2 \le \cdots \le \lambda_k \le \cdots$ the eigenvalues of the following nonlocal operator eigenvalue problem:

$$\begin{aligned} -\mathscr{L}_{K_1} u &= \lambda u \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega. \end{aligned} \tag{11}$$

Similarly, denote by $0 < \mu_1 < \mu_2 \le \cdots \le \mu_k \le \cdots$ the eigenvalues of the following nonlocal operator eigenvalue problem:

$$\begin{aligned} -\mathscr{L}_{K_2} v &= \mu v \quad \text{in } \Omega, \\ v &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega. \end{aligned} \tag{12}$$

We say that $(u, v) \in E$ is a weak solution of system (1) if, for every $(\phi, \psi) \in E$, one has

$$\begin{aligned} &M_1\left(\|u\|_{Z_1}^2\right) \int_{\mathbb{R}^{2n}} (u(x) - u(y))(\phi(x) - \phi(y)) \\ &\quad \times K_1(x - y)\, dx\, dy + M_2\left(\|v\|_{Z_2}^2\right) \\ &\quad \times \int_{\mathbb{R}^{2n}} (v(x) - v(y))(\psi(x) - \psi(y)) K_2(x - y)\, dx\, dy \\ &\quad - \int_\Omega f(x, v)\phi(x)\, dx - \int_\Omega g(x, u)\psi(x)\, dx = 0. \end{aligned} \tag{13}$$

The fractional Laplacian and nonlocal operators of elliptic type arise in both pure mathematical research and concrete applications, since these operators occur in a quite natural way in many different contexts. For an elementary introduction to this topic, see [2] and the references therein. Recently, some elliptic boundary problems driven by the nonlocal integrodifferential operator $\mathscr{L}_K$ have been studied in the works [3–8].

Recently, problems involving Kirchhoff type operators have been studied in many papers; we refer to [9–13] in which the authors have used the variational method and topological method to get the existence of solutions.

In this paper, motivated by the above mentioned works, we will use Morse theory to investigate the multiplicity of solutions of problem (1). To the best of our knowledge, there is no effort being made in the literature to study the existence of solutions for problem (1). This paper will make some contribution to this research field.

In order to establish solutions for problem (1), we make the following assumptions.

(H1) $M_i : \mathbb{R}^+ \to \mathbb{R}^+$ $(i = 1, 2)$ are two continuous functions, and there exist constants $m_1, m_2, M_1, M_2 > 0$ such that

$$m_i \le M_i(t) \le M_i, \quad i = 1, 2,\ \forall t \ge 0. \tag{14}$$

(H2) $f(x, v)$ and $g(x, u)$ are two continuous functions with the subcritical growth; that is, there exist some positive constants C_1, C_2 such that

$$\begin{aligned} |f(x, v)| &\le C_1\left(1 + |v|^{p-1}\right), \\ |g(x, u)| &\le C_2\left(1 + |u|^{q-1}\right), \\ &\forall x \in \Omega, \quad u, v \in \mathbb{R} \end{aligned} \tag{15}$$

hold, where $1 < p < 2_{s_1}^* = 2n/(n - 2s_1)$, $1 < q < 2_{s_2}^* = 2n/(n - s_2)$.

(H3) There exists $r > 0$, $\overline{\lambda} \in (\lambda_1, \lambda_2)$ and $\overline{\mu} \in (\mu_1, \mu_2)$ such that $M_1\lambda_1 < m_1\overline{\lambda}$, $M_2\mu_1 < m_2\overline{\mu}$, and $|u|, |v| \le r$ implies

$$\frac{1}{2}M_2\mu_1 v^2 \le F(x,v) \le \frac{1}{2}m_2\overline{\mu}v^2,$$
$$\frac{1}{2}M_1\lambda_1 u^2 \le G(x,u) \le \frac{1}{2}m_1\overline{\lambda}u^2, \qquad (16)$$
$$\text{a.e. } x \in \Omega.$$

(H4) $\lim_{|v|\to\infty}(F(x,v)/v^2) < (1/2)m_2\mu_1$, $\lim_{|u|\to\infty}(G(x,u)/u^2) < (1/2)m_1\lambda_1$, uniformly for all a.e. $x \in \overline{\Omega}$.

The main result of this paper is as follows.

Theorem 1. *If (H1)–(H4) hold, then the problem* (1) *has at least two nontrivial weak solutions in E.*

2. Preliminaries

For each $(u, v) \in E$, we define the functional $\mathcal{J} : E \to \mathbb{R}$ as follows:

$$\mathcal{J}(u,v) = \frac{1}{2}\widehat{M_1}\left(\|u\|_{Z_1}^2\right) + \frac{1}{2}\widehat{M_2}\left(\|v\|_{Z_2}^2\right) - \int_\Omega F(x,v)\,dx - \int_\Omega G(x,u)\,dx, \qquad (17)$$

where

$$\widehat{M_i}(t) = \int_0^t M_i(\tau)\,d\tau, \quad i = 1,2, \quad t \ge 0,$$
$$F(x,v) = \int_0^v f(x,s)\,ds, \qquad G(x,u) = \int_0^u g(x,s)\,ds. \qquad (18)$$

It is easy to check that (u, v) is a weak solution of problem (1) which is equivalent to being a critical point of the functional $\mathcal{J}$.

First let us recall the definition of the local linking which plays an important role in our paper.

Definition 2. Let X be a Banach space with a direct sum decomposition $X = X^1 \oplus X^2$. The functional $f \in C^1(X,\mathbb{R})$ has a local linking at 0 with respect to (X^1, X^2) if there is $r > 0$ such that

$$f(u) \ge 0, \quad \forall u \in X^1 \text{ with } \|u\| \le r$$
$$f(u) \le 0, \quad \forall u \in X^2 \text{ with } \|u\| \le r. \qquad (19)$$

Lemma 3. *Assume that (H1) and (H4) hold; then the functional $\mathcal{J}$ is coercive in E; that is, $\mathcal{J}(u,v) \to +\infty$ as $\|(u,v)\| \to \infty$.*

Proof. From (H4) and the continuity of the potentials F and G we have that, for some $\epsilon > 0$, there exists a positive constant C_3 such that

$$F(x,t) \le \frac{m_2}{2}(\mu_1 - \epsilon)|t|^2 + C_3,$$
$$G(x,t) \le \frac{m_1}{2}(\lambda_1 - \epsilon)|t|^2 + C_3, \qquad (20)$$
$$\forall t \in \mathbb{R}, \quad \text{a.e. } x \in \Omega.$$

Thus, by the Sobolev inequality [1] and (H1), for $(u, v) \in E$, we obtain

$$\begin{aligned}\mathcal{J}(u,v) &\ge \frac{m_1}{2}\|u\|_{Z_1}^2 + \frac{m_2}{2}\|v\|_{Z_2}^2 - \frac{m_1(\lambda_1-\epsilon)}{2}\int_\Omega u^2 dx \\ &\quad - \frac{m_2(\mu_1-\epsilon)}{2}\int_\Omega v^2 dx - 2C_3|\Omega| \\ &\ge \frac{m_1}{2}\left(1 - \frac{\lambda_1-\epsilon}{\lambda_1}\right)\|u\|_{Z_1}^2 + \frac{m_2}{2}\left(1 - \frac{\mu_1-\epsilon}{\mu_1}\right)\|v\|_{Z_2}^2 \\ &\quad - 2C_3|\Omega| \longrightarrow +\infty,\end{aligned} \qquad (21)$$

as $\|(u,v)\| \to \infty$. Hence, we have that $\mathcal{J}$ is coercive in E. □

Lemma 4. *If (H1), (H2), and (H4) hold, then $\mathcal{J}$ satisfies the (P.S.) condition.*

Proof. Let $\{z_n = (u_n, v_n)\}$ be a (PS) sequence of $\mathcal{J}$; then $\{(u_n, v_n)\}$ must be bounded by Lemma 3. Passing to a subsequence if necessary, there exists $z = (u, v) \in E$ such that $(u_n, v_n) \rightharpoonup (u, v)$ weakly in E. Thus, there exists a strictly decreasing subsequence ϵ_n, $\lim_{n\to\infty}\epsilon_n = 0$, such that

$$\left|\mathcal{J}'(u_n, v_n)(u_n - u, 0)\right| \le \epsilon_n \|u_n - u, 0\|. \qquad (22)$$

In particular,

$$\begin{aligned}&\Big|M_1\left(\|u_n\|_{Z_1}^2\right) \\ &\quad \times \int_{\mathbb{R}^{2n}} (u_n(x) - u_n(y)) \\ &\quad\quad \times ((u_n - u)(x) - (u_n - u)(y))K_1(x-y)\,dx\,dy \\ &\quad - \int_\Omega f(x, v_n)(u_n - u)\,dx\Big| \le \epsilon_n \|(u_n - u, 0)\|.\end{aligned} \qquad (23)$$

Since the potential F satisfies (H2) and by remark (3.2.24) in [14] we have

$$\int_\Omega f(x, v_n)(u_n - u)\,dx \longrightarrow 0. \qquad (24)$$

Combining (23) with (24), we obtain

$$\begin{aligned} m_1 \Big| \iint_{\mathbb{R}^{2n}} & (u_n(x) - u_n(y))((u_n - u)(x) - (u_n - u)(y)) \\ & \times K_1(x-y)\,dx\,dy \Big| \\ \le & \left| M_1\left(\|u_n\|_{Z_1}^2\right) \int_{\mathbb{R}^{2n}} (u_n(x) - u_n(y)) \right. \\ & \quad \times ((u_n - u)(x) - (u_n - u)(y)) \\ & \quad \left. \times K_1(x-y)\,dx\,dy \right| \\ \longrightarrow & \ 0. \end{aligned} \tag{25}$$

On the other hand, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^{2n}} (u(x) - u(y))((u_n - u)(x) - (u_n - u)(y)) \times K_1(x-y)\,dx\,dy = 0. \tag{26}$$

Adding (25) to (26), we conclude that

$$0 = \lim_{n\to\infty}\left[\int_{\mathbb{R}^{2n}} (u_n(x) - u_n(y))^2 K_1(x-y)\,dx\,dy - \int_{\mathbb{R}^{2n}} (u(x) - u(y))^2 K_1(x-y)\,dx\,dy\right], \tag{27}$$

which implies $\|u_n\|_{Z_1}^2 \to \|u\|_{Z_1}^2$. So, $\|u_n\|_{Z_1} \to \|u\|_{Z_1}$.

Similarly, we can obtain that $\|v_n\|_{Z_2} \to \|v\|_{Z_2}$. The uniform convexity of E yields that $\{z_n\}$ converges strongly to z in E.

Thanks to the fact that $L^{2_{s_1}^*}(\Omega) \hookrightarrow L^{p_1}(\Omega)$ $(2 < p_1 < 2_{s_1}^*)$ continuously, we get by Lemma 6 in [2] and (4) that

$$\begin{aligned} \|u\|_{L^{p_1}(\Omega)} &\le |\Omega|^{(2_{s_1}^* - p_1)/(p_1 2_{s_1}^*)} \|u\|_{L^{2_{s_1}^*}(\Omega)} \\ &\le |\Omega|^{(2_{s_1}^* - p_1)/(p_1 2_{s_1}^*)} \\ &\quad \times \sqrt{c_1}\left(\int_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s_1}}\,dx\,dy\right)^{1/2} \\ &\le C_4 \|u\|_{Z_1}, \end{aligned} \tag{28}$$

where $C_4 = |\Omega|^{(2_{s_1}^* - p_1)/(p_1 2_{s_1}^*)}\sqrt{c_1/\theta_1}$. Similarly, for $2 < q_1 < 2_{s_2}^*$, there exists a constant $C_5 > 0$ such that

$$\|v\|_{L^{q_1}(\Omega)} \le C_5 \|v\|_{Z_2}. \tag{29}$$

In the following, set $U = \operatorname{span}\{\varphi_1\} \times \operatorname{span}\{\psi_1\} := \langle\varphi_1\rangle \times \langle\psi_1\rangle$, where $\varphi_1 > 0$ with $\|\varphi_1\|_{Z_1} = 1$ is the corresponding eigenfunction of λ_1 and $\psi_1 > 0$ with $\|\psi_1\|_{Z_2} = 1$ is the corresponding eigenfunction of μ_1. Eigenvalues λ_1 and μ_1 are as in (11) and (12), respectively. Taking

$$V = \left\{(u,v) \in E : u \in \langle\varphi_1\rangle^{\perp},\ v \in \langle\psi_1\rangle^{\perp}\right\}, \tag{30}$$

we can easily know that V is complementary subspace of U. Hence we have the following direct sum:

$$E = U \oplus V. \tag{31}$$

If $(u,v) \in U$, from Proposition 9 in [4], we get

$$\|u\|_{Z_1}^2 = \lambda_1 \int_\Omega |u(x)|^2 dx, \qquad \|v\|_{Z_2}^2 = \mu_1 \int_\Omega |v(x)|^2 dx. \tag{32}$$

Moreover, if $(u,v) \in V$, by Proposition 9 in [4], we have

$$\|u\|_{Z_1}^2 \ge \lambda_2 \int_\Omega |u(x)|^2 dx, \qquad \|v\|_{Z_2}^2 \ge \mu_2 \int_\Omega |v(x)|^2 dx. \tag{33}$$

□

Lemma 5. *Assume that (H1)–(H3) hold. Then the functional $\mathcal{J}$ has a local linking at the origin with respect to $E = U \oplus V$.*

Proof. (i) Let $(u,v) \in U$. Since

$$\|(u,v)\| \longrightarrow 0 \Longrightarrow \int_\Omega |u(x)|^2 dx \longrightarrow 0,\ \int_\Omega |v(x)|^2 \longrightarrow 0 \tag{34}$$

by (32), we have that, for given $r > 0$, there is some $\rho > 0$ small enough such that

$$(u,v) \in U,\ \|(u,v)\| \le \rho \Longrightarrow |u(x)| \le r,\ |v(x)| \le r, \quad \text{a.e. } x \in \Omega. \tag{35}$$

Now on U, we have by (H1) and (H3) that, for $(u,v) \in U$ with $\|(u,v)\| \le \rho$,

$$\begin{aligned} \mathcal{J}(u,v) &= \frac{1}{2}\widehat{M_1}\left(\|u\|_{Z_1}^2\right) + \frac{1}{2}\widehat{M_2}\left(\|v\|_{Z_2}^2\right) - \int_\Omega F(x,v)\,dx \\ &\quad - \int_\Omega G(x,u)\,dx \\ &\le \frac{M_1}{2}\lambda_1 \int_\Omega |u|^2 dx + \frac{M_2}{2}\mu_1 \int_\Omega |v|^2 dx \\ &\quad - \int_\Omega F(x,v)\,dx - \int_\Omega G(x,u)\,dx \\ &= \int_{|u|\le r}\left(\frac{1}{2}M_1\lambda_1|u|^2 - G(x,u)\right)dx \\ &\quad + \int_{|v|\le r}\left(\frac{1}{2}M_2\mu_1|v|^2 - F(x,v)\right)dx \le 0. \end{aligned} \tag{36}$$

(ii) Let $(u,v) \in V$. By (33), similar to (34) and (35), we obtain by (H1)–(H3) that, for $(u,v) \in V$ with $\|(u,v)\| \le \rho$,

$$\begin{aligned}
\mathcal{J}(u,v) = {} & \frac{1}{2}\widehat{M_1}\left(\|u\|_{Z_1}^2\right) + \frac{1}{2}\widehat{M_2}\left(\|v\|_{Z_2}^2\right) - \frac{1}{2}m_1\overline{\lambda}\int_\Omega u^2 dx \\
& - \frac{1}{2}m_2\overline{\mu}\int_\Omega v^2 dx \\
& - \int_{\{|v|\le r\}}\left(F(x,v) - \frac{1}{2}m_2\overline{\mu}|v|^2\right)dx \\
& - \int_{\{|v|>r\}}\left(F(x,v) - \frac{1}{2}m_2\overline{\mu}|v|^2\right)dx \\
& - \int_{\{|u|\le r\}}\left(G(x,u) - \frac{1}{2}m_1\overline{\lambda}|u|^2\right)dx \\
& - \int_{\{|u|>r\}}\left(G(x,u) - \frac{1}{2}m_1\overline{\lambda}|u|^2\right)dx \\
\ge {} & \frac{m_1}{2}\left(1 - \frac{\overline{\lambda}}{\lambda_2}\right)\|u\|_{Z_1}^2 + \frac{m_2}{2}\left(1 - \frac{\overline{\mu}}{\mu_2}\right)\|v\|_{Z_2}^2 \\
& - \int_{\{|v|>r\}}\left(F(x,v) - \frac{1}{2}m_2\overline{\mu}|v|^2\right)dx \\
& - \int_{\{|u|>r\}}\left(G(x,u) - \frac{1}{2}m_1\overline{\lambda}|u|^2\right)dx \\
\ge {} & \frac{m_1}{2}\left(1 - \frac{\overline{\lambda}}{\lambda_2}\right)\|u\|_{Z_1}^2 + \frac{m_2}{2}\left(1 - \frac{\overline{\mu}}{\mu_2}\right)\|v\|_{Z_2}^2 \\
& - C_6\int_{\{|v|>r\}}|v|^{p_2}dx - C_7\int_{\{|u|>r\}}|u|^{q_2}dx \\
\ge {} & \frac{m_1}{2}\left(1 - \frac{\overline{\lambda}}{\lambda_2}\right)\|u\|_{Z_1}^2 + \frac{m_2}{2}\left(1 - \frac{\overline{\mu}}{\mu_2}\right)\|v\|_{Z_2}^2 \\
& - C_8\|u\|_{Z_1}^{p_2} - C_9\|v\|_{Z_2}^{q_2}, \quad (\text{by (28)-(29)}),
\end{aligned} \tag{37}$$

where C_i $(i = 6, \ldots, 9)$ are positive constants, $2 < p_2 < 2^*_{s_1}$, and $2 < q_2 < 2^*_{s_2}$. Thus, (37) implies that $\mathcal{J}(u,v) > 0$ for $0 < \|(u,v)\| \le \rho$ with $\rho > 0$ is small enough. The proof is complete. □

Let X be a real Banach space and $f \in C^1(X,\mathbb{R})$. Suppose p is an isolated critical point of f with $f(p) = c$ and U is a neighborhood of p, containing the unique critical point; the group

$$C_q(f,p) = H_q\left(f_c \cap U, f_c \cap U \setminus \{p\}\right), \quad q = 0,1,2,\ldots, \tag{38}$$

is called the qth critical group of f at p, where $f_c = \{u \in X : f(u) \le c\}$ and $H_q(\cdot,\cdot)$ is the qth singular relative homology group with integer coefficients.

Lemma 6 (see [15]). *Let E be a Banach space and $f : E \to \mathbb{R}$ a C^1-functional satisfying the (P.S) condition. Assume that f has a local linking to the decomposition $E = U \oplus V$ near the origin, where $\dim U = m < \infty$. If $0 \in E$ is the unique critical point of f in B_ρ, then*

$$C_m(f,0) = H_m\left(f_c \cap B_\rho, f_c \cap B_\rho \setminus \{0\}\right) \ne 0. \tag{39}$$

3. The Proof of Theorem 1

We say that u is a homological nontrivial critical point of f if at least one of its critical groups is nontrivial. By [16], we have the following abstract critical point theorem.

Lemma 7 (see [16]). *Let X be a real Banach space and let $\Phi \in C^1(X,\mathbb{R})$ satisfy the (P.S) condition and be bounded from below. If Φ has a critical point that is homologically nontrivial and is not the minimizer of Φ, then Φ has at least three critical points.*

From the proof of Lemma 3, we can conclude that $(0,0) \in E$ is the unique critical point of our $\mathcal{J}$ in a ball that is small enough. Since $\dim U = \dim\langle\varphi_1\rangle \times \langle\psi_1\rangle = 2 < \infty$, by Lemmas 5 and 6, we have the following lemma.

Lemma 8. *Let (H1)–(H3) hold. Then $(0,0)$ is a critical point of $\mathcal{J}$ and $C_2(\mathcal{J},(0,0)) \ne 0$.*

Proof of Theorem 1. By Lemmas 3 and 4, $\mathcal{J}$ is coercive and satisfies the (P.S) condition. Hence $\mathcal{J}$ is bounded below. By Lemma 8, $(0,0) \in E$ is homologically nontrivial critical point of $\mathcal{J}$ but not a minimizer. Then the conclusion follows from Lemma 7. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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